Inspiralling compact objects with generic deformations

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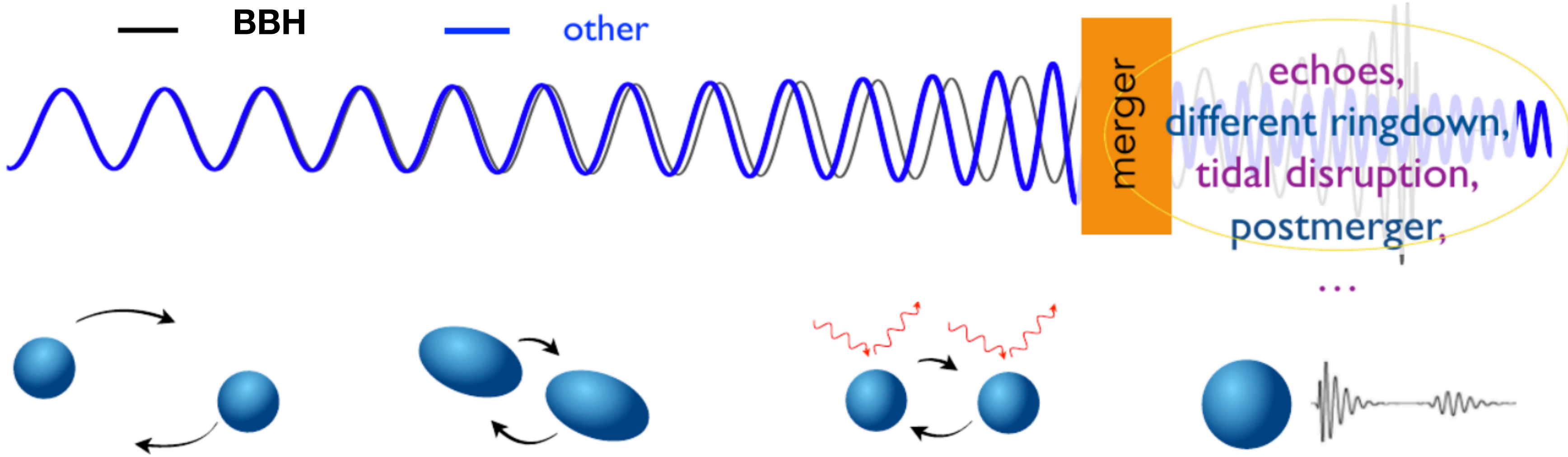
Nature of Compact Objects

Credit: Paolo Pani (slide concept by T. Hinderer and A. Maselli)



~ point masses: same signal for all objects

tidal effects + bodies' multipole moments

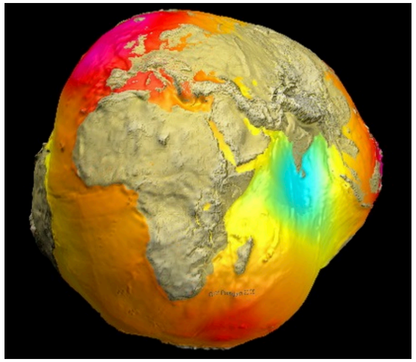
This talk

absorption effects: tidal heating

Nature of compact objects **imprinted** in gravitational waveform.

Multipolar structure of isolated bodies

- Structure of a self-gravitating object can be characterised through its **multipole moments**.



Credit: GFZ Potsdam

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi\rho(\mathbf{x})$$

$$\Phi_{\rm ext}(\mathbf{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{M_{\ell m}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell m}(\theta, \varphi)\,, \qquad M_{\ell m} = \sqrt{\frac{4\pi}{2\ell+1}} \int_V \rho(\mathbf{x})\, r^\ell Y^*_{\ell m} d^3x$$

Or in terms of **symmetric trace-free** tensors $M^{<i_1 \ldots i_\ell>}$:

$$\Phi_{\rm ext}(\mathbf{x}) = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \frac{(2\ell-1)!!}{\ell\,!} M^{<i_1 \ldots i_\ell>} n^i \ldots n^\ell\,, \qquad M^{<L>} = N_L \sum_{m=-\ell}^{\ell} \mathscr{Y}^{*<L>}_{\ell m} M_{\ell m}$$

(See e.g. Poisson & Will, “Gravity”)

Relativistic multipole moments

Geroch '70; Hansen '74; Thorne '80

- Stationary, asymptotically flat spacetimes can be described through a set of relativistic ***mass multipole moments*** $M_{\ell m}$ and ***current multipole moments*** $S_{\ell m}$.

$$g_{tt} + 1 \sim \sum_{\ell,m} \frac{M_{\ell m}}{r^{\ell+1}} Y_{\ell m}\,, \qquad g_{ti} \sim \sum \frac{S_{\ell m}}{r^{\ell+1}} Y^{B}_{i,\ell m}$$

(K. Thorne '80)

- **Axial symmetry** $\implies M_{\ell m} = 0\,,\ S_{\ell m} = 0$ if $m \neq 0$

- **Axial symmetry** and **equatorial symmetry**:

