Review
Features of near-extremal black holes

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[Based on 1203.3561, 1509.07637, 1712.07130, 1901.05370,...]

1. How near-extremal are astrophysical (Kerr) black holes ?

AGNs are candidate High Spin Supermassive Black Holes

AGN	a	log M	$L_{\rm bol}/L_{\rm Edd}$	Host
MCG–6-30-15[a]	$\geq +0.98$	$6.65^{+0.17}_{-0.17}$	$0.40^{+0.13}_{-0.13}$	E/S0
Fairall 9[b]	$+0.52^{+0.19}_{-0.15}$	$8.41^{+0.11}_{-0.11}$	$0.05^{+0.01}_{-0.01}$	Sc
SWIFT J2127.4+5654[c]	$+0.6^{+0.2}_{-0.2}$	$7.18^{+0.07}_{-0.07}$	$0.18^{+0.03}_{-0.03}$	—
1 H0707–495[d]	$\geq +0.98$	$6.70^{+0.40}_{-0.40}$	$\sim 1.0_{-0.6}$	—
Mrk 79[e]	$+0.7^{+0.1}_{-0.1}$	$7.72^{+0.14}_{-0.14}$	$0.05^{+0.01}_{-0.01}$	SBb
Mrk 335[f]	$+0.70^{+0.12}_{-0.01}$	$7.15^{+0.13}_{-0.13}$	$0.25^{+0.07}_{-0.07}$	S0a
NGC 3783[g]	$\geq +0.98$	$7.47^{+0.08}_{-0.08}$	$0.06^{+0.01}_{-0.01}$	SB(r)ab
Ark 120[h]	$+0.94^{+0.1}_{-0.1}$	$8.18^{+0.05}_{-0.05}$	$0.04^{+0.01}_{-0.01}$	Sb/pec
3C 120[i]	≥ 0.95	$7.74^{+0.20}_{-0.22}$	$0.31^{+0.20}_{-0.19}$	S0
1 H0419–577[j]	$\geq +0.88$	$8.18^{+0.12}_{-0.12}$	$1.27^{+0.42}_{-0.42}$	—
Ark 564[j]	$+0.96^{+0.01}_{-0.06}$	≤ 6.90	≥ 0.11	SB
Mrk 110[j]	$\geq +0.99$	$7.40^{+0.09}_{-0.09}$	$0.16^{+0.04}_{-0.04}$	—
SWIFT J0501.9-3239[j]	$\geq +0.96$	—	—	SB0/a(s) pec
Ton S180[j]	$+0.91^{+0.02}_{-0.09}$	$7.30^{+0.60}_{-0.40}$	$2.15^{+3.21}_{-1.61}$	—
RBS 1124[j]	$\geq +0.98$	8.26	0.15	—
Mrk 359[j]	$+0.66^{+0.30}_{-0.54}$	6.04	0.25	pec
Mrk 841[j]	$\geq +0.52$	7.90	0.44	E
IRAS 13224-3809[j]	$\geq +0.995$	7.00	0.71	—
Mrk 1018[j]	$+0.58^{+0.36}_{-0.74}$	8.15	0.01	S0
IRAS 00521-7054[l]	$\geq +0.84$	—	—	—
NGC 4051[m]	$\geq +0.99$	6.28	0.03	SAB(rs)bc
NGC 1365[k]	$+0.97^{+0.01}_{-0.04}$	$6.60^{+1.40}_{-0.30}$	$0.06^{+0.06}_{-0.04}$	SB(s)b

7 out of 22 AGNs observed in X-rays have $a > 0.98M$

[Brenneman, 2013]

(Controversy)

How High can the Spin be?

- Cosmic censorship conjecture: $a/M < 100\%$
- Thin disk accretion: $a/M < 99{,}8\%$

  [Thorne, 1974]
- Accretion supported by magnetic fields: $< 100\%$

  [Abramowicz, Lasota, 1980]

[0.9994 in Sadowski et al, 2011]

- Capture of compact objects: $< 100\%$

  [Colleoni, Barack, Shah, van de Meent, 2015]

Population models and High Spin Supermassive black hole

For

$$M > 10^9 M_\odot$$

"Anisotropic accretion
spins up the supermassive black hole to
the Thorne bound"

[Berti, Volonteri, 2008]

$\longrightarrow$ Possibility for a High Spin Supermassive BH

2. Features <u>at</u> and <u>close to</u> extremality

2. Features at and close to extremality

Definition: at: $T_H = 0$

close to: $MT_H \ll 1$

Features of exactly extremal black holes

- $T_H = 0$
- Inner and outer horizons coincide.
  NB: The inner horizon is unstable and singular.
  [Marolf, "The dangers of extremes", 2010]
  Breakdown of EFT: higher derivative and string corrections?
  [Horowitz, Kolanowski, Remmen, Santos, 2023 and 2024]
- Most known are stationary, but non-stationary exist
  [Murata, Reall, Tanahashi, 2013]
- No physical process is known that would make an extremal black hole from a non-extremal one. Third law of thermodynamics. [Mathematical fine-tuning is possible]
  The reverse is possible.
- Stationarity implies axisymmetry assuming Einstein gravity + weak energy condition

Features of stationary extremal black holes

- Angular velocity from horizon generator: $\xi = \partial_t + \Omega_J \partial_\phi$
- Electrostatic potential: $\Phi_e = -\xi^\mu A_\mu|_{r=r_+}$
- If spin: ergoregion: ∂_t spacelike
- Superradiance: $0 < \omega < m\Omega_J$ (amplification due to lack of a global Killing vector). Negative greybody factor: $\sigma_{\text{abs}} = \dfrac{dE_{\text{abs}}/dt}{dE_{\text{in}}/dt} < 0$ in superradiant range.
  Electromagnetic analogue in the range $0 < \omega < q_e\Phi_e$.
- No Hawking radiation but spontaneous emission of superradiant waves. Quantum decay.

$$\Gamma = \frac{1}{e^{\frac{\omega - m\Omega_J - q_e\Phi_e}{T_H}} - 1}\sigma_{\text{abs}} \longrightarrow \Gamma_{\text{ext}} = -\Theta(-\omega + m\Omega_J + q_e\Phi_e)\sigma_{\text{abs}}$$

Extremal limits: the 4d Kerr example

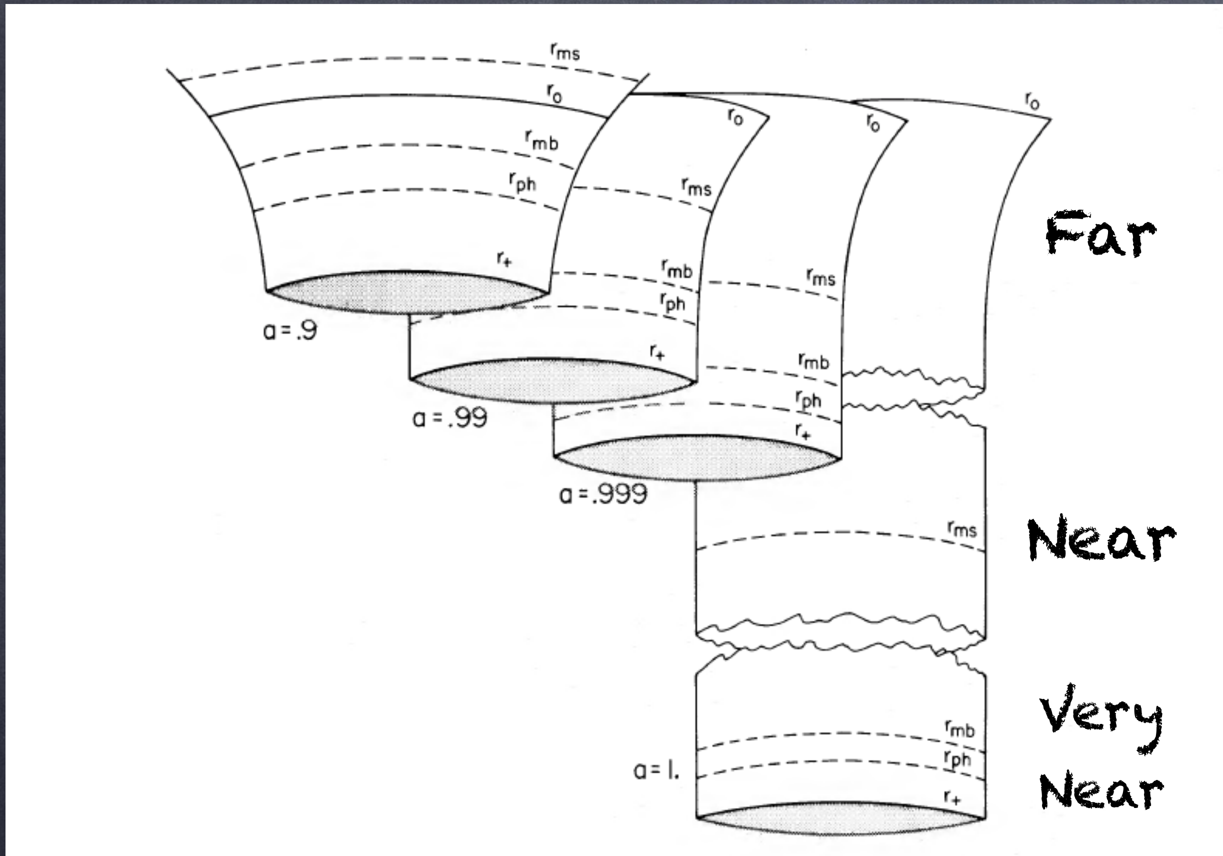
"Several spacetimes can be obtained in a limit from a given spacetime, depending upon which coordinates are kept fixed during the limiting process"
[Geroch, Limits of spacetimes, 1970]

$$\begin{aligned} ds^2 &= -(1-\frac{2M\hat{r}}{\Sigma})d\hat{t}^2 + \frac{\Sigma}{\Delta}d\hat{r}^2 + \Sigma d\theta^2 + (\hat{r}^2 + a^2 + \frac{2Ma^2\hat{r}\sin^2\theta}{\Sigma})\sin^2\theta d\hat{\phi}^2 \\ & \quad -\frac{4Ma\hat{r}\sin^2\theta}{\Sigma}d\hat{t}d\hat{\phi} \end{aligned}$$

$$\Delta \equiv \hat{r}^2 - 2M\hat{r} + a^2, \qquad \Sigma \equiv \hat{r}^2 + a^2\cos^2\theta.$$

LIMIT 1) In Boyer-Lindquist coordinates, $\lambda \mapsto 0$ gives the Extremal Kerr metric.

Near-extremality
=> Opening up of "Near" and "Very Near" regions



[Bardeen, Press, Teukolsky, 1972]
[Bardeen, Horowitz, 1999]

- 1 Far region: extremal Kerr geometry
- 2 Near-horizon regions "Near" and "Very Near": NHEK geometry

The ISCO lies in the "Near" region

Feature: $\hat{r}_{ISCO} = M + 2^{1/3}\lambda^{2/3}M + O(\lambda)$

Change of coordinates to the "co-rotating near-horizon frame" :

$$\begin{aligned} T &= \frac{\hat{t}}{2M}\lambda^{2/3}, \\ R &= \frac{\hat{r}-\hat{r}_+}{M}\lambda^{-2/3}, \\ \Phi &= \hat{\phi} - \Omega_{ext}\hat{t}, \qquad \Omega_{ext} \equiv \frac{1}{2M}. \end{aligned}$$

LIMIT 2) Take $\lambda \mapsto 0$ in that frame: obtain the near-horizon extremal Kerr geometry (NHEK) :

$$ds^2 = 2M^2\Gamma(\theta)\left(-R^2dT^2 + \frac{dR^2}{R^2} + d\theta^2 + \Lambda^2(\theta)(d\Phi + RdT)^2\right)$$

$$\Gamma(\theta) = \frac{1+\cos^2\theta}{2}, \qquad \Lambda(\theta) = \frac{2\sin\theta}{1+\cos^2\theta}$$

NB: "ISCO is everywhere in NHEK" and "ISCO is at $R = 2^{1/3}$"

Feature: $\hat{r}_{IBCO} = M + 2^{1/2}\lambda M + o(\lambda)$

Change of coordinates to the "co-rotating very near-horizon frame" :

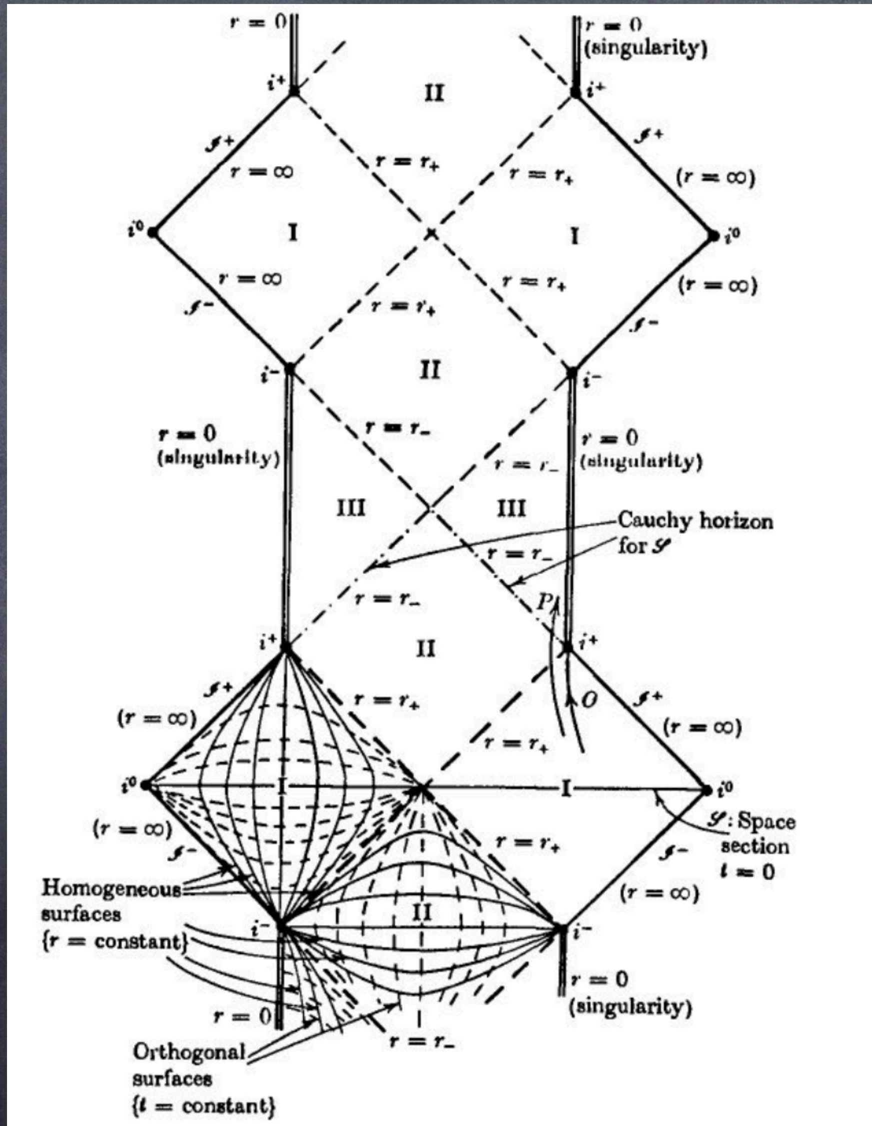
$$
\begin{aligned}
t &= \frac{\hat{t}}{2M\kappa}\lambda, \\
r &= \kappa\frac{\hat{r}-\hat{r}_+}{M\lambda}, \\
\phi &= \hat{\phi} - \frac{\hat{t}}{2M},
\end{aligned}
$$

LIMIT 3) Take $\lambda \mapsto 0$ in that frame: obtain the very near-horizon extremal Kerr geometry (near-NHEK) :

$$
ds^2 = 2M^2\Gamma(\theta)\left(-r(r+2\kappa)dt^2 + \frac{dr^2}{r(r+2\kappa)} + d\theta^2 + \Lambda^2(\theta)(d\phi + (r+\kappa)dt)^2\right)
$$

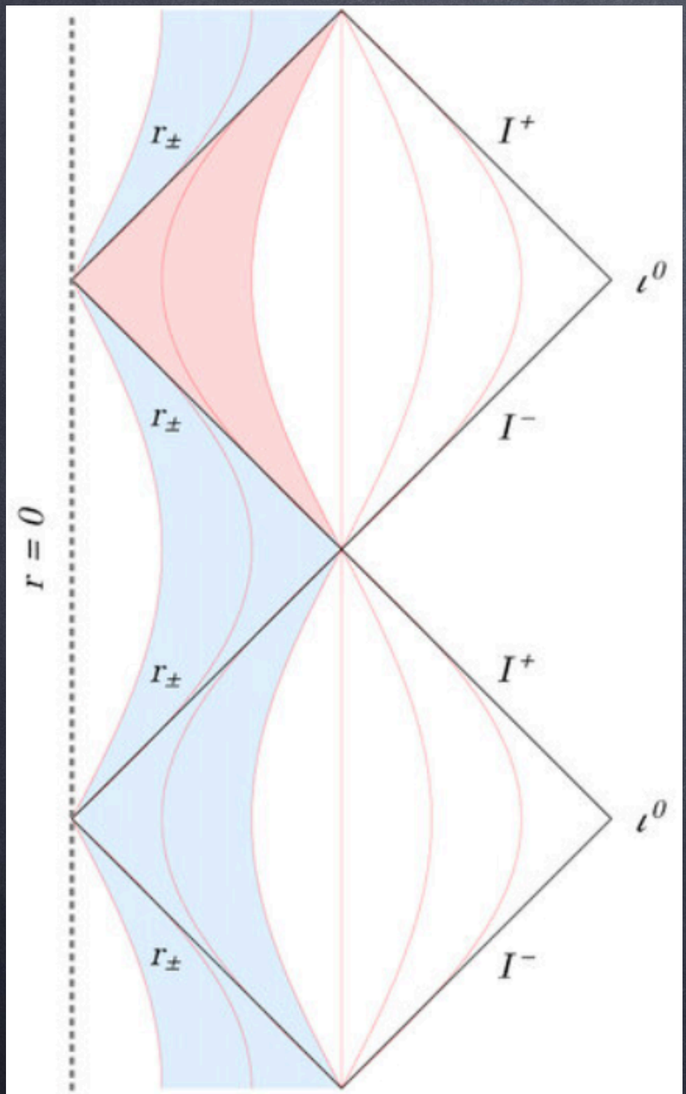
[Bredberg, Hartman, Song, Strominger, 2009]

Carter-Penrose diagram of non-extremal Kerr



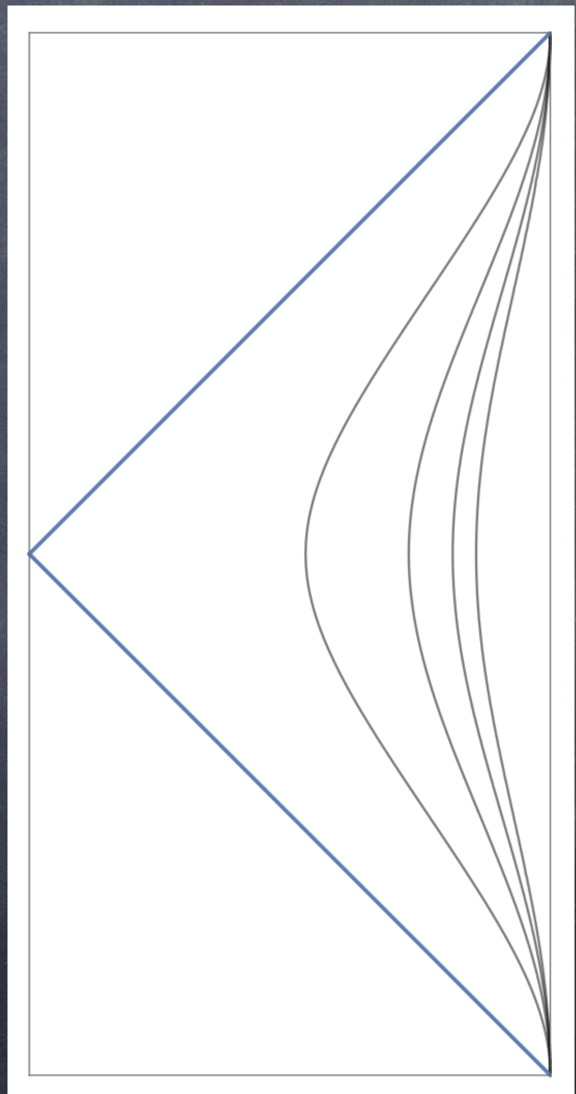
Penrose-Carter diagram of the 3 limiting spacetimes

Extremal Kerr



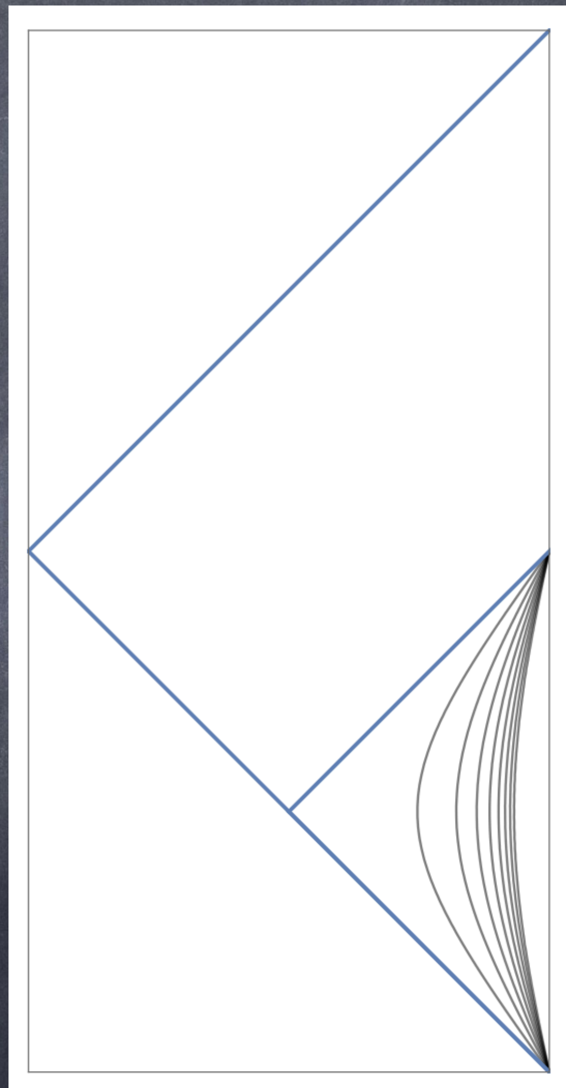
Exterior geodesics

NHEK



ISCO

near-NHEK



IBCO

Two finite diffeomorphisms

1. Finite diffeomorphism between NHEK and near-NHEK

$$
\begin{aligned}
R &= \frac{1}{\kappa} e^{\kappa t}\sqrt{r(r+2\kappa)},\\
T &= -e^{-\kappa t}\frac{r+\kappa}{\sqrt{r(r+2\kappa)}},\\
\Phi &= \phi - \frac{1}{2}\log\frac{r}{r+2\kappa}.
\end{aligned}
$$

2. Finite diffeomorphism between NHEK and global NHEK coordinates

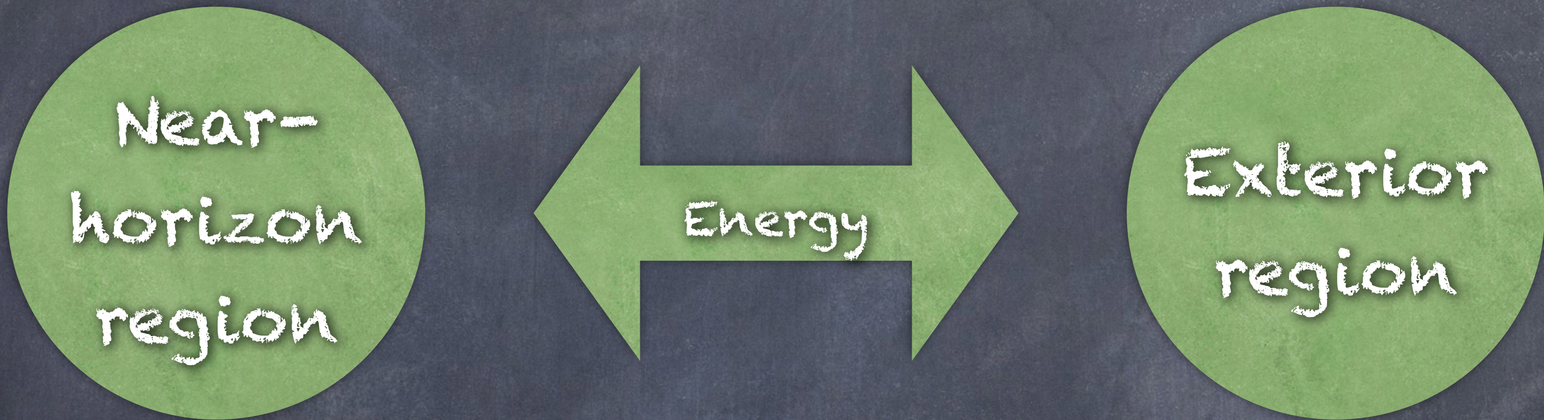
$$
\begin{aligned}
R &= \sqrt{1+y^2}\cos\tau + y,\\
T &= \sqrt{1+y^2}\sin\tau\frac{1}{R},\\
\Phi &= \varphi + \log\frac{\cos\tau + y\sin\tau}{1+\sqrt{1+y^2}\sin\tau},
\end{aligned}
$$

$$
ds^2 = 2M^2\Gamma(\theta)\left(-(1+y^2)d\tau^2 + \frac{dy^2}{1+y^2} + d\theta^2 + \Lambda^2(\theta)(d\phi + y d\tau)^2\right)
$$

$-\infty < \tau < \infty,\ -\infty < y < \infty$

Main feature of near-extremal black holes

Non-suppressed interactions between the near-horizon region and the environment



No Hamiltonian alone describes the evolution of the near-horizon region.

[Amsel, Horowitz, Marolf, Roberts, 2009]
[Dias, Reall, Santos, 2009]

Generic 4d near-horizon limits

Static Near-horizon limit

$$t \to \frac{r_0\, t}{\lambda}\,, \qquad r \to r_+ + \lambda r_0\, r\,,$$

+ Electric case: gauge transformation

Enhanced symmetry (by construction): $\zeta_0 = r\partial_r - t\partial_t$

Further enhancement to $SL(2,\mathbb{R})$ or iso(1,1) [Kunduri, Lucietti, 2009, 2013]

Assuming strong energy condition, $SL(2,\mathbb{R})$!

Solution: $AdS_2 \times S^2$

$$ds^2 = v_1(-r^2\, dt^2 + \frac{dr^2}{r^2}) + v_2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

$$\chi^A = \chi^A_\star\,, \qquad A^I = e_I r\, dt - \frac{p^I}{4\pi}\cos\theta d\phi\,,$$

Susy & non-susy: attractor mechanism: q^L, p^I rule it all.

May be walls of marginal stability in scalar moduli space.

Spinning Near-horizon Limit

$$t \to r_0 \frac{t}{\lambda}\,, \qquad r \to r_+ + \lambda r_0\, r\,,$$

$$\phi \to \phi + \Omega_J^{\rm ext} \frac{r_0\, t}{\lambda}\,, \qquad A \to A + \frac{\Phi_e^{\rm ext}}{\lambda} r_0\, dt\,,$$

+ Spin case: corotate
+ Electric case: gauge transformation

Enhanced symmetry (by construction): $\zeta_0 = r\partial_r - t\partial_t$

Further enhancement to $SL(2,\mathbb{R})$ or iso(1,1)
[Kunduri, Lucietti, 2009, 2013]

Assuming strong energy condition, $SL(2,\mathbb{R})$!

Solution: warped $AdS_2 \times S^2$

$$\begin{aligned} ds^2 &= \Gamma(\theta)\left[-r^2 dt^2 + \frac{dr^2}{r^2} + \alpha(\theta)^2 d\theta^2 + \gamma(\theta)^2 (d\phi + k\, r dt)^2\right], \\ \chi^A &= \chi^A(\theta)\,, \qquad A^I = f^I(\theta)(d\phi + k\, r dt) - \frac{e_I}{k} d\phi\,, \end{aligned}$$

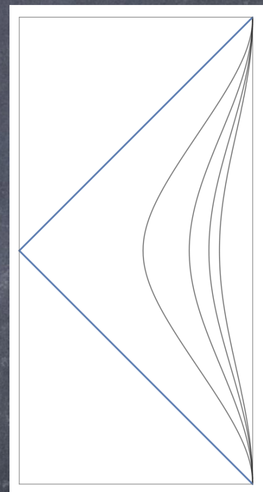
non-trivial gauge

Important feature: $SL(2,\mathbb{R}) \times U(1)$ symmetry

$$\zeta_{-1} = \partial_t, \qquad \zeta_0 = t\partial_t - r\partial_r\,,$$
$$\zeta_1 = \left(\frac{1}{2r^2} + \frac{t^2}{2}\right)\partial_t - tr\partial_r - \frac{k}{r}\partial_\phi\,, \qquad L_0 = \partial_\phi\,.$$

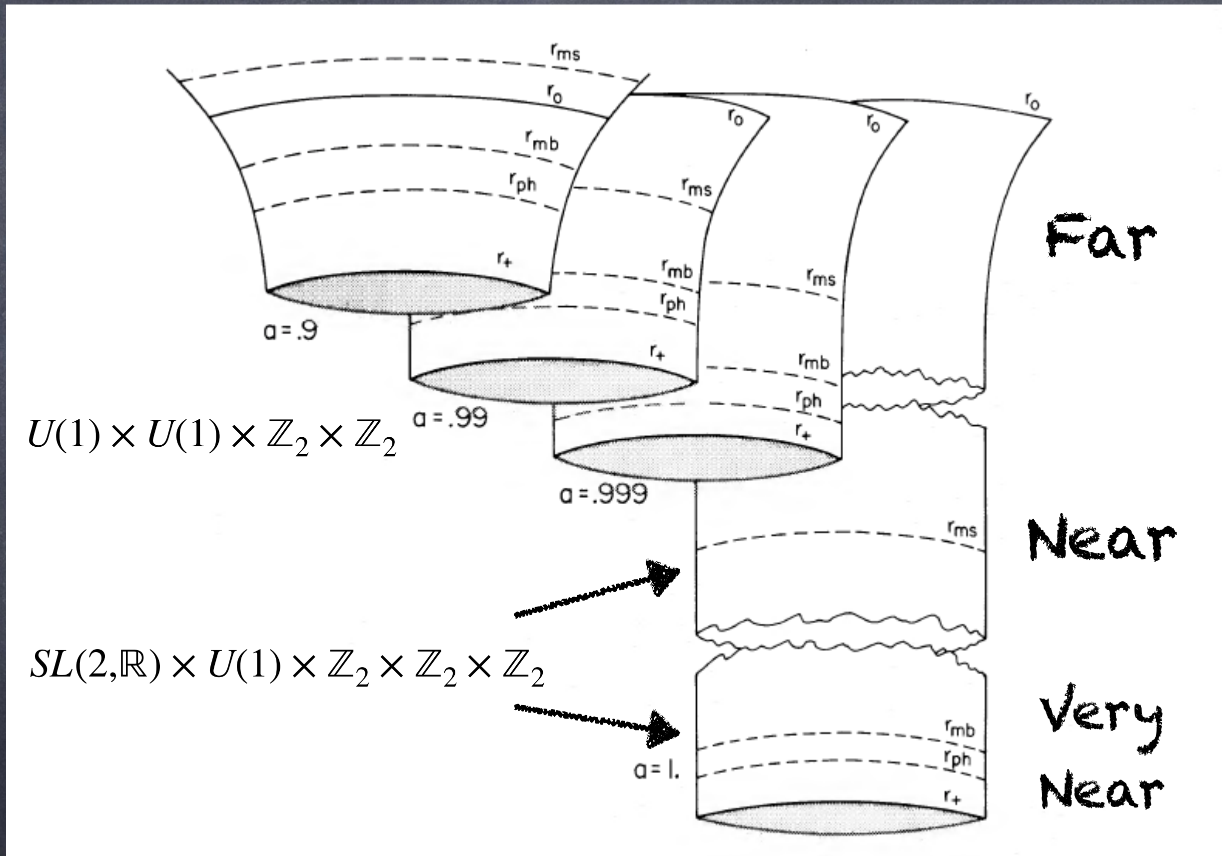
In addition: discrete $(\mathbb{Z}_2)^3 = P \times r\phi \times t\phi$ symmetry

$$ds^2 = \Gamma(\theta)\left[-r^2dt^2 + \frac{dr^2}{r^2} + \alpha(\theta)^2 d\theta^2 + \gamma(\theta)^2(d\phi + k\,r dt)^2\right],$$
$$\chi^A = \chi^A(\theta)\,, \qquad A^I = f^I(\theta)(d\phi + k\,rdt) - \frac{e_I}{k}d\phi\,,$$