$\implies \quad M_{\ell 0} = 0$ if ℓ is **odd** $\quad \& \quad S_{\ell 0} = 0$ if ℓ is **even**

Testing the spacetime symmetries

- For a Kerr black hole (which is **axisymmetric** and **equatorially symmetric**) non-vanishing multipole moments **only depend** on its mass M and angular momentum J:

$$M^{\rm Kerr}_{\ell=2n\,m=0} = (-1)^n M\left(\frac{J}{M}\right)^{2n}\,,\quad S^{\rm Kerr}_{\ell=2n+1\,m=0} = (-1)^n M\left(\frac{J}{M}\right)^{2n+1}\,,\quad n=0,1,\ldots$$

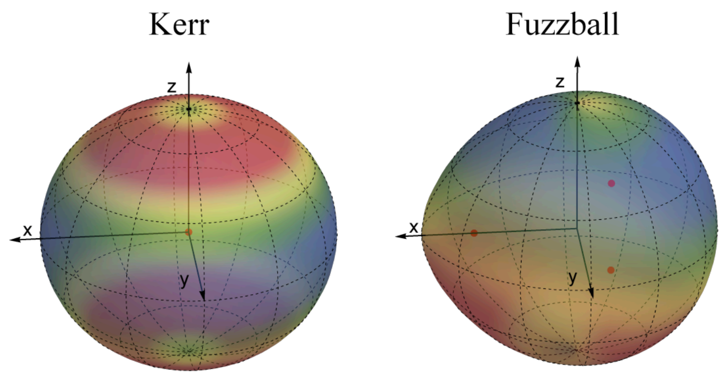
- Multipolar structure can be **much richer** for exotic objects and in beyond GR theories, with solutions breaking the symmetries of the Kerr metric. [fuzzballs: Bianchi+'20;Bena+ '20-'21; boson stars: Herdeiro+ '20] [See P. Pani's talk]

Breaking **equatorial symmetry**, e.g.

$$S_{20} \neq 0\,,\; M_{30} \neq 0$$

Breaking **axial symmetry**, e.g.

$$M_{2\pm1} \neq 0\,,\; M_{2\pm2} \neq 0$$



Credits: G. Raposo

Ryan's theorem

- Multipolar structure of orbiting bodies **imprinted** in gravitational waves emitted by a binary system. (F. Ryan, '95,'97)

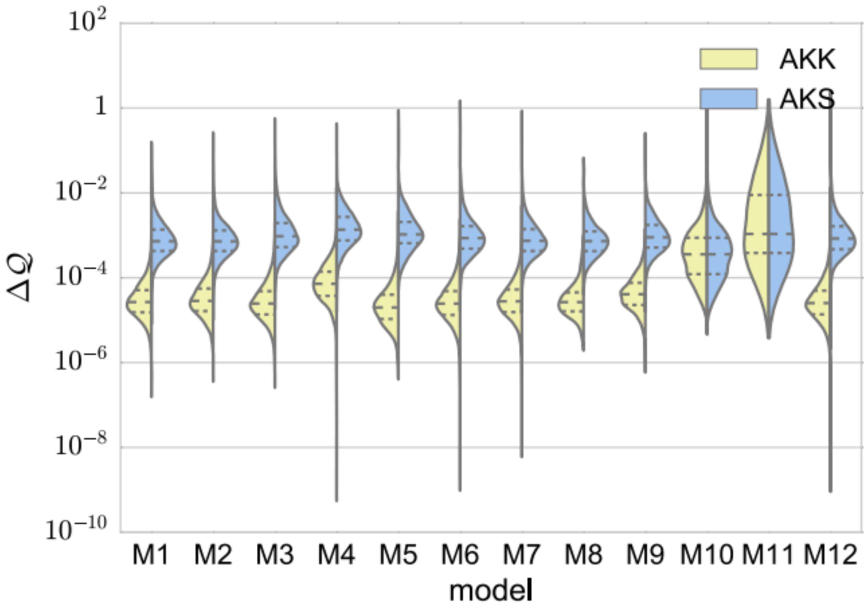
- Inspiral of a small body with mass μ orbiting an **axisymmetric** and **equatorially symmetric** massive body ($\tilde{u} \equiv (\pi M f)^{1/3} \ll 1$):

current dipole (i.e. spin)

mass quadrupole

$$f\,\frac{dE_{\rm GW}}{df} = \mu\left[\frac{1}{3}\tilde{u}^2 - \frac{1}{2}\tilde{u}^4 + \frac{20}{9}\frac{S_{10}}{M^2}\tilde{u}^5 + \left(-\frac{27}{8} + \frac{M_{20}}{M^3}\right)\tilde{u}^6 + \ldots\right]$$

higher-order PN terms (including higher multipoles)



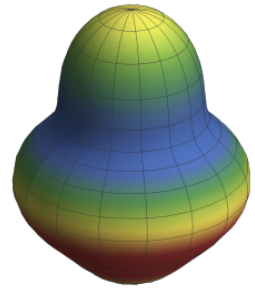
From: S. Babak *et al*, PRD95, 103012 (2017)

Deviations away from Kerr mass quadrupole $\Delta\mathcal{Q} \equiv (M_{20} - M_{20}^{\rm Kerr})/M^3$ measurable with accuracies $\sim 10^{-4} - 10^{-2}$ with **EMRIs**.