- Geodesic completeness expected (proven for Kerr)
- Global time function exists: no CTCs $\quad g^{ab}\partial_a\tau\partial_b\tau = g^{\tau\tau} = -(1+y^2)^{-1}\Gamma^{-1}(\theta) < 0$
- No global Killing vector in general. No QFT vacuum [see Aggarwal's talk]

Extremality => Conformal symmetry



Other example: BTZ black hole

Extremal limit: $M\ell = \pm J$

The near-horizon limit gives

$$ds^2 = \frac{l^2}{4}\left(-r^2\,dt^2 + \frac{dr^2}{r^2} + 2|\mathcal{J}|(d\phi + \frac{r}{\sqrt{2|\mathcal{J}|}}dt)^2\right)$$

This is the self-dual AdS_3 orbifold

[Coussaert, Henneaux, 1994]

There also exist singular near-horizon geometries

- Obtained in the limit $T_H \mapsto 0$ and $A_{BH} \mapsto 0$ keeping A_{BH}/T_H fixed.
- Contain an AdS_3 factor: either a null self-dual orbifold or a pinching orbifold

[Bardeen, Horowitz, 1999][Balasubramanian, de Boer, Jejjala, Simon, 2008][de Boer, Sheikh-Jabbari, Simon, 2010]

Thermodynamics at extremality

Stationary black hole entropy formula [Iyer-Wald, 1994]:

$$\mathcal{S} = \frac{1}{4G_N\hbar}\int_\Sigma vol(\Sigma),$$

$$\mathcal{S} = -\frac{2\pi}{\hbar}\int_\Sigma \frac{\delta^{\text{cov}} L}{\delta R_{abcd}}\epsilon_{ab}\epsilon_{cd} vol(\Sigma)$$

Attractor mechanism: the entropy is the extremum of

$$\mathcal{E}(\Gamma(\theta), \alpha(\theta), \gamma(\theta), f^I(\theta), \chi^A(\theta), k, e_I) \quad = \quad 2\pi(k\mathcal{J} + e_I\mathcal{Q}^I - \int_\Sigma d\theta d\phi\sqrt{-g}\mathcal{L}),$$

$$\frac{\delta\mathcal{E}}{\delta\Gamma(\theta)} = \frac{\delta\mathcal{E}}{\delta\alpha(\theta)} = \frac{\delta\mathcal{E}}{\delta\gamma(\theta)} = \frac{\delta\mathcal{E}}{\delta f^I(\theta)} = \frac{\delta\mathcal{E}}{\delta\chi^A(\theta)} = \frac{\delta\mathcal{E}}{\delta k} = \frac{\delta\mathcal{E}}{\delta e_I} = 0$$

Entropy is a function of the charges (only discontinuous dependency on scalar moduli) :

$$\mathcal{S} = \mathcal{S}_{\text{ext}}(\mathcal{J}, \mathcal{Q}_e^I, \mathcal{Q}_m^I)$$

[Sen, 2005]

Logarithmic corrections can be computed (one method: the quantum entropy action formalism) [Sen, 2011]

Such quantum corrections might lift the vacuum degeneracy at extremality in the absence of protective symmetries [Page, 2000] [See talk of Swapnamay Mondal]

Some diffeomorphisms were identified in the near-horizon limit become zero modes that contribute to the gravitational path integral [Ghosh, Maxfield, Turiaci, 2019]
Such modes are computed using an IR regulator T_q .

For Kerr this gives [Rakic, Rangamani, Turiaci, 2024] [Kapec, Sheta, Strominger, Toldo, 2023]

$$S = S_0 + \frac{154}{180}\log S_0 + 4\pi^2\frac{T}{T_q} + \frac{3}{2}\log\frac{T}{T_q} + \mathcal{O}(T^2)\,.$$

$$e^{-S_0}T_q \ll T \ll T_q$$

The zero modes lower the entropy, which might be consistent with a zero entropy in the extremal limit.

Extremal first law of thermodynamics

Given the entropy function $\mathcal{S} = \mathcal{S}_{\rm ext}(\mathcal{J}, \mathcal{Q}_e^I, \mathcal{Q}_m^I)$ we define the chemical potentials:

$$\frac{1}{T_\phi} = \left(\frac{\partial \mathcal{S}_{\rm ext}}{\partial \mathcal{J}}\right)_{\mathcal{Q}_{e,m}}, \quad \frac{1}{T_e} = \left(\frac{\partial \mathcal{S}_{\rm ext}}{\partial \mathcal{Q}_e}\right)_{\mathcal{J},\mathcal{Q}_m}, \quad \frac{1}{T_m} = \left(\frac{\partial \mathcal{S}_{\rm ext}}{\partial \mathcal{Q}_m}\right)_{\mathcal{J},\mathcal{Q}_e}$$

They obey

$$\delta \mathcal{S}_{\rm ext} = \frac{1}{T_\phi}\delta\mathcal{J} + \frac{1}{T_e}\delta\mathcal{Q}_e + \frac{1}{T_m}\delta\mathcal{Q}_m$$

This is the extremal limit of the first law :

$$\delta\mathcal{S} = \frac{1}{T_H}\left(\delta\mathcal{M} - (\Omega_J\delta\mathcal{J} + \Phi_e\delta\mathcal{Q}_e + \Phi_m\delta\mathcal{Q}_m)\right)$$

$$\delta\mathcal{M} = \delta\mathcal{M}_{\rm ext}(\mathcal{J}, \mathcal{Q}_e, \mathcal{Q}_m)$$

The chemical potentials are now obtained from a limit:

$$\begin{aligned} T_\phi &= \lim_{T_H\to 0}\frac{T_H}{\Omega_J^{\rm ext} - \Omega_J} = -\left.\frac{\partial T_H/\partial r_+}{\partial\Omega_J/\partial r_+}\right|_{r_+=r_{\rm ext}}, \\ T_{e,m} &= \lim_{T_H\to 0}\frac{T_H}{\Phi_{e,m}^{\rm ext} - \Phi_{e,m}} = -\left.\frac{\partial T_H/\partial r_+}{\partial\Phi_{e,m}/\partial r_+}\right|_{r_+=r_{\rm ext}}, \end{aligned}$$

Laws of thermodynamics of extremal black holes

- The extremal limit of the first law

$$\delta \mathcal{S}_{\text{ext}} = \frac{1}{T_\phi}\delta \mathcal{J} + \frac{1}{T_e}\delta \mathcal{Q}_e + \frac{1}{T_m}\delta \mathcal{Q}_m \,.$$

- The zero law (with $SL(2,\mathbb{R})$ symmetry)

$$T_\phi = \frac{1}{2\pi k}, \qquad T_e = \frac{1}{2\pi e}$$

- "Classical" extremal entropy = extremum of the entropy function

$$\mathcal{E} \equiv \frac{\mathcal{J}}{T_\phi} + \frac{\mathcal{Q}_e}{T_e} + \frac{\mathcal{Q}_m}{T_m} - 2\pi \int_\Sigma d\theta d\phi \sqrt{-g}\mathcal{L},$$

[Astefanesei, Goldstein, Jena, Sen, Trivedi, 2006]
[Hajian, Seraj, Sheikh-Jabbari, 2013]

Kerr/CFT entropy match

1) Ansatz: asymptotic symmetry generator

$$\zeta_\epsilon = \epsilon(\varphi)\partial_\varphi - r\epsilon'(\varphi)\partial_r \ .$$

$$\epsilon_n(\varphi) \ = \ -e^{-in\varphi}$$

$$\zeta_n = \zeta(\epsilon_n)$$

$$i[\zeta_m, \zeta_n]_{L.B.} = (m-n)\zeta_{m+n}$$

2) Phase space: empty, except finite diffeomorphisms !

3) Associated asymptotic symmetry algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{J}{\hbar}m(m^2-1)\delta_{m+n,0}.$$

$$c_L = \frac{12J}{\hbar}$$

4) "Thermal" Cardy formula $S = \frac{\pi^2}{3}c_L T_L$

5) Entropy matching $S_{micro} = \frac{2\pi J}{\hbar} = S_{BH}$

[Guica, Hartman, Song, Strominger, 2008]

Extremal entropy matching

$$\frac{\pi^2}{3} c_L T_L = S_{BH}$$

Matching is (surprisingly!) very general:

- Gauge fields, scalar fields
- Higher dimensions
- Supergravities
- Higher derivative corrections

Higher derivative corrections to black hole entropy

A diffeomorphic covariant Lagrangian for the metric can be written as

$$\boldsymbol{L} = \star f(g_{ab}, R_{abcd}, \nabla_{e_1} R_{abcd}, \nabla_{(e_1}\nabla_{e_2)} R_{abcd}, \ldots, \nabla_{(e_1} \ldots \nabla_{e_k)} R_{abcd}),$$

$$\begin{aligned}\boldsymbol{L} = \star[& f(g_{ab}, \mathbb{R}_{abcd}, \mathbb{R}_{abcd|e_1}, \ldots, \mathbb{R}_{abcd|e_1 \ldots e_k}) + Z^{abcd}(R_{abcd} - \mathbb{R}_{abcd}) \\ & + Z^{abcd|e_1}(\nabla_{e_1}\mathbb{R}_{abcd} - \mathbb{R}_{abcd|e_1}) + Z^{abcd|e_1 e_2}(\nabla_{(e_2}\mathbb{R}_{abcd|e_1)} - \mathbb{R}_{abcd|e_1 e_2}) \\ & + \cdots + Z^{abcd|e_1 \ldots e_k}(\nabla_{(e_k}\mathbb{R}_{abcd|e_1 \ldots e_{k-1})} - \mathbb{R}_{abcd|e_1 \ldots e_k})].\end{aligned}$$

[Lee, Iyer, Wald, 1990]

The stationary black hole entropy is

$$S = -\frac{2\pi}{\hbar}\int_{\Sigma} \frac{\delta^{\mathrm{cov}} L}{\delta R_{abcd}} \epsilon_{ab}\epsilon_{cd} \mathrm{vol}(\Sigma).$$

$$\frac{\delta^{\mathrm{cov}}}{\delta R_{abcd}} = \sum_{i=0} (-1)^i \nabla_{(e_1} \ldots \nabla_{e_i)} \frac{\partial}{\partial \nabla_{(e_1} \ldots \nabla_{e_i)} R_{abcd}}.$$

[Iyer-Wald, 1993]

Higher derivative corrections to black hole entropy

Metric with $SL(2,\mathbb{R}) \times U(1) \times t - \phi$ symmetry

$$ds^2 = A(\theta)^2 \left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + d\theta^2 + B(\theta)^2 (d\varphi + krdt)^2.$$

Ansatz for asymptotic symmetry:

$$\xi_n = -e^{-in\varphi}(\partial_\varphi + inr\partial_r),$$

$$i[\xi_m, \xi_n] = (m-n)\xi_{m+n}.$$

Representation by charges:

$$i\{H_m, H_n\} = (m-n)H_{m+n} + \frac{c}{12} m(m^2+a)\delta_{m,-n},$$

$$\frac{c}{12} m(m^2+a)\delta_{m,-n} = i \int_\Sigma \boldsymbol{k}_{\xi_m}[\delta_{\xi_n} g; g],$$

The Barnich-Brandt definition for $\mathbf{k}_{\xi_m}$ reproduces the formula

$$\frac{\pi^2}{3} cT = S \ .$$

[Azeyanagi, Compère, Ozawa, Tachikawa, Terashima, 2009]

Puzzle: non-invariance under field redefinition. Boundary terms?

[Krishnan, Kuperstein, 2009] [H-S. Liu, H. Lü, 2021]

Puzzle: non-uniqueness of Virasoro central charge

Ansatz for asymptotic symmetries:

$$\boxed{\overline{\chi}[\epsilon(\vec{\varphi})] = \epsilon\vec{k}\cdot\vec{\partial}_\varphi - \vec{k}\cdot\vec{\partial}_\varphi\epsilon\,(\frac{b}{r}\partial_t + r\partial_r)}$$

Regularity of the constant t, r surfaces fixes $b = \pm 1$ instead of $b = 0$. This leads to a smooth phase space of geometries.

The asymptotic symmetry algebra is

$$\{H_{\vec{m}}, H_{\vec{n}}\} = -i\vec{k}\cdot(\vec{m}-\vec{n})H_{\vec{m}+\vec{n}} + C_{\vec{m},\vec{n}}$$

$$\begin{aligned} iC_{\vec{m},\vec{n}} = &(\vec{k}\cdot\vec{m})^3\left((1-b(b+\Delta))\frac{A_{\mathcal{H}}}{8\pi G} + 2b(b+\Delta)\vec{k}\cdot\vec{J}\right)\delta_{\vec{m}+\vec{n},0} \\ &+ (\vec{k}\cdot\vec{m})(2\vec{k}\cdot\vec{J})\delta_{\vec{m}+\vec{n},0} \end{aligned}$$

[Compère, Hajian, Seraj, Sheikh-Jabbari, 2015]

Observational features of High Spin

Caveat: Extremely high spin

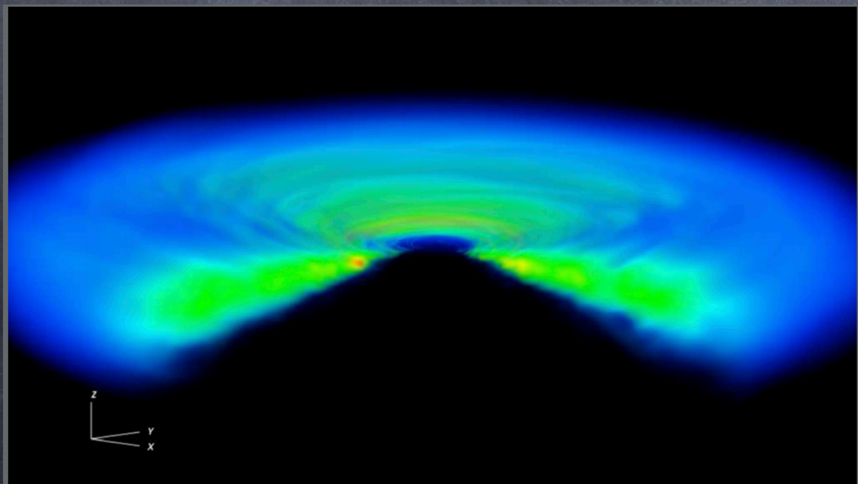
$$a > 0.9999\,M$$

is generally required in order that the NHEK features dominate at least part of the signal

1) Cold accretion disk

The standard Novikov-Thorne model assumes :

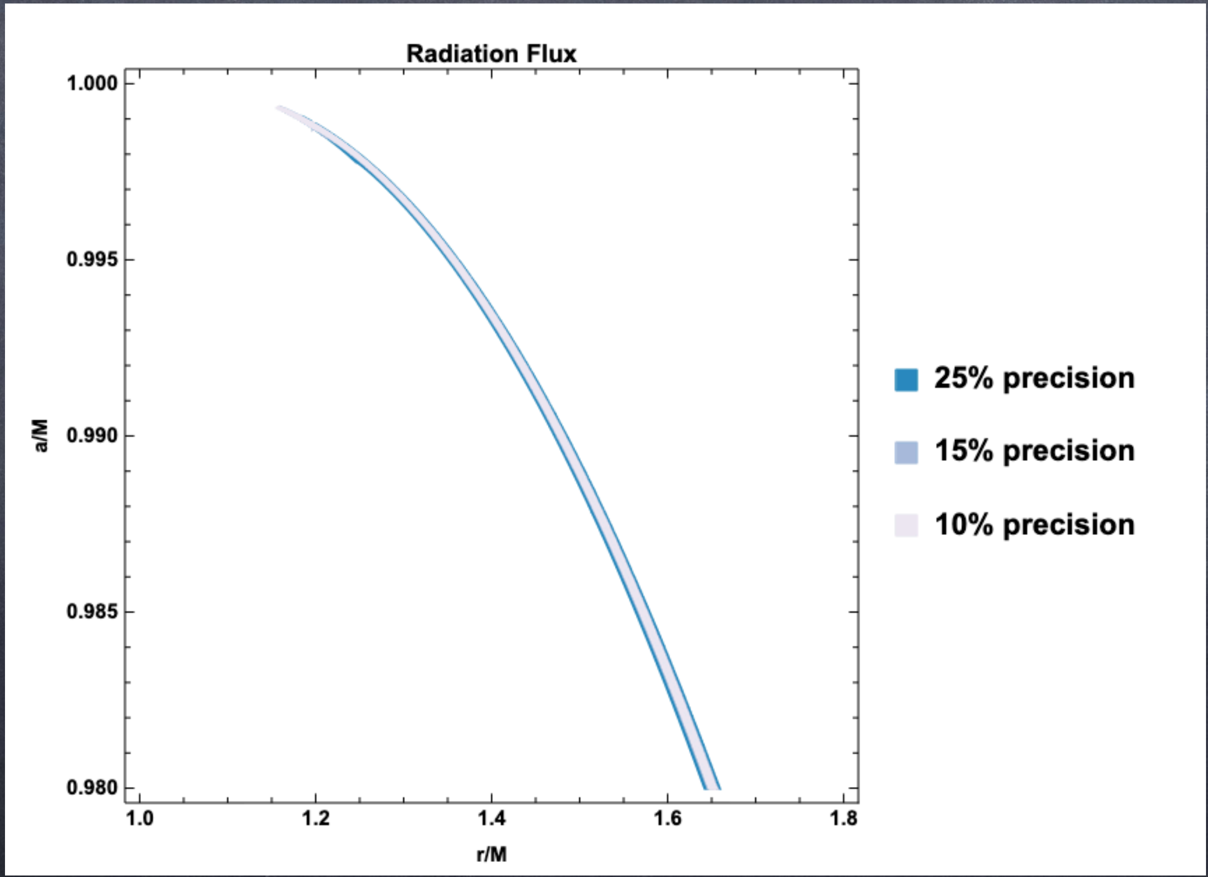
- Relativistic viscous fluid flow (relativistic Navier-Stokes)
- Thermal equilibrium ($T \sim 10^4 - 10^7 K$ lower than virial T)
- Geometrically thin disk
- Optically thick disk (free-free electrons absorption and electron scattering) with radiating EM flux
- Radiation pressure and gas pressure



Volume visualization of the logarithmic density in the accretion disk (Teixeira et al. 2013)

The extremal limit of the Novikov-Thorne model is a self-similar solution.