(Barack and Cutler, '06; Babak *et al*, 2017)

Constraining equatorial symmetry with LISA

K. Fransen & D.R. Mayerson, arXiv:2201.03569

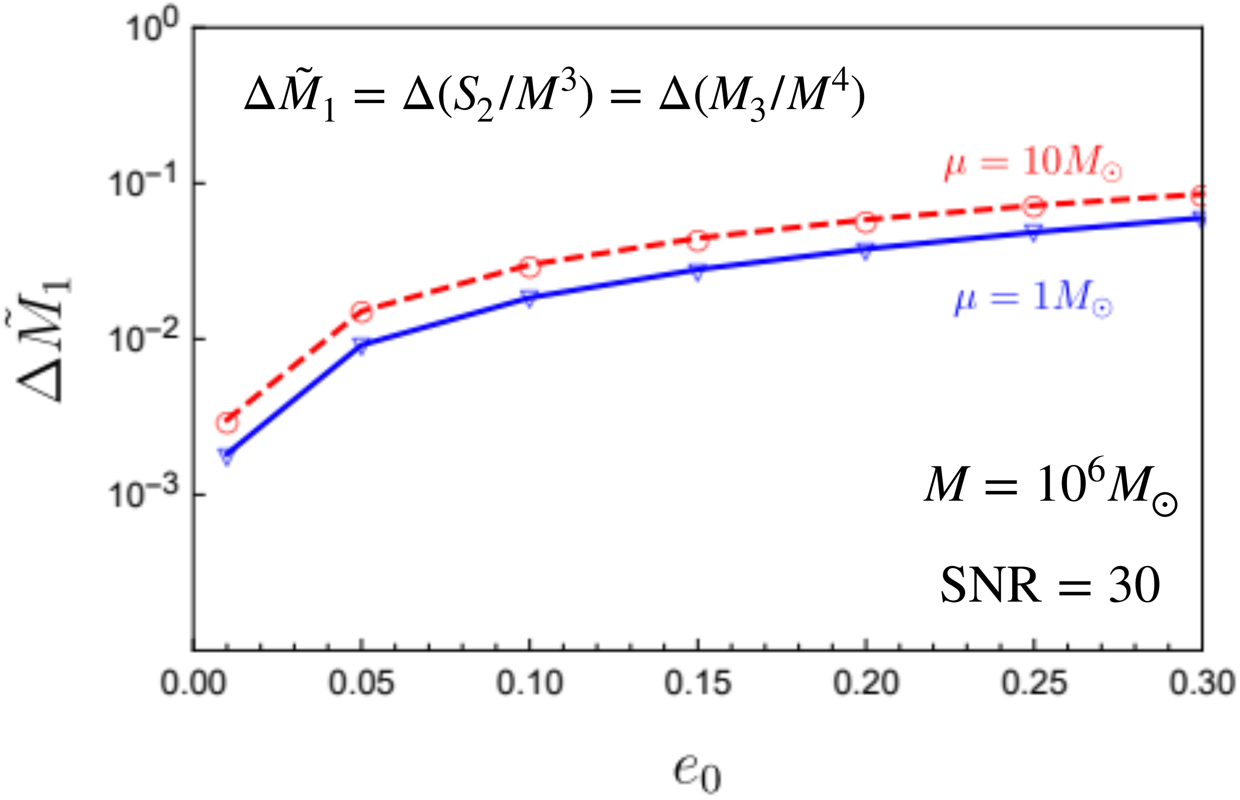


Credits: Raposo+ arXiv:1812.07615

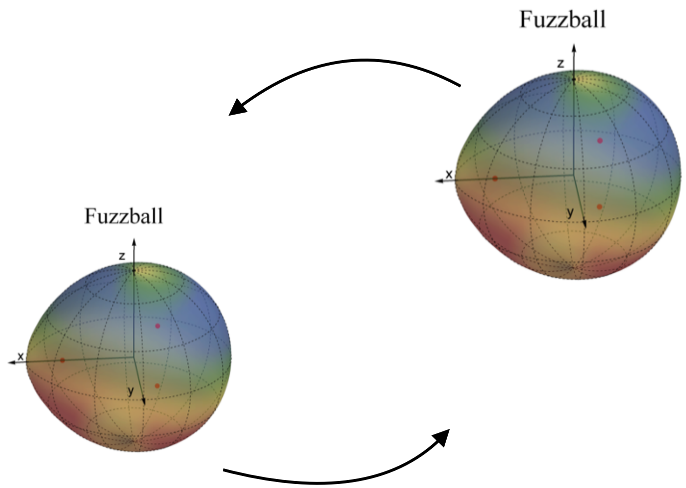
- EMRI around **non-equatorially symmetric** but **axisymmetric** body with $S_{20} \neq 0$ and $M_{30} \neq 0$ using extension of analytic kludge model by Barack&Cutler [arXiv:0310125 ;arXiv:0612029].

LISA could **constrain** deviations away from equatorial symmetry with accuracies $\mathcal{O}(1\%)$.