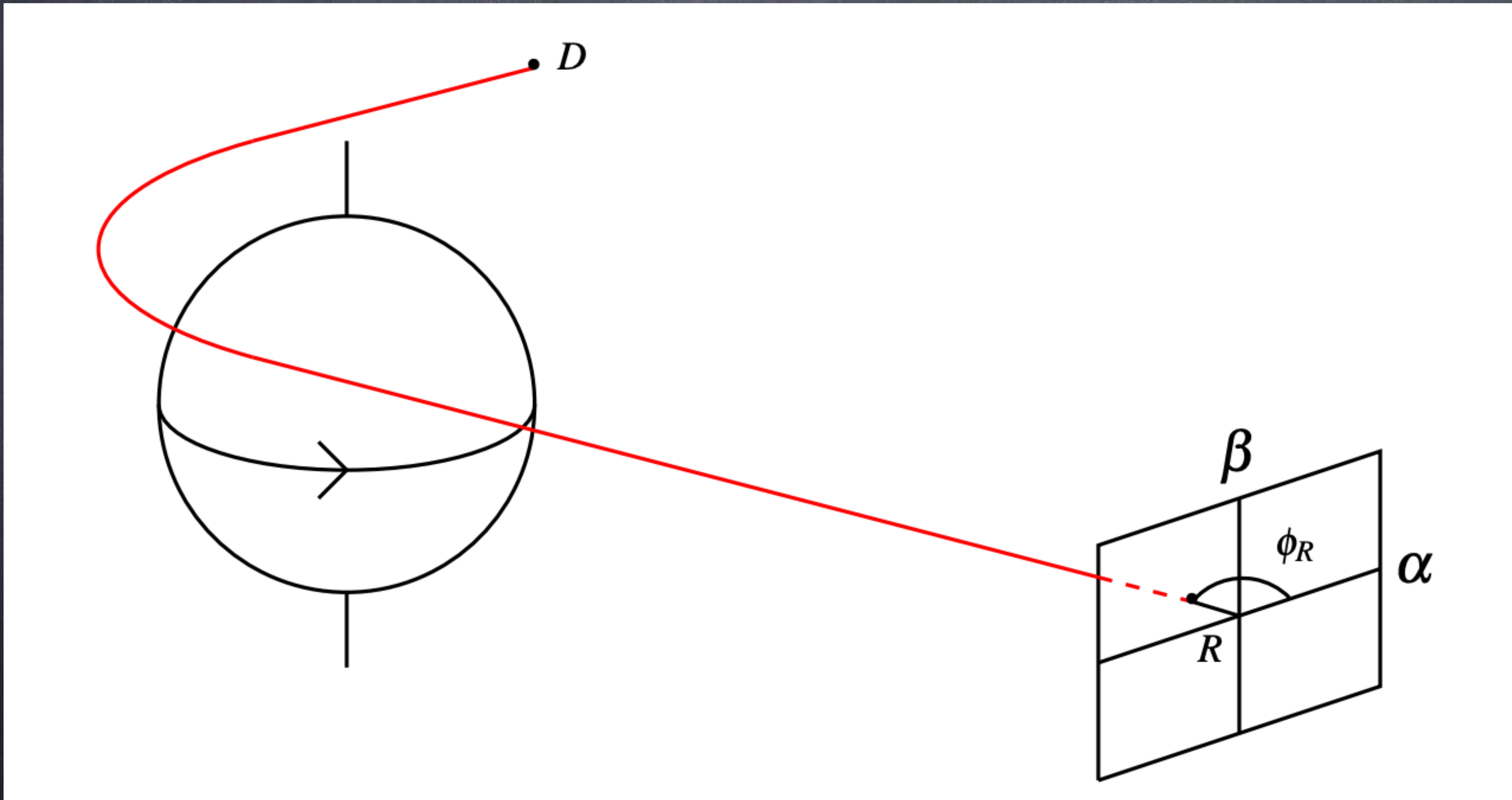
However, the NHEK features are barely visible



[Compère, Oliveri, 2017]

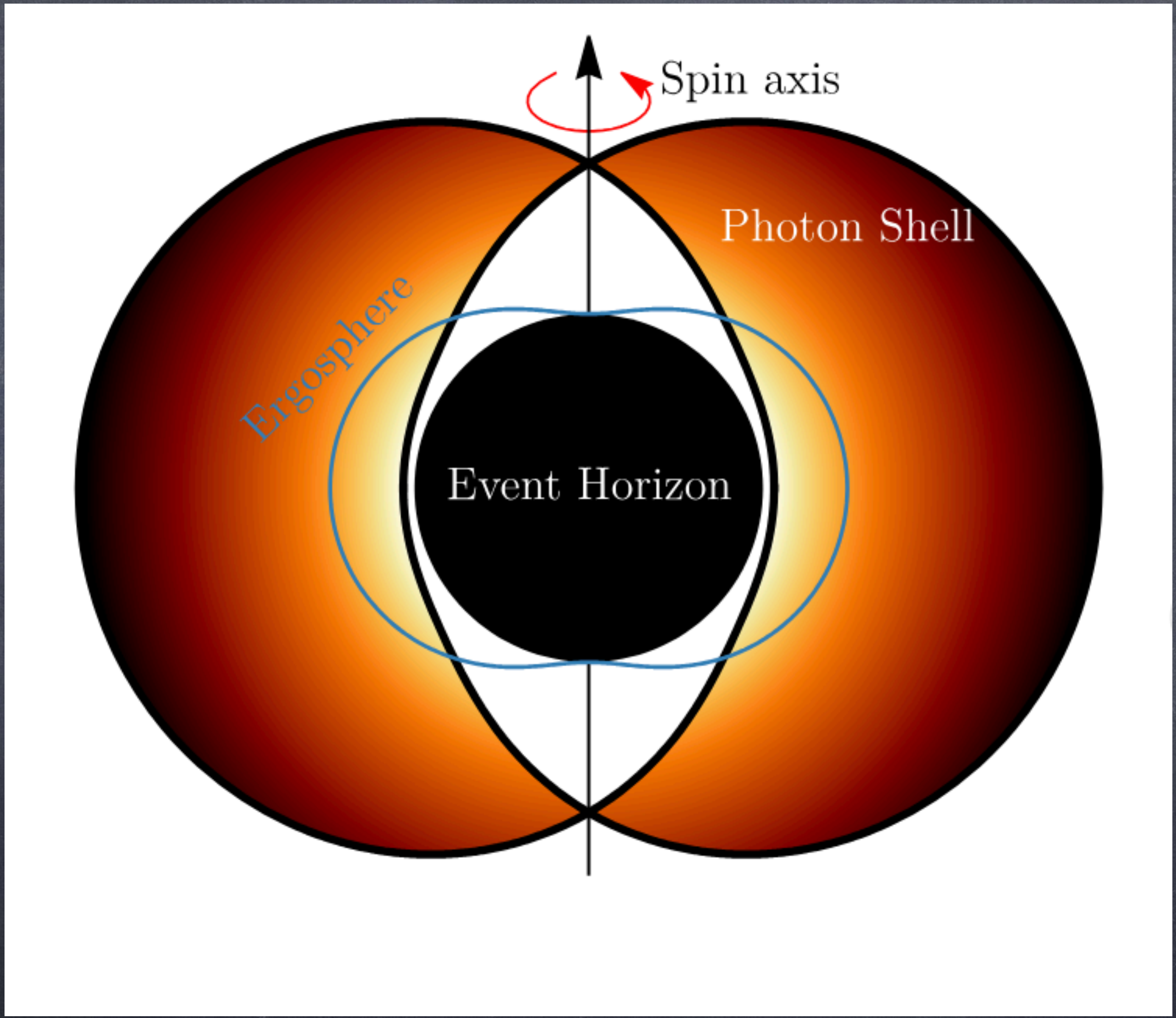
2) Black hole imaging

Bardeen screen



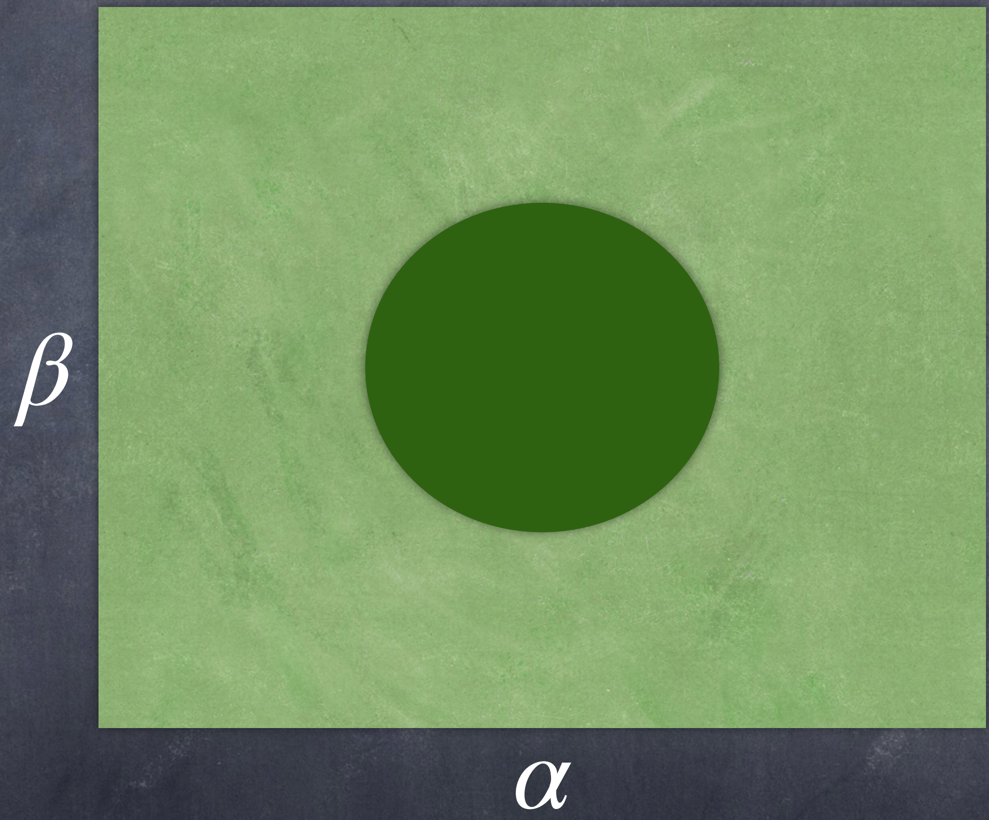
[Bardeen, 1973]

Photon shell

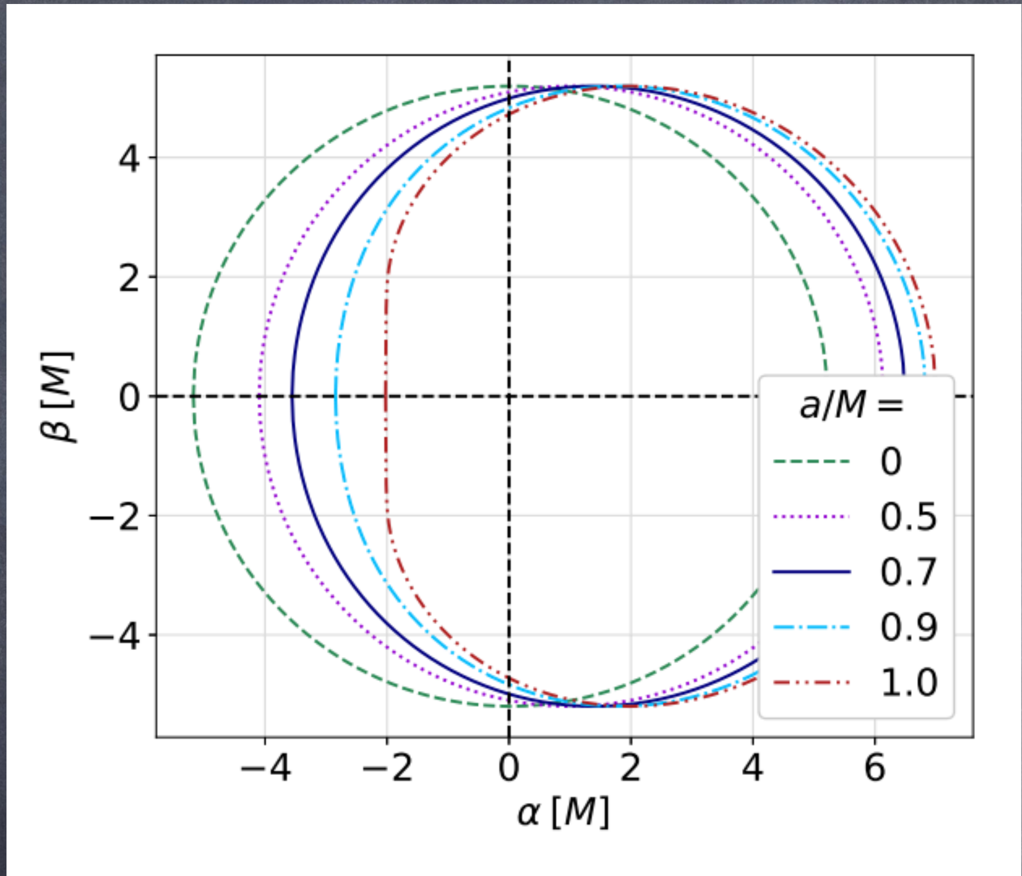


[Gralla, Lupsasca, 2019]

Critical curve

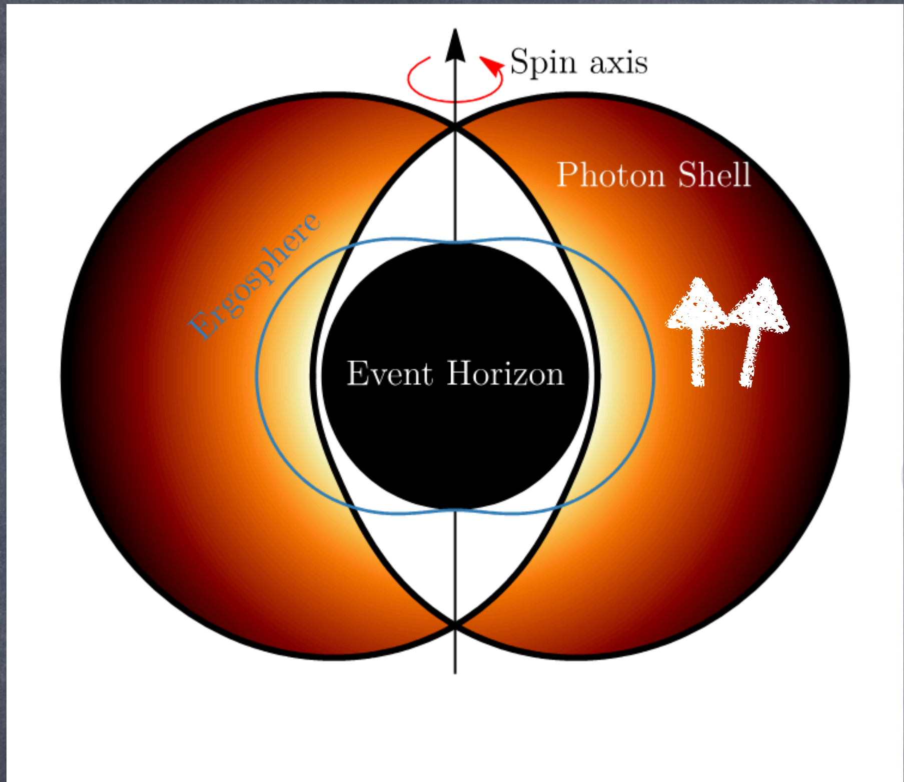


At extremality, the critical curve contains the NHEKline



[Bardeen, 1973] [Gralla, Lupsasca, Strominger, 2017] [Lupsasca, Porfyriadis, Shi, 2017] [Lupsasca, Mayerson, Ripperda, Staelens, 2024]

Lyapunov exponents of the critical curve

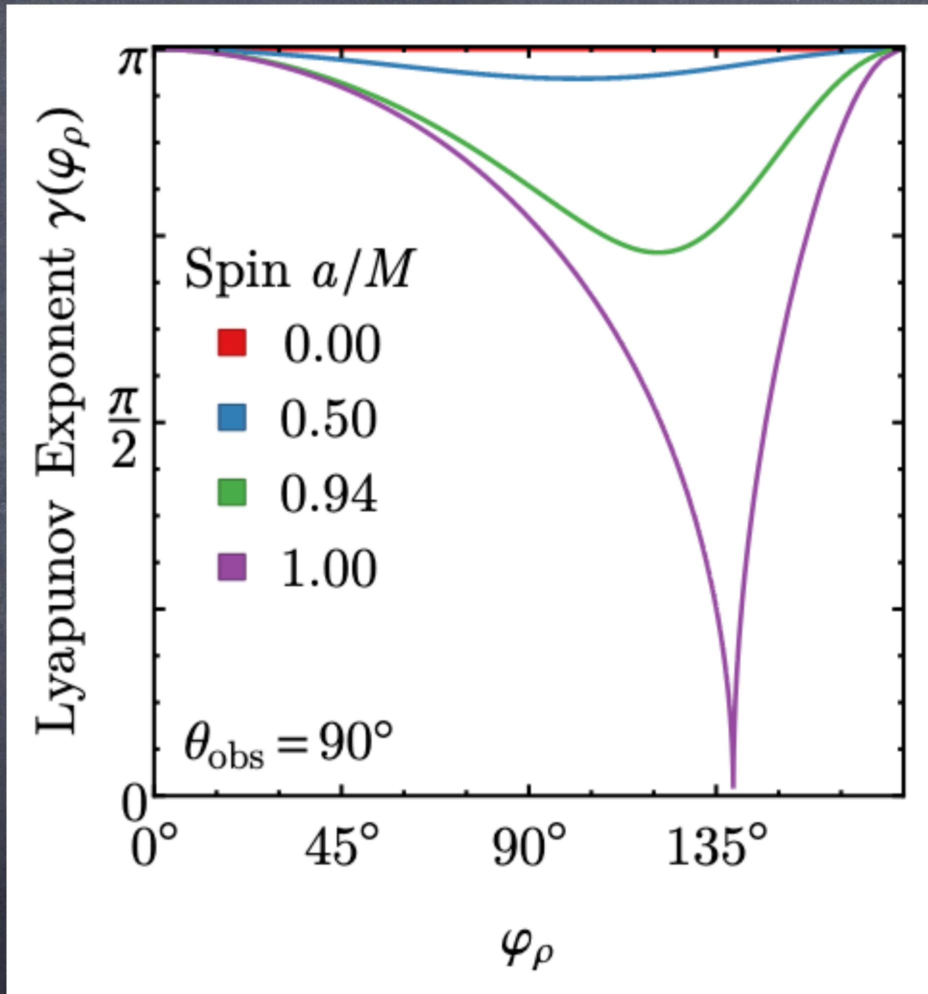


Nearly critical null geodesics have exponential deviation in their radius:

$$\delta r_n = \delta r e^{\gamma n}$$

where n is the number of half-orbits and $\gamma(a, r)$ is a critical exponent.

Lyapunov exponents of the critical curve

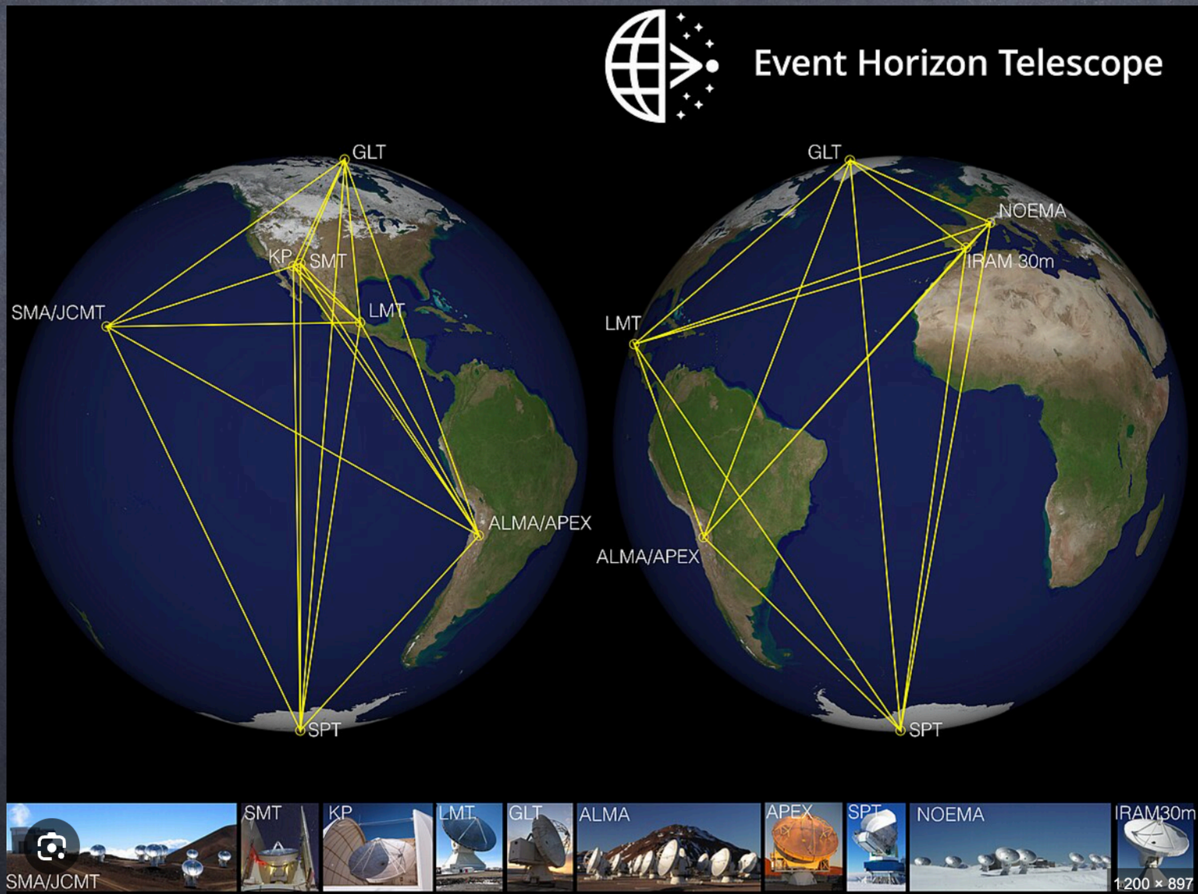


There are 3 other characteristic Lyapunov exponents

[Lupsasca, Porfyriadis, Shi, 2017] [Johnson et al. 2019]

Observability

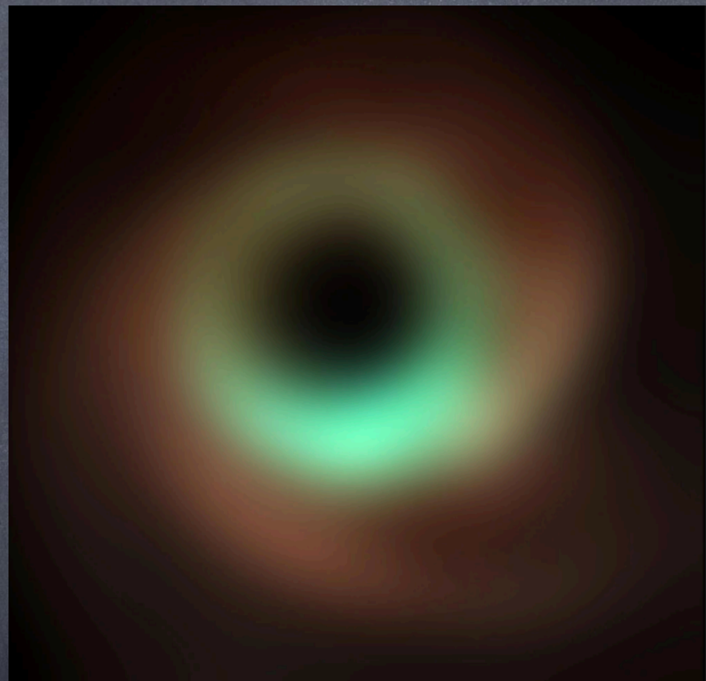
Event Horizon Telescope



Observability

Only a handful of sources :

- Sgr A*
- M87*
- ...



345 GHz

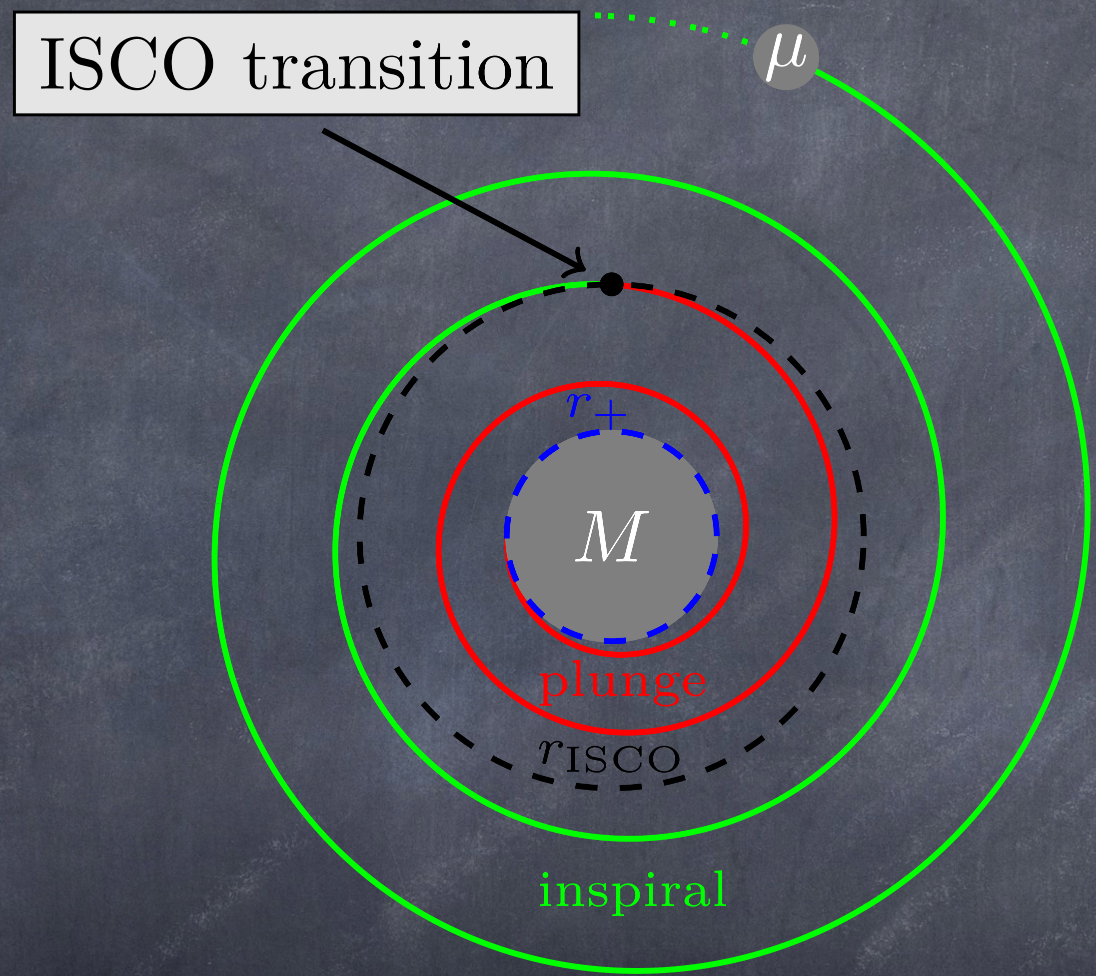
Low probability of high spin $a > 0.999$

3) Gravitational waves

Binary coalescences:

- 2G - LVK : LIGO-Virgo-Kagra
- 3G - Einstein Telescope
- 3G - Cosmic Explorer
- 3G - LISA

Stages of a coalescence around a High Spin black hole



Smoking Gun #1
Locked oscillation timescale

- The ISCO is in the Near Horizon region. The adiabatically evolved inspiral/merger as well.
- The Near Horizon region is exactly co-rotating with the Black hole
- Hence, the GW oscillation timescale of inspirals/plunges in NHEK is fixed to the inverse extremal Kerr angular velocity:

$$\hat{t}_{oscillation} = \Omega_{Ext}^{-1} = 2GM$$

[Porfyriadis, Strominger, 2014] [Gralla, Hughes, Warburton, 2016]

Smoking Gun #2
Redshift suppression

- The ISCO is in the Near Horizon Region which is redshifted.
- The GW amplitude is therefore suppressed. The scaling is universally

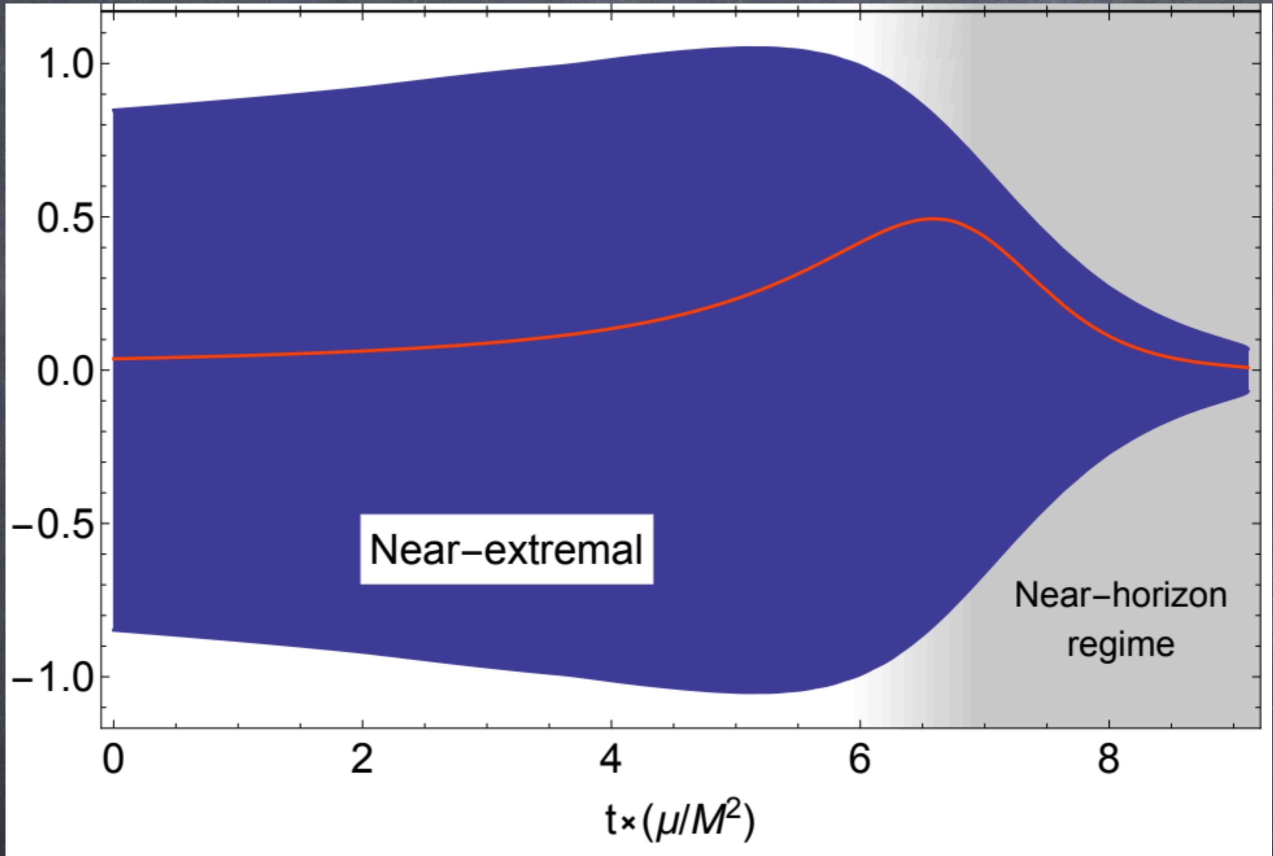
$$|h_+ + ih_\times| \sim \lambda^{1/3} e^{-t/\tau}$$

$$\tau = 0.451\mu \left(\frac{M}{\mu}\right)^2 \qquad \lambda = \sqrt{1 - \frac{a^2}{m^2}}$$