See: K. Fransen & D.R. Mayerson, arXiv:2201.03569 for details

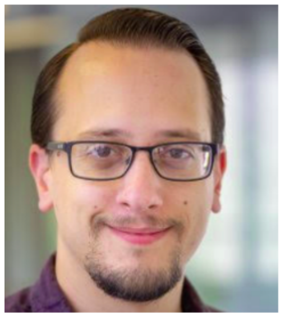


Inspiral of two bodies with no symmetries



Let's be even more generic and consider the inspiral of two deformed bodies with **no symmetries.**

See: Nick Loutrel, RB, Andrea Maselli & Paolo Pani, arXiv: 2203.01725



Conservative orbital dynamics

- Consider a two-body system with masses (M_1, M_2), each one endowed with a **generic mass quadrupole moment** (Q_1^{ij}, Q_2^{ij}).

- Conservative orbital dynamics can be described through a Lagrangian formulation. To leading-order in the mass quadrupole contribution:

[J. E Vines & É. Flanagan, Phys. Rev. D 88, 024046 (2013)]

$$n^{ij} = n^i n^j$$

$$\mu = \frac{M_1 M_2}{M}$$

$$M = M_1 + M_2$$

$$\eta_A = M_A / M$$

$$Q_{\rm eff}^{ij} = \eta_2 Q_1^{ij} - \eta_1 Q_2^{ij}$$

$$Q_A^{ij} \equiv M_A^{ij}$$

$$A = 1,2$$

point-particle spin contributions mass quadrupole

$$\mathscr{L} = \mathscr{L}_{\rm pp} + \mathscr{L}_{\rm spin} + \mathscr{L}_{\rm quad} + \dots$$

$$\mathscr{L}_{\rm pp} = \mathscr{L}_{\rm N} + c^{-2}\mathscr{L}_{\rm 1PN} + c^{-4}\mathscr{L}_{\rm 2PN} + \mathscr{O}\left(c^{-6}\right)$$

$$\mathscr{L}_{\rm N} = \frac{1}{2}\mu v^2 + \frac{\mu M}{r}, \qquad \mathscr{L}_{\rm quad} = \frac{3M}{2r^3} Q_{\rm eff}^{ij} n^{<ij>} + O\left(c^{-2}\right)$$

Conserved orbital energy: $E_{\rm orb} = \frac{1}{2}\mu v^2 - \frac{\mu M}{r} - \frac{3M}{2r^3} Q_{\rm eff}^{ij} n^{<ij>} \dots$

Equations of motion & Osculating orbits

(See e.g. Poisson & Will, "Gravity")

- Schematically equations of motion can be written as:

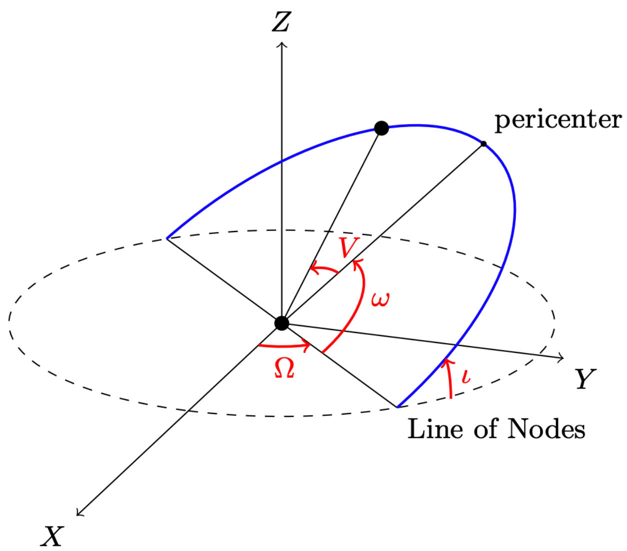
$$\frac{d}{dt}\vec{r}(t) = \vec{v}\,, \qquad \frac{d}{dt}\vec{v}(t) = \vec{f}_{\rm Newton} + \vec{f}_{\rm pert}$$

- $\vec{f}_{\rm pert} = 0 \implies$ Keplerian motion which in the "body" frame (X, Y, Z) can be parameterised by a set of five constants of motion: $\mu^a = [p, e, \iota, \omega, \Omega]$.

- Treat the problem as Keplerian orbits **perturbed** by a small additional "force" $\vec{f}_{\rm pert} = \vec{f}_{\rm quad}$.

- **Method of osculating orbits -** At each time t take orbit to be Keplerian with orbital elements $\mu^a \to \mu^a(t)$:

$$\vec{r}(t) = \vec{r}_{\rm Kepler}\left(t, \mu_a(t)\right)\,, \qquad \vec{v}(t) = \vec{v}_{\rm Kepler}\left(t, \mu_a(t)\right)$$



p - semi-latus rectum
e - Newtonian eccentricity
ι - inclination
ω - longitude of pericenter
Ω - longitude of ascending node

Secular variation

- $\vec{f}_{\text{quad}}$ introduces a new timescale in the problem, **longer** than orbital timescale, which lead to *secular changes* in the orbital elements: use *multiscale analysis*.