[Porfyriadis, Strominger, 2014] [Gralla, Hughes, Warburton, 2016]

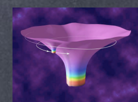
Resulting inspiral waveform

h_+



$a/M = 1 - 10^{-9}$

Face-on



[Gralla, Hughes, Warburton, 2016]

GW fluxes for a NHEK circular orbit: spin and finite size corrections

$$\left(\frac{dE}{d\hat{t}}\right)^{\infty} = q^2\hat{x}_0\left(a^{\infty}_{(0)} + a^{\infty}_{(1)}\chi q + (a^{\infty}_{(2)} + \tilde{a}^{\infty}_{(2)}\kappa_{S^2})(\chi q)^2 + O(q^3)\right);$$

$$\left(\frac{dE}{d\hat{t}}\right)^{H} = q^2\hat{x}_0\left(a^{H}_{(0)} + a^{H}_{(1)}\chi q + (a^{H}_{(2)} + \tilde{a}^{H}_{(2)}\kappa_{S^2})(\chi q)^2 + O(q^3)\right)$$

where $\hat{x}_0 = (2\lambda^2)^{1/3}$ and

$$\begin{aligned} a^{\infty}_{(0)} &= 0.987\,; & a^{H}_{(0)} &= -0.13285\,;\\ a^{\infty}_{(1)} &= -0.409\,; & a^{H}_{(1)} &= 0.28780\,;\\ a^{\infty}_{(2)} &= 0.784\,; & a^{H}_{(2)} &= -0.03169\,;\\ \tilde{a}^{\infty}_{(2)} &= 2.889\,; & \tilde{a}^{H}_{(2)} &= -0.70616\,. \end{aligned}$$

$$q \equiv \frac{\mu}{M} \ll 1.$$

$$\chi \equiv \frac{S}{\mu^2}$$

Point Particle: [Gralla, Hughes, Warburton, 2016]

Spin-spin couplings and spin-induced quadrupole coupling: [Chen, Compère, Liu, Long, Zhang, 2019]

Observability by LISA

We need an intermediate mass black hole coalescing into a high spin supermassive BH within (1 Gpc)^3

$$D^{max} \approx 1Gpc \left(\frac{M}{10^7 M_\odot}\right)^{1/2} \left(\frac{\mu}{100 M_\odot}\right) \left(\frac{15}{SNR}\right)$$

[Gralla, Hughes, Warburton, 2016]

[Compère, Fransen, Hertog, Long, 2018]

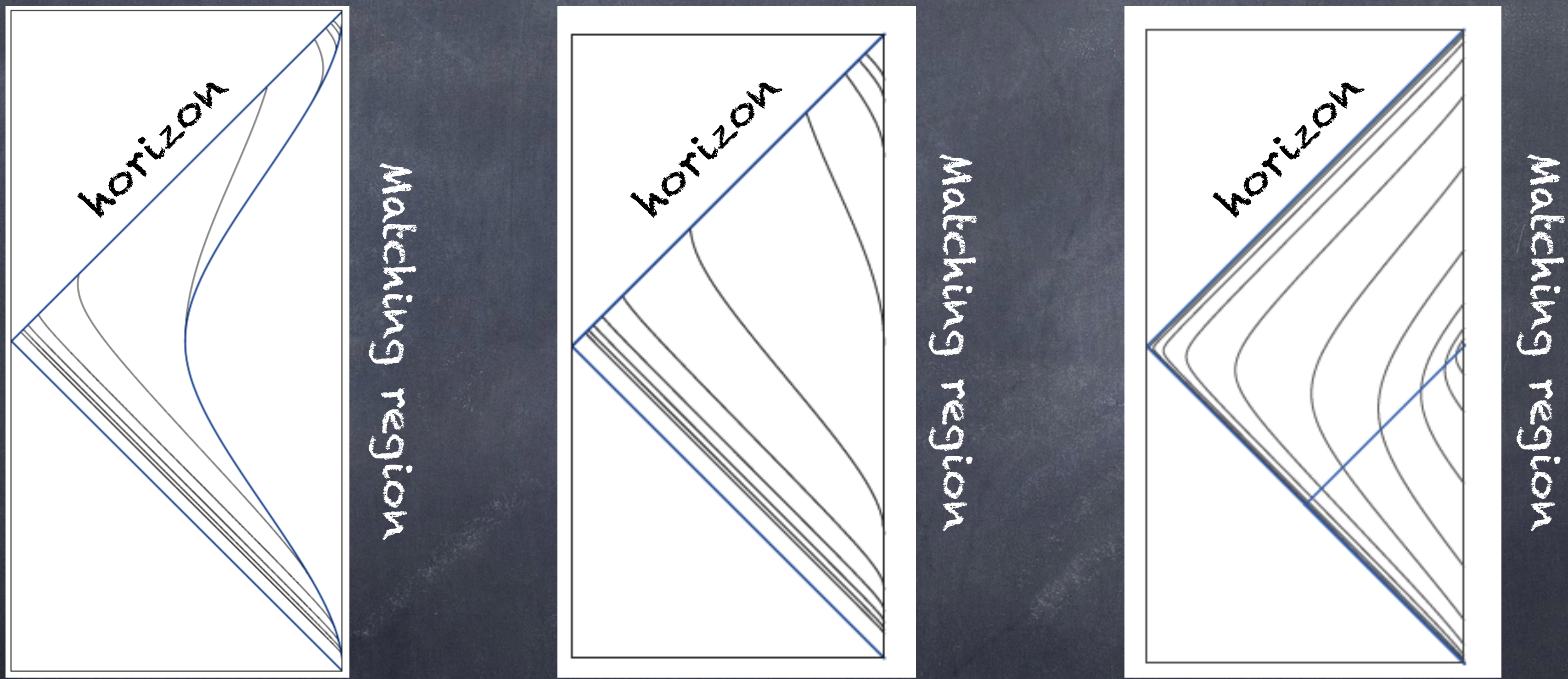
[Burke, Gair, Simon, Edwards, 2020]

Motivation: A small black hole orbiting a supermassive black hole follows a sequence of geodesics.

Near-horizon Kerr Geodesics

Feature: Critical proper angular momentum

$$\ell_* = \frac{\hat{E}_{ISCO}}{\Omega_{ext}} = \frac{2}{\sqrt{3}}M$$



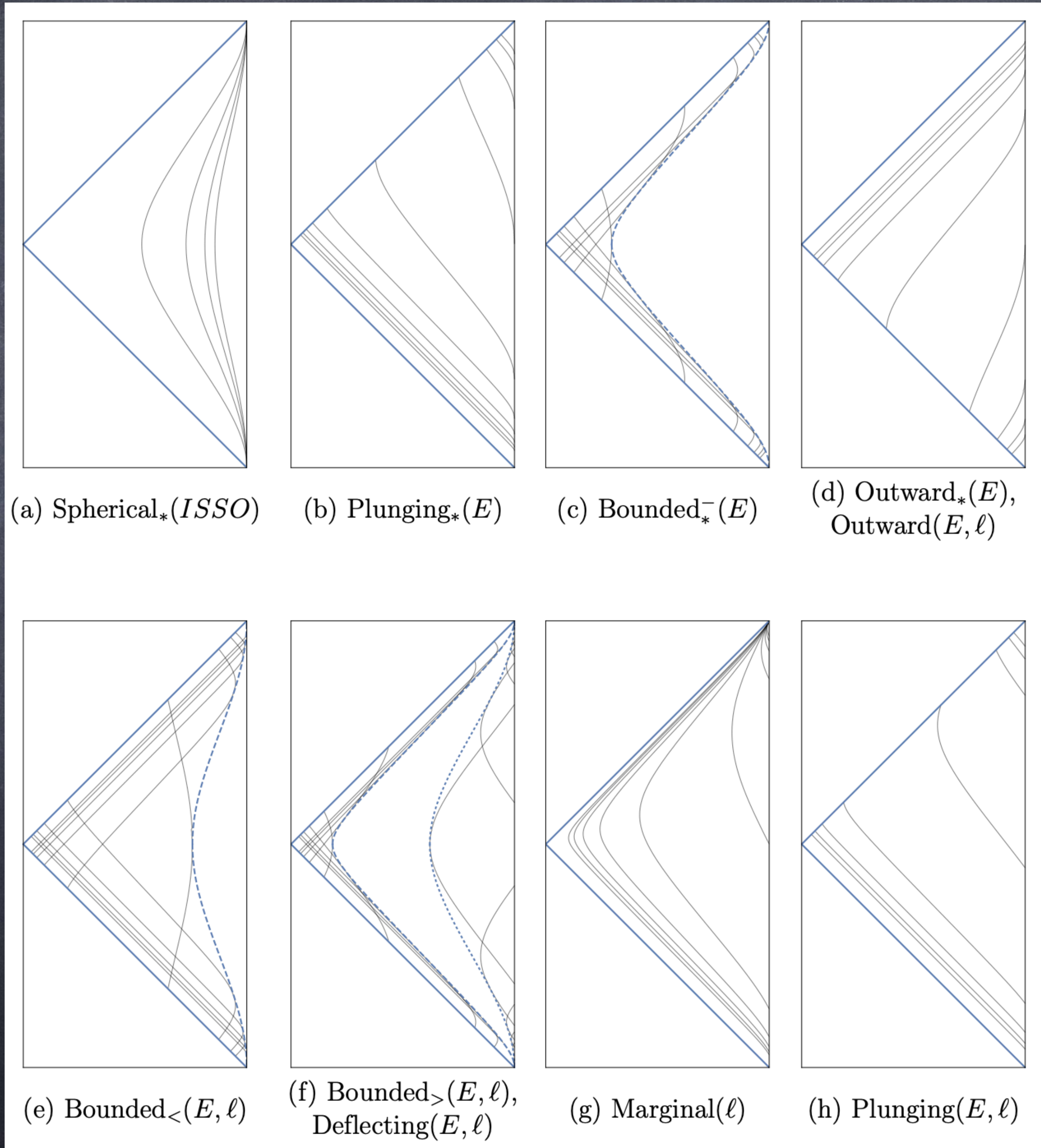
Subcritical orbits $\ell < \ell_*$

Critical orbits $\ell = \ell_*$

Supercritical orbits $\ell > \ell_*$

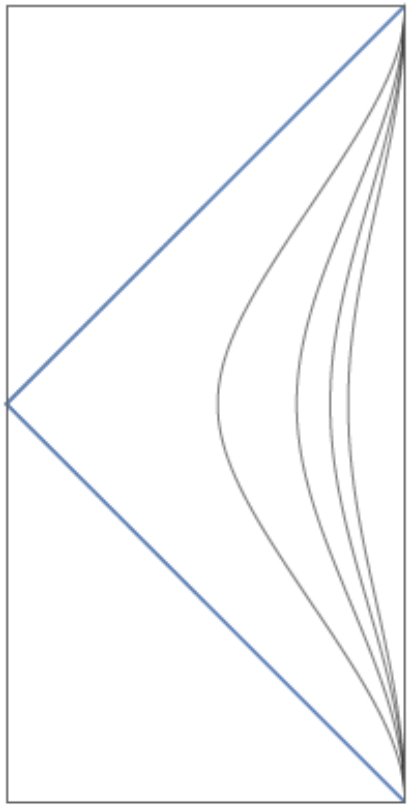
[Compère, Fransen, Hertog, Long, 2018]

Geodesics can be classified (radially)



[Porfyriadis, Strominger, 2014] [Hadar, Porfyriadis, Strominger, 2014] [Kapec, Lupsasca, 2019] [Compère, Druart, 2020]

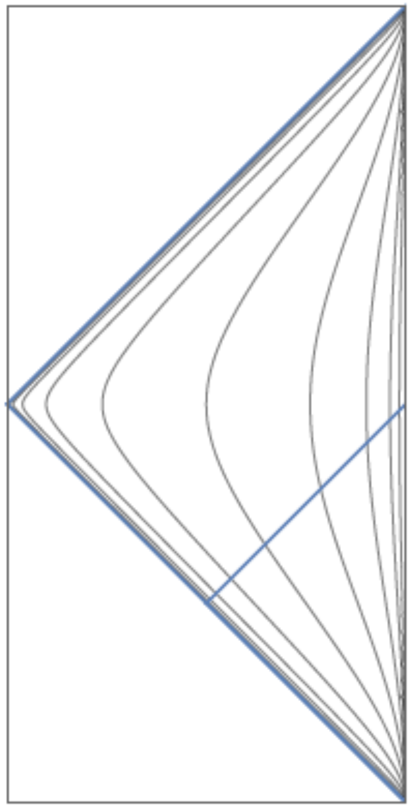
$SL(2,\mathbb{R})$ map of orbits



(a) Circular$_*$ (ISCO)

$SL(2,\mathbb{R})$ transformation

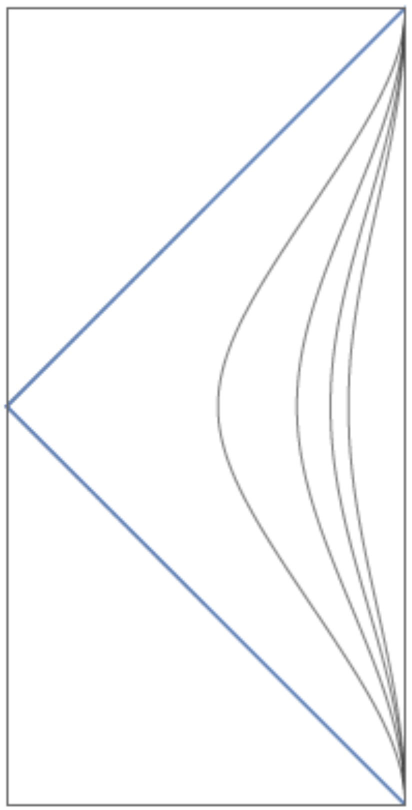
$$R = \frac{1}{\kappa} e^{\kappa t} \sqrt{r(r+2\kappa)},$$
$$T = -e^{-\kappa t} \frac{r+\kappa}{\sqrt{r(r+2\kappa)}},$$
$$\Phi = \phi - \frac{1}{2} \log \frac{r}{r+2\kappa}.$$



(a) Plunging$_*$$(e=0)$

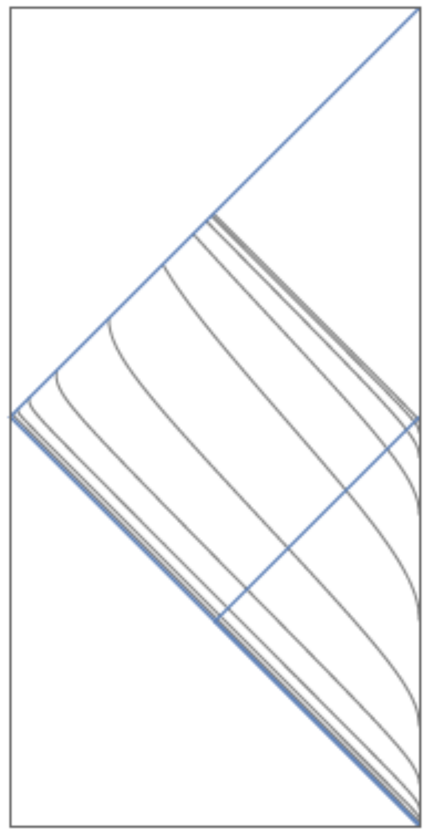
[Hadar, Porfyriadis, Strominger, 2015]

$SL(2,\mathbb{C}) \times U(1)$ map of orbits



(a) Circular$_*$ (ISCO)