(See e.g. Poisson & Will, “Gravity”)

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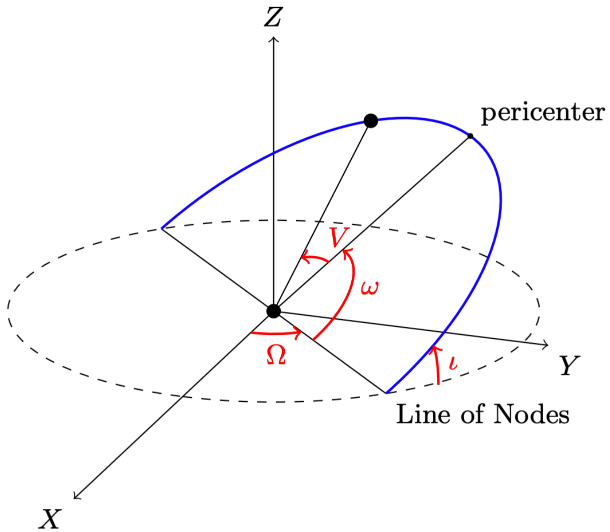
$$\left\langle \frac{dp}{dt} \right\rangle = \left\langle \frac{de}{dt} \right\rangle = 0$$

No secular changes for p, e
(but have periodic corrections on orbital timescale)

Secular variation



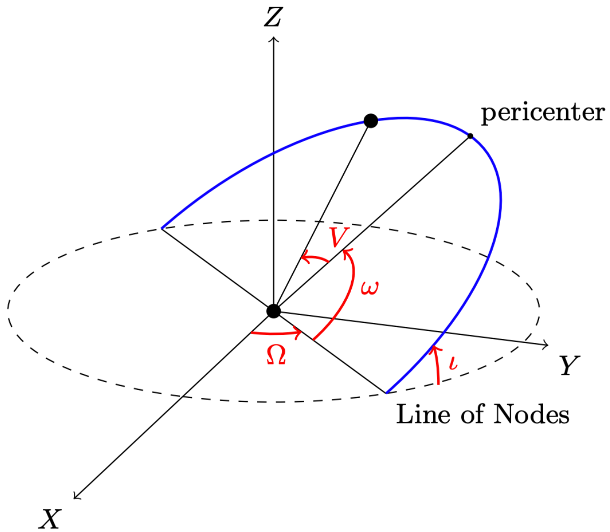
$$\left\langle \frac{dp}{dt} \right\rangle = \left\langle \frac{de}{dt} \right\rangle = 0$$

$$\left\langle \frac{d\iota}{dt} \right\rangle = \left(\frac{6\pi}{5} \right)^{1/2} \frac{(1-e^2)^{3/2}}{\nu M^{1/2} p^{7/2}} \left\{ \cos\iota(-Q^R_{+1}\cos\Omega + Q^I_{+1}\sin\Omega) + \sin\iota[Q^R_{+2}\sin(2\Omega) + Q^I_{+2}\cos(2\Omega)] \right\}$$

Nutation due to non-axisymmetric
mass quadrupole moment

$$Q_{21} = Q^R_{+1} + iQ^I_{+1}\,, \qquad Q_{22} = Q^R_{+1} + iQ^I_{+1}\,, \qquad Q_0 \equiv Q_{20}$$

Secular variation



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$$\left\langle \frac{d\omega}{dt} \right\rangle = \frac{1}{4}\left(\frac{\pi}{5} \right)^{1/2} \frac{(1-e^2)^{3/2}}{\nu M^{1/2} p^{7/2}} \left\{ -3Q_0\left[3+5\cos(2\iota)\right] + 2\sqrt{6}\left[3-5\cos(2\iota)\right]\cot\iota\left(Q^R_{+1}\sin\Omega + Q^I_{+1}\cos\Omega\right) + \sqrt{6}\left[1-5\cos(2\iota)\right]\left(Q^R_{+2}\cos(2\Omega) - Q^I_{+2}\sin(2\Omega)\right) \right\}$$

$$\left\langle \frac{d\Omega}{dt} \right\rangle = \left(\frac{\pi}{5} \right)^{1/2} \frac{(1-e^2)^{3/2}}{\nu M^{1/2} p^{7/2}} \left\{ \cos\iota[3Q_0 + \sqrt{6}Q^R_{+2}\cos(2\Omega) - \sqrt{6}Q^I_{+2}\sin(2\Omega)] + \sqrt{6}\cos(2\iota)\csc\iota(Q^R_{+1}\sin\Omega + Q^I_{+1}\cos\Omega) \right\}$$

Pericenter advance due to mass quadrupole moment

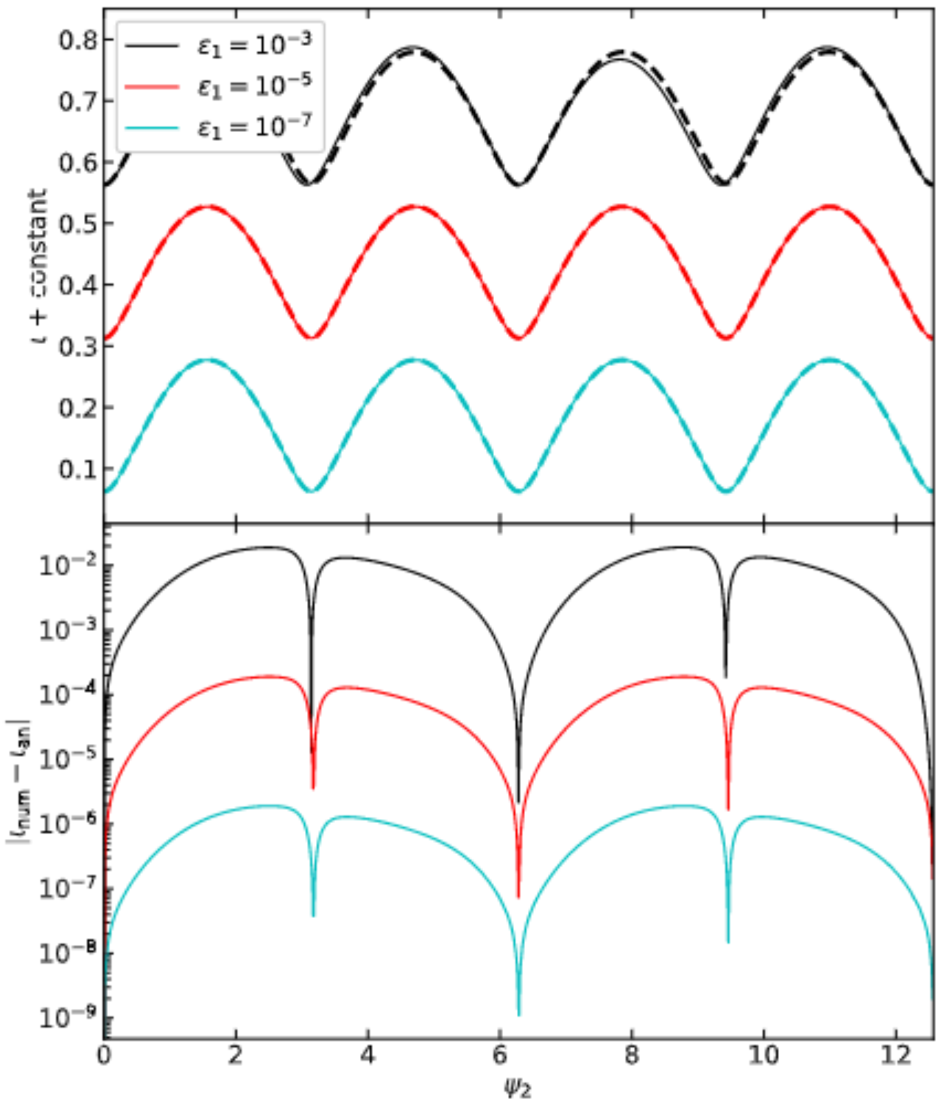
$$Q_{21} = Q^R_{+1} + iQ^I_{+1}\,, \qquad Q_{22} = Q^R_{+1} + iQ^I_{+1}\,, \qquad Q_0 \equiv Q_{20}$$

Secular mass quadrupole effects

- Closed-form analytic solutions for particular cases ($Q_{21} = 0$), whereas for generic mass quadrupole equations can be solved perturbatively assuming $Q_{21} \ll Q_{20}$

precession phase:

$$\boxed{\frac{d\psi_2}{dt} = 3\left(\frac{\pi}{5}\right)^{1/2} \frac{Q_0(1-e^2)^{3/2}}{\nu M^{1/2} p^{7/2}} \sqrt{1-\epsilon_2^2}\cos\iota}$$



$$\epsilon_m = \frac{1}{Q_0}\left[\frac{2}{3}\left((Q^R_{+m})^2 + (Q^I_{+m})^2\right)\right]^{1/2}$$

$$\alpha_m = \frac{1}{2}\tan^{-1}(Q^I_{+m}/Q^R_{+m})$$

Radiation reaction

- Gravitational-wave emission becomes important on radiation reaction timescale $t_{\rm RR} \gg t_{\rm prec.} \gg t_{\rm orb}$.

- Focusing on circular orbits, leading-order radiation reaction effects can be computed using:

Quadrupole formula

$$\mathscr{P} = \frac{1}{5c^5} \dddot{I}^{<ij>} \dddot{I}^{<ij>}$$

$$I^{ij} = \mu x^i x^j$$

Flux-balance law

$$\frac{dE_{\rm orb}}{dt} = -\langle \mathscr{P} \rangle$$

- Use hierarchy of timescales $t_{\rm RR} \gg t_{\rm prec.} \gg t_{\rm orb}$, to average over precession timescale (solved analytically for $\epsilon_1, \epsilon_2 \ll 1$), e.g. :

$$\boxed{\phi(\tilde{u}) = \phi_c + M^{-1} \int d\tilde{u}\ \tilde{u}^3 \left(\left\langle \frac{d\tilde{u}}{dt} \right\rangle_{\psi_2} \right)^{-1} = \phi_c - \frac{1}{32\nu \tilde{u}^5} \left[1 - \frac{Q_0 \tilde{u}^4}{8M^3 \nu} \sqrt{\frac{\pi}{5}} \epsilon_1^p \epsilon_2^q\ \mathscr{U}_{pq} \right]}$$