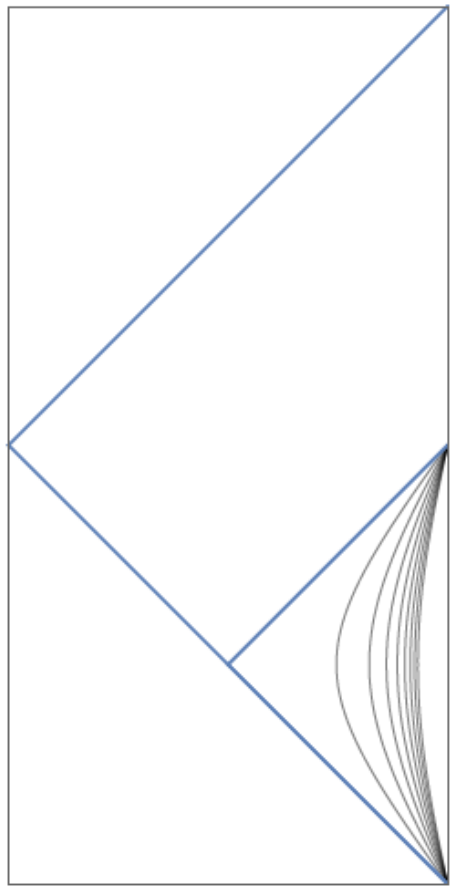
$SL(2,\mathbb{C}) \times U(1)$
transformation



(d) Plunging$_*(e > \kappa\ell^*)$

[Compère, Fransen, Hertog, Long, 2018]

$SL(2,\mathbb{R}) \times PT$ map of orbits



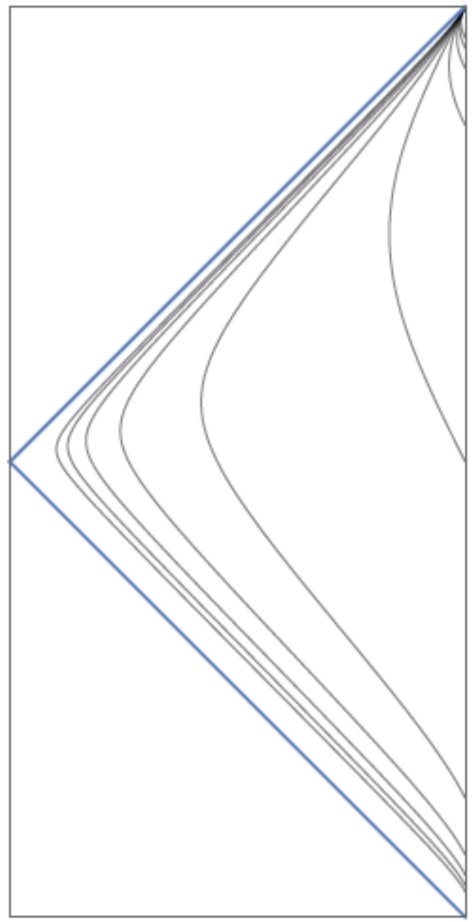
(b) Circular(ℓ)

$SL(2,\mathbb{R}) \times PT$

transformation

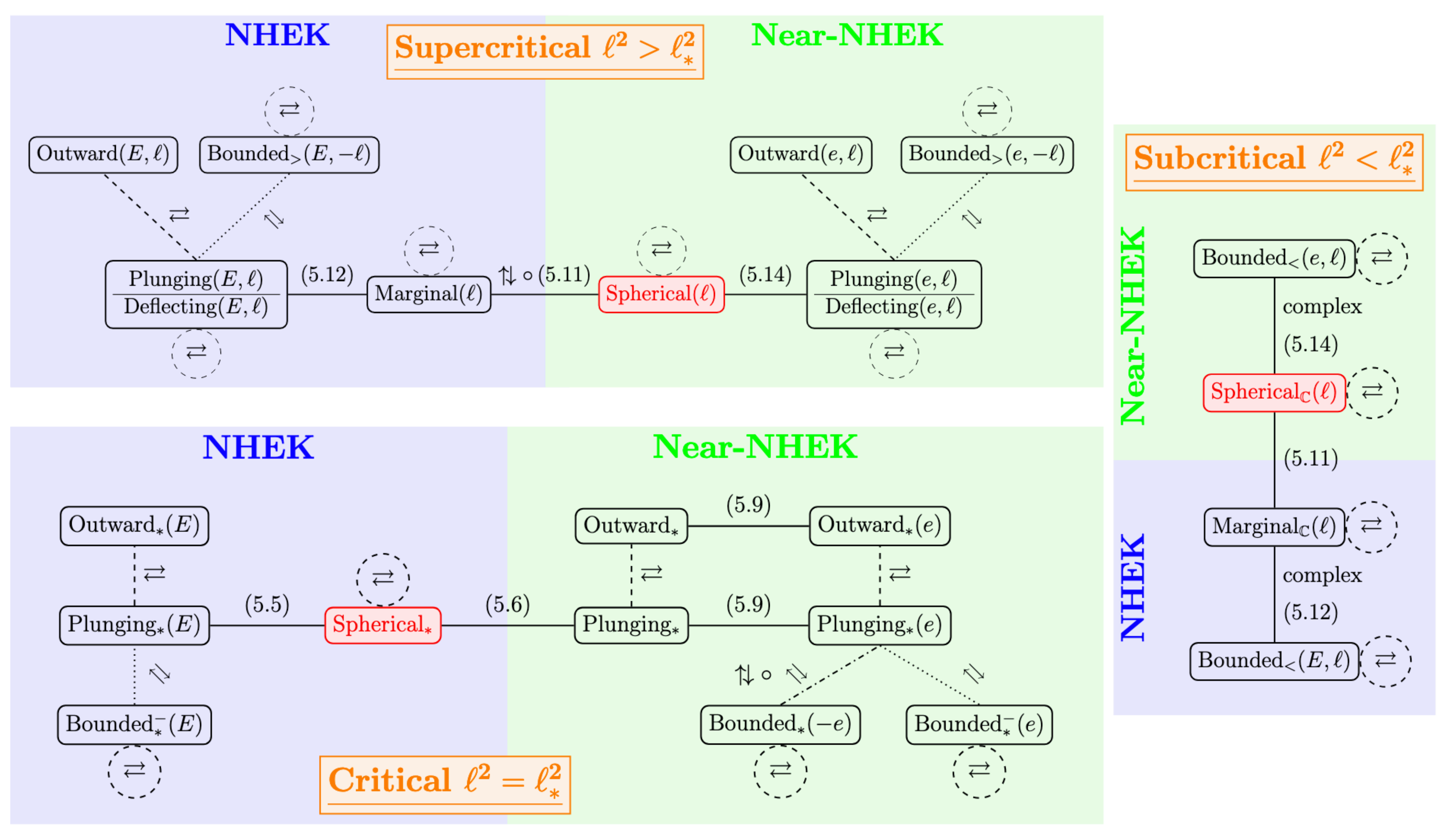
$$\begin{aligned} r &= \kappa(-RT-1), \\ t &= \frac{1}{\kappa}\log\frac{R}{\sqrt{R^2T^2-1}}, \\ \phi &= \Phi + \frac{1}{2}\log\frac{-RT-1}{-RT+1}. \end{aligned}$$

$T \to -T$ $\quad$ $\Phi \to -\Phi$



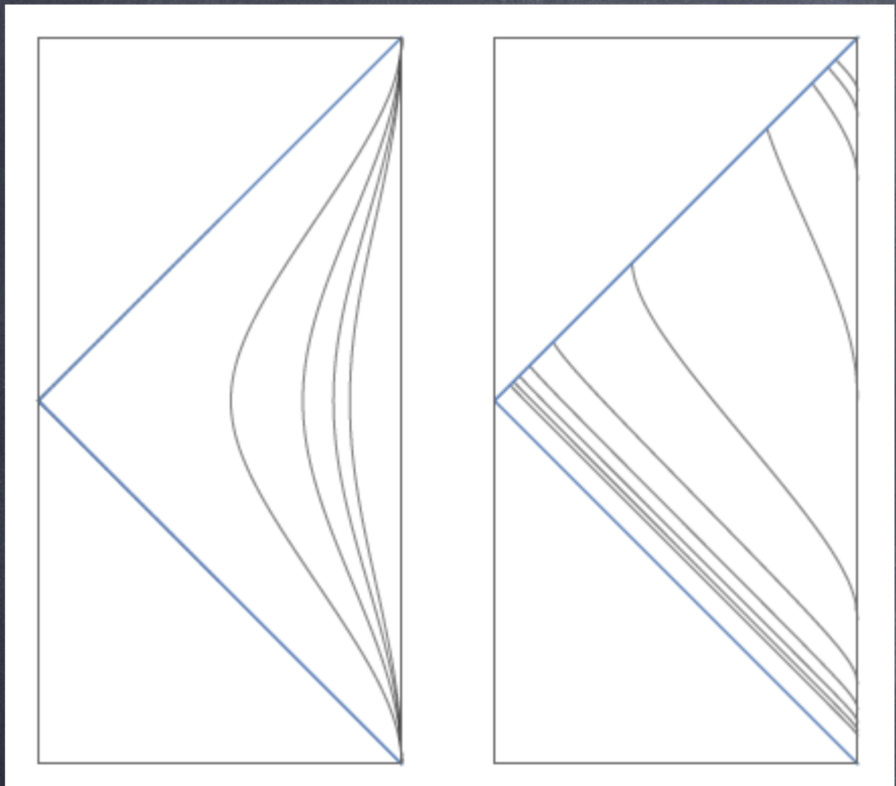
(d) Marginal(ℓ)

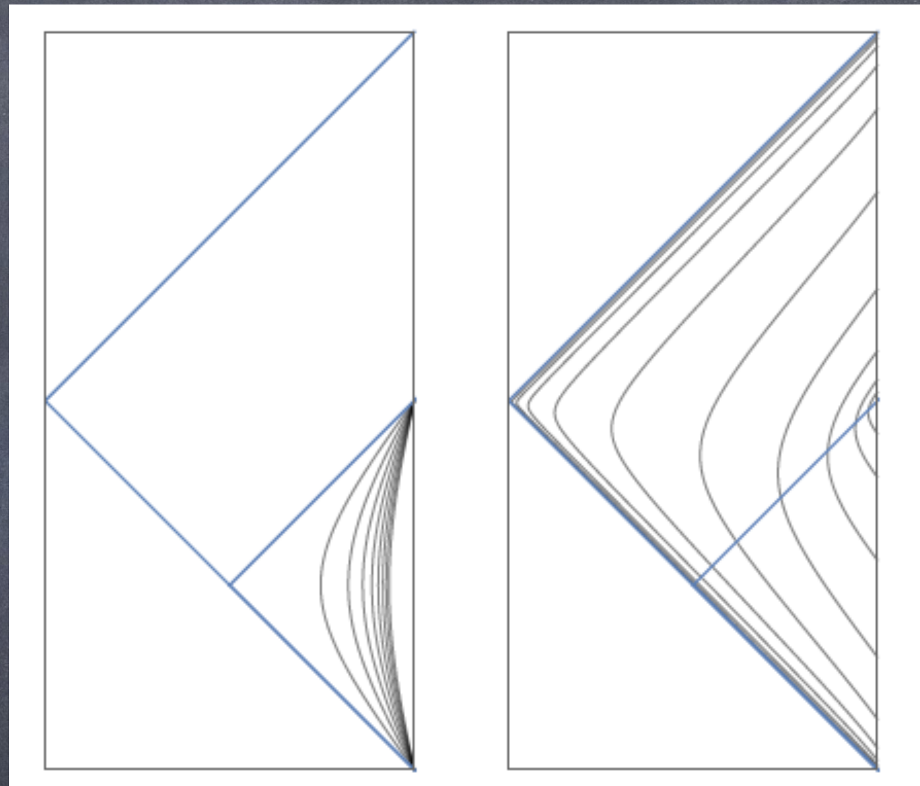
Geodesics are related by symmetry to (only!) 3 classes of spherical geodesics



[Compère, Druart, 2020]

Feature #3: All incoming equatorial orbits belong to either of 2 conformal classes of orbits





Critical Conformal class

$$\ell = \ell_*$$

Supercritical Conformal class

$$\ell > \ell_*$$

Contains circular NHEK orbit

Contains circular near-NHEK orbit

[Compère, Fransen, Hertog, Long, 2018]

$SL(2,\mathbb{R}) \times U(1) \times (Z_2)^3$ symmetry leads to the "conformal method" to obtain analytic waveforms for plunges

- 1. Solve GW emission for circular orbits by brute force (Teukolsky method)
- 2. Apply symmetry transformations to obtain the GW waveforms for all geodesic plunges
- 3. Analytically resolve the conformal map of Dirichlet-Neumann boundary conditions (hypergeometric functions)
- 4. Analytically solve at late times in the QNM approximation (overtone sum over hypergeometric functions : results in polynomials and exponentials)

Feature #4: Bifurcation of Quasi-Normal Modes into: Zero damped (Near) and Damped modes (Far)

Zero damped QNM overtones have same real part of frequency and an equally spaced imaginary part

$$\hat{\omega}_{Nlm} = \frac{1}{2M}(m - i\lambda(N + h)) + o(\lambda)$$

Quasinormal modes (QNM)
of a Kerr black hole
for l=2 through 12
and first 8 overtones

This leads to Polynomial Quasi-normal Mode Ringing !

- Zero damped QNM overtones have same real part of frequency and an equally spaced imaginary part

$$\hat{\omega}_{Nlm} = \frac{1}{2M}(m - i\lambda(N+h)) + o(\lambda)$$

- This might lead to a transient Polynomial Ringing due to coherent stacking of overtones

$$\sum_{N=0}^{\infty} e^{-N\lambda t} = \frac{1}{1-e^{-\lambda t}} \approx \frac{1}{1-(1-\lambda t)} \approx \frac{1}{\lambda t} + O(\lambda t)$$

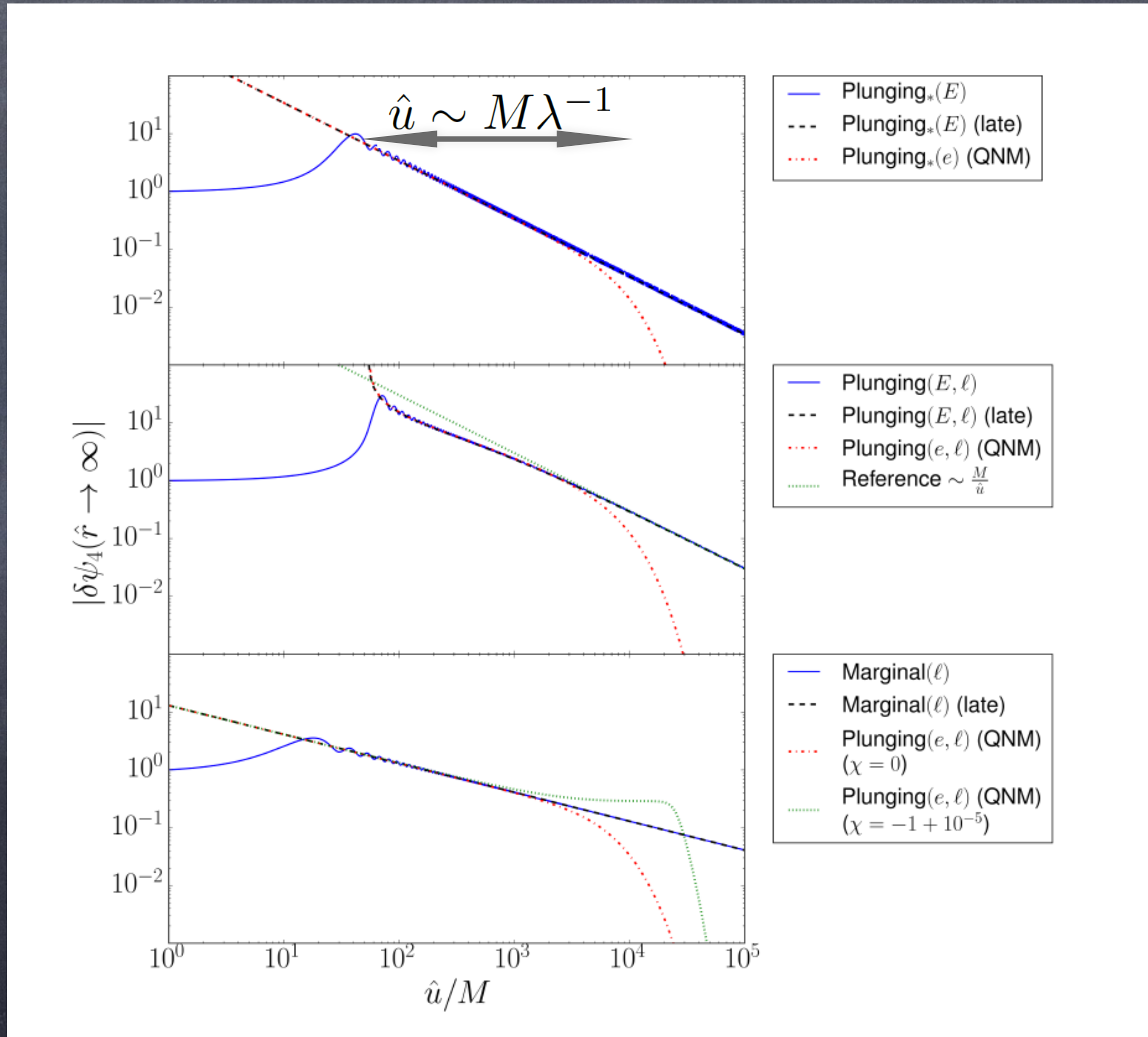
[Yang, Zimmerman^2, Zenginoglu, Zhang, Berti, Chen, 2013]