$$\tilde{u} = (2\pi M F)^{1/3}\,, \quad F = 1/T_{\rm orb}$$

Gravitational waveform

- Leading-order corrections to metric perturbation can be computed using quadrupole approximation:

$$\boxed{h_{ij} = \frac{2}{c^4 D_L} \ddot{I}_{\langle ij \rangle}}$$

- Time-domain waveform found by projecting h_{ij} into TT gauge:

$$\boxed{h = h_+ - i h_\times = \frac{\nu M}{D_L} \tilde{u}^2 \sum_{mn} A_{m,n}(\iota, \Omega) e^{in\phi} {}_{-2}Y_{2m}(\theta_N, \phi_N)}$$

$$n = \pm 2$$

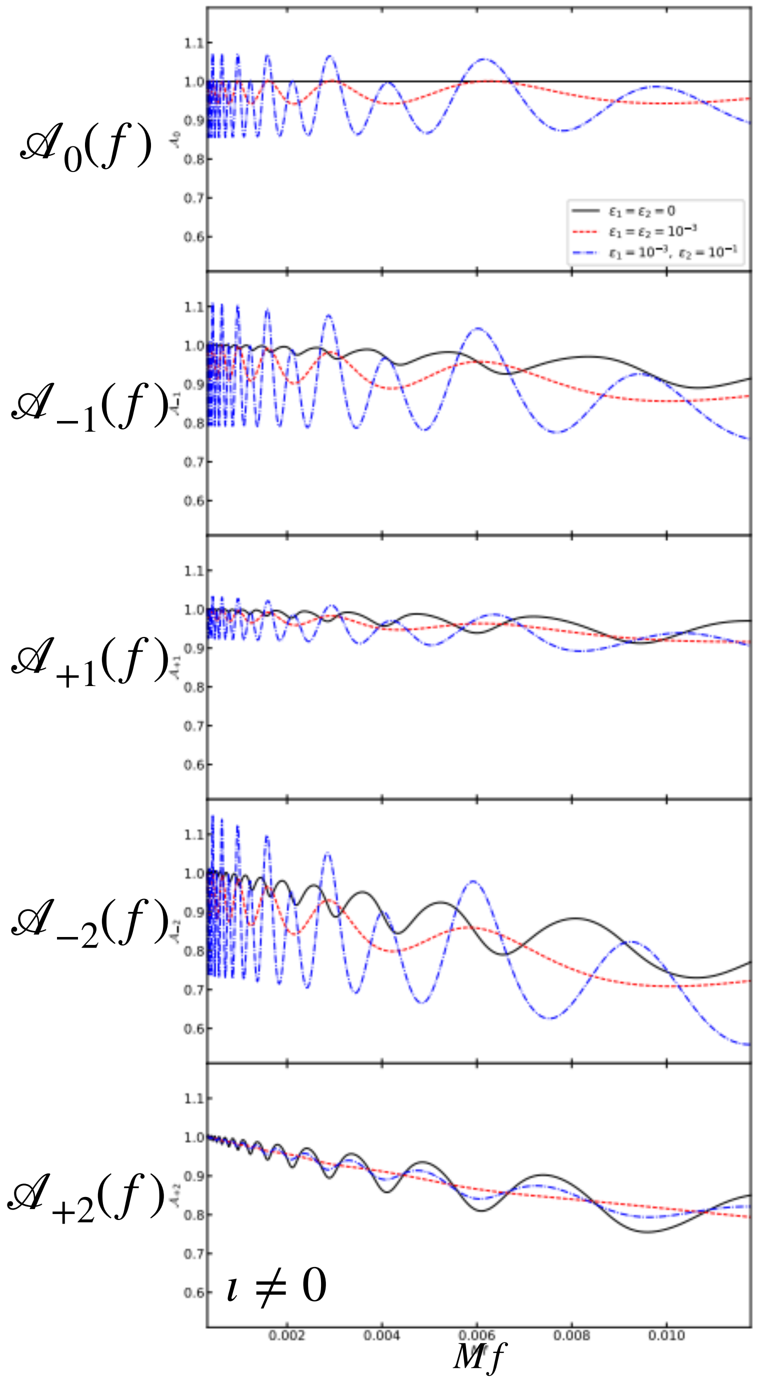
$$|m| \leq 2$$

- Frequency-domain waveform waveform obtained applying “SPA” and “SUA”:

[A. Klein, N. Cornish & N. Yunes, PRD90, 124029 (2014)]

$$\boxed{\tilde{h}(f) = \sqrt{\frac{5}{96}} \frac{\mathscr{M}^{5/6}}{\pi^{2/3} D_L} f^{-7/6} e^{i\tilde{\Psi}_F} \sum_m \mathscr{A}_m(f)\, {}_{-2}Y_{2m}(\theta_N, \phi_N)}$$

Amplitude modulations



Amplitudes $\mathscr{A}_m(f)$ generically **modulated** due to precession of orbital angular momentum.

Black: $\epsilon_1 = \epsilon_2 = 0$

Red: $\epsilon_1 = \epsilon_2 = 10^{-3}$

Blue: $\epsilon_1 = 10^{-3}, \epsilon_2 = 10^{-1}$

Gravitational-wave phase

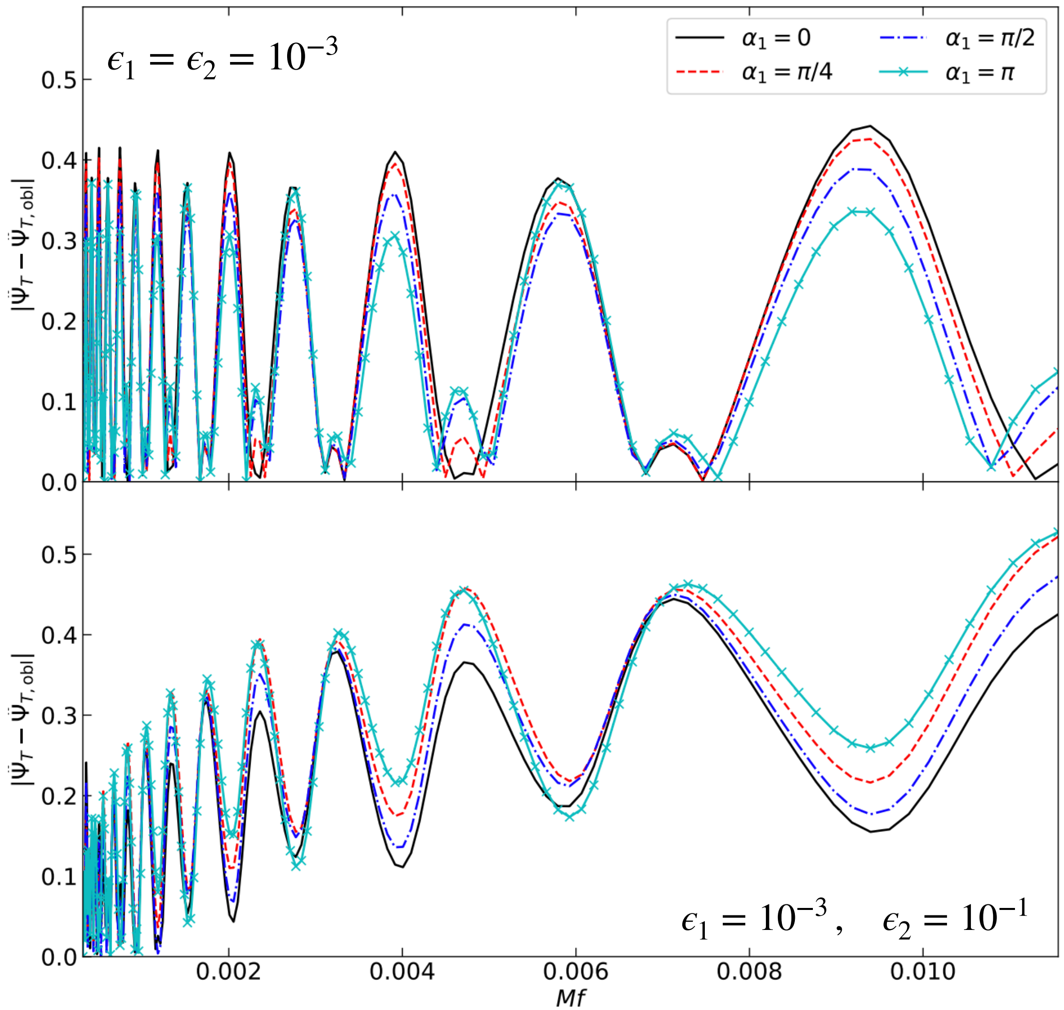
$$\tilde{\Psi}_F = 2\pi f t_c - 2\phi_c - \frac{\pi}{4} + \frac{3}{128\nu(\pi M f)^{5/3}}\left[1 - \frac{Q_0}{8M^3\nu}\sqrt{\frac{\pi}{5}}\epsilon_1^p \epsilon_2^q\, \mathscr{U}_{pq}\left(\pi M f\right)^{4/3}\right]$$

$$\tilde{\Psi}_T(f) = \tilde{\Psi}_F(f) + \arg\left[\sum_m \mathscr{A}_m(f)\right]$$

Total phase difference between **axisymmetric** case and **generically deformed bodies** can be $\gtrsim 0.1$.

Detectability?

Mapping to realistic exotic compact objects?



Conclusions

Detecting **non-axisymmetry** or **non-equatorial symmetry** through measurements of the **multipole moments** of a compact object would be **smoking-gun** from departures from Kerr geometry.

- To do: Fisher Matrix analysis for different systems and detectors
- Degeneracies? Degeneracy between non-axisymmetric and axisymmetric pieces?
- EMRIs/IMRIs in LISA are in general **best systems** to constrain multipole moments, but **accurate modelling** requires going beyond PN and kludge models…

Thank you!