Smoking Gun #3: Polynomial Ringdown

- GW Amplitude of plunges follows a late time power law continuously between $1/\sqrt{\hat{u}}$ and $1/\hat{u}$ before exponential decay

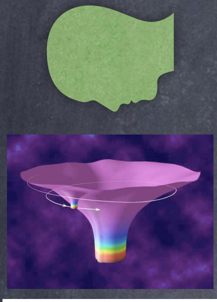
- The power analytically depends upon the impact parameters

- Time spent in polynomial ringdown : $\hat{u} \sim \lambda^{-1}$



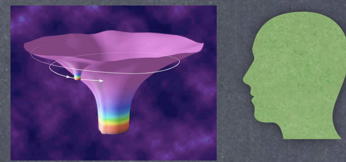
[Compère, Fransen, Hertog, Long, 2018]

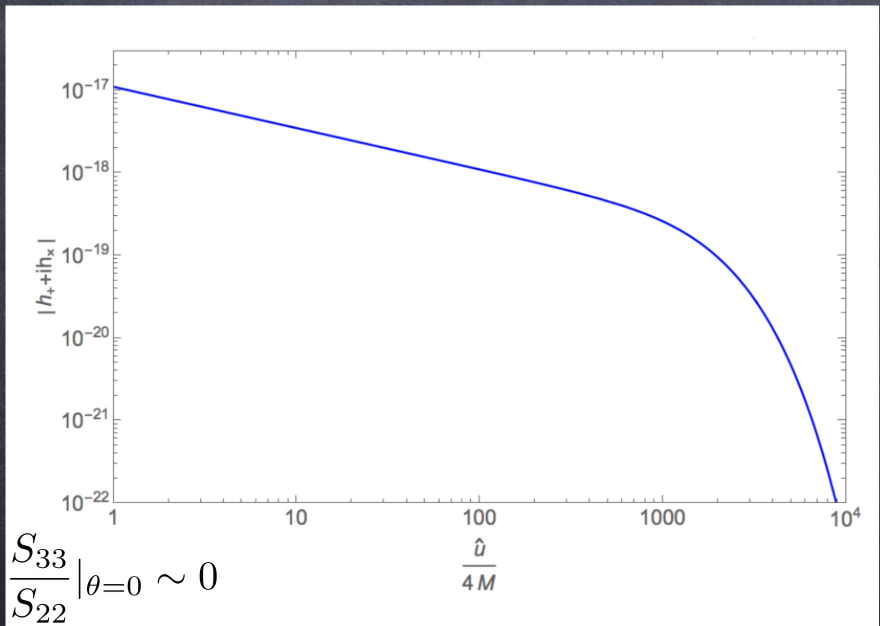
Smoking Gun #4
Emission of higher multipoles

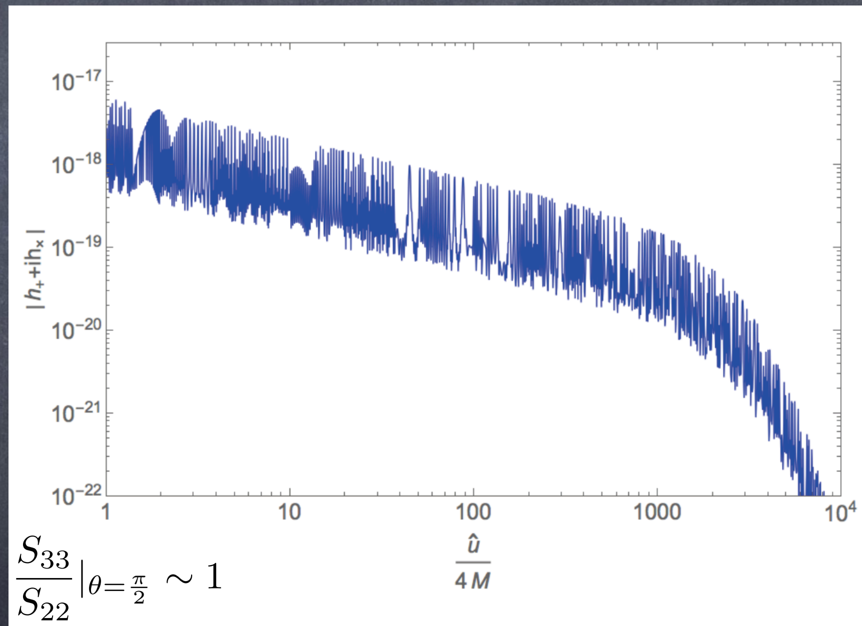


Face-On

Edge-On







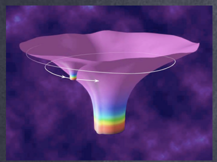
$m = \ell = 2$ dominant

All m, ℓ up to 20 contribute

-> Enhanced sky localization of the source

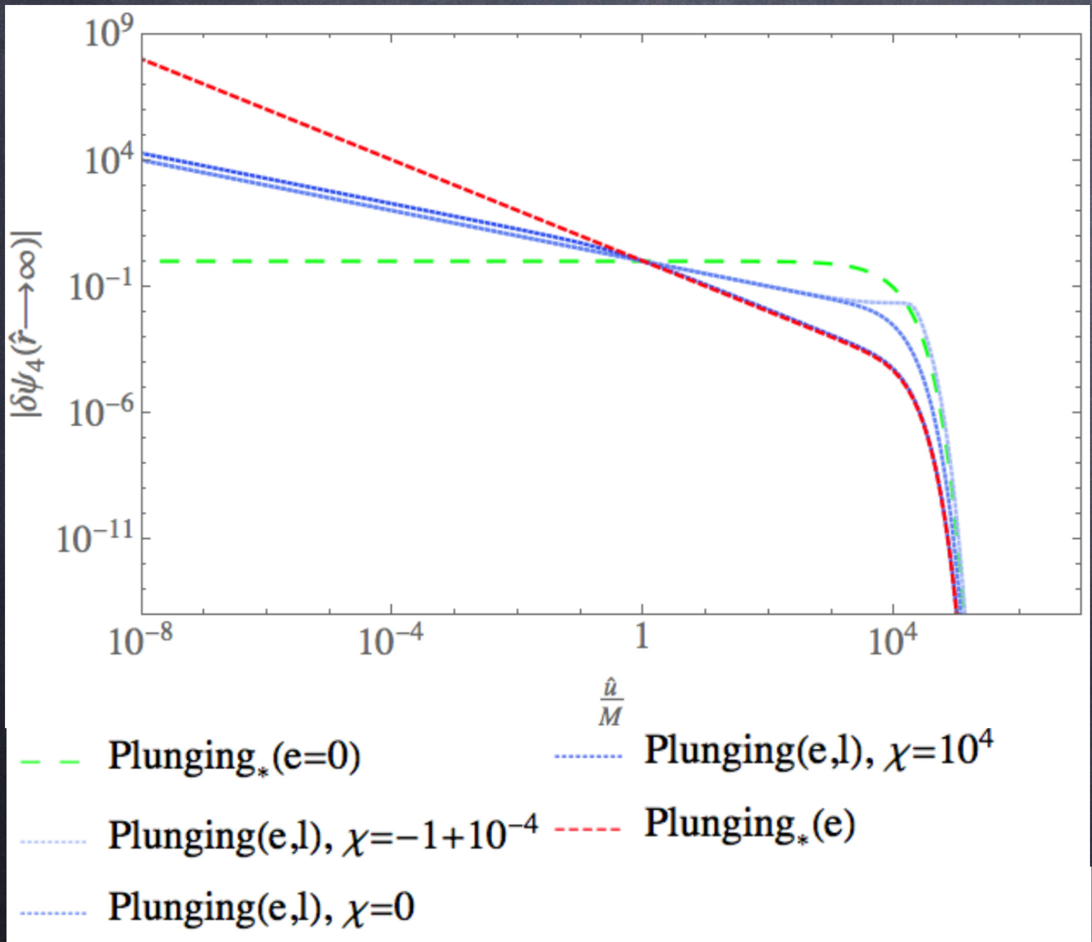
[Gralla, Hughes, Warburton, 2016] [Compère, Fransen, Hertog, Long, 2018]

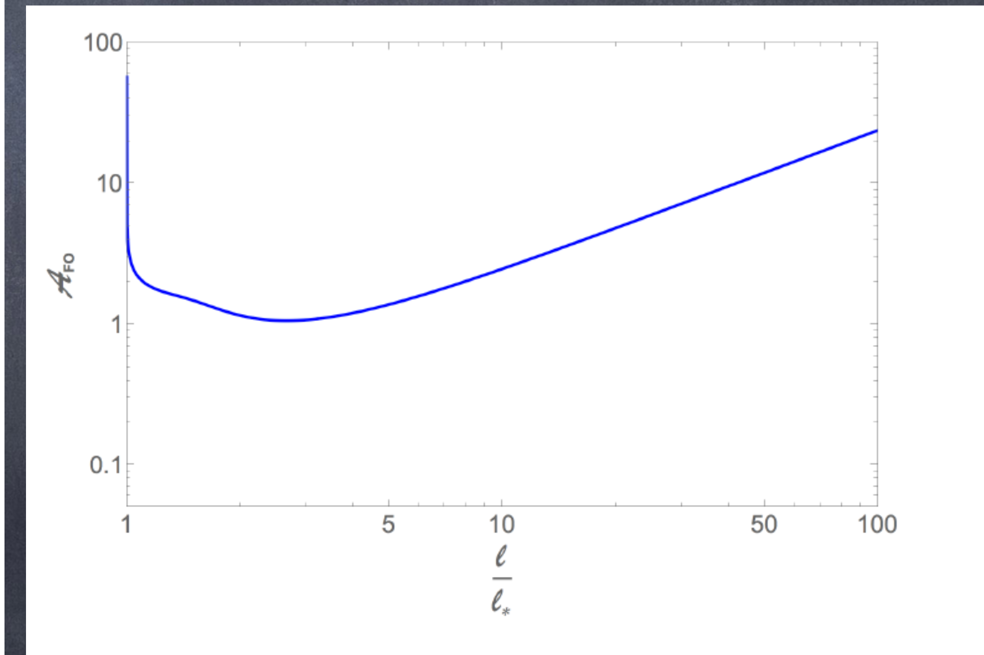
Example: Face-on plunge



$$|h_+ + i h_\times| = \mathcal{A}_{FO}\left(\frac{\ell}{\ell_*}\right)\frac{m_0}{D}\frac{\sqrt{\lambda}}{(\sinh\frac{\lambda\hat{u}}{4M})^{1/2}(\cosh\frac{\lambda\hat{u}}{4M}+\chi\sinh\frac{\lambda\hat{u}}{4M})^{1/2}}$$

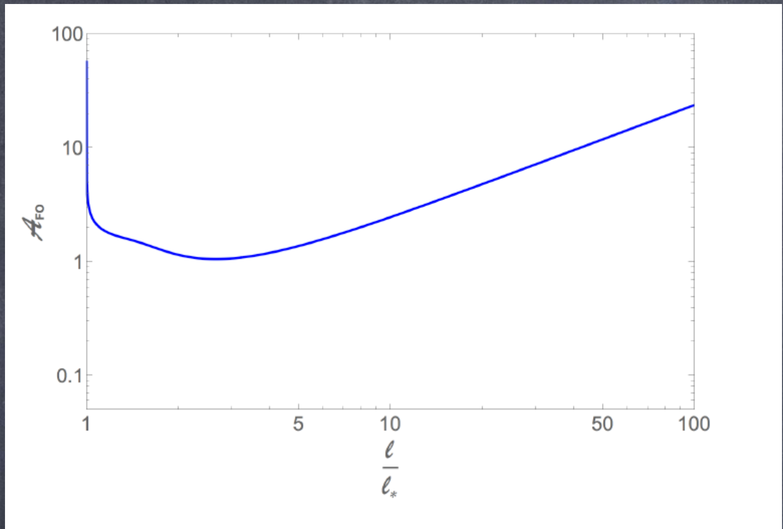
$m = \ell = 2$





[Compère, Fransen, Hertog, Long, 2018]

Feature #5: Critical behaviour

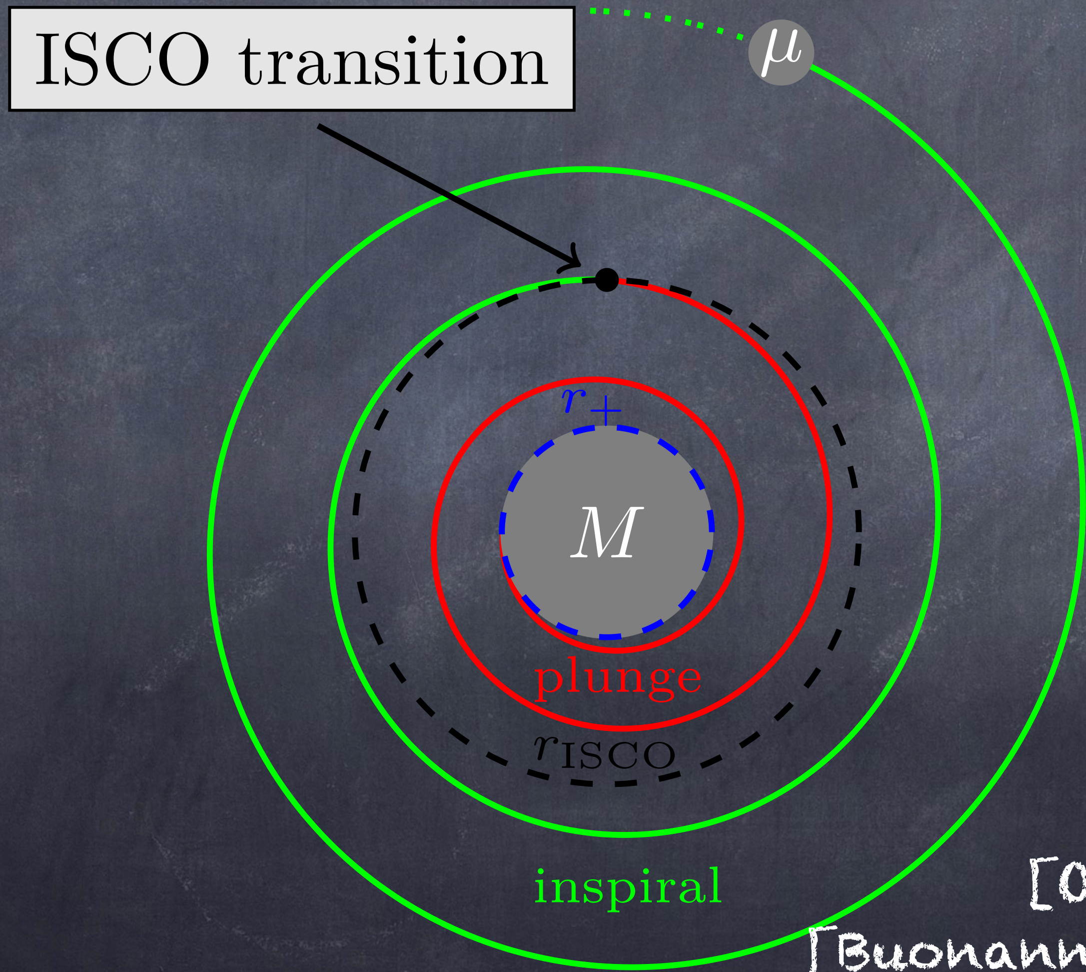


$$\begin{aligned} \mathcal{A}_{FO}\left(\frac{\ell}{\ell_*} \to 1\right) &= 1.1\left(\frac{\ell}{\ell_*} - 1\right)^{-\frac{1}{4}}, \\ \mathcal{A}_{FO}\left(\frac{\ell}{\ell_*} \to \infty\right) &= 0.2\frac{\ell}{\ell_*}. \end{aligned}$$

- Critical behaviour is a typical feature at near-extremality
- Divergence is capped by the match with the flat region: This is the physics of a capped AdS2.

$$\mathcal{A}_{FO}\left(\frac{\ell}{\ell_*} \to 1\right) \sim \left(\frac{\lambda}{\sqrt{1-\frac{\ell_*^2}{\ell^2}}}\right)^{1/2} \sim \sqrt{\frac{\hat{r}_{ISCO} - \hat{r}_+}{M}} \sim \lambda^{1/3}$$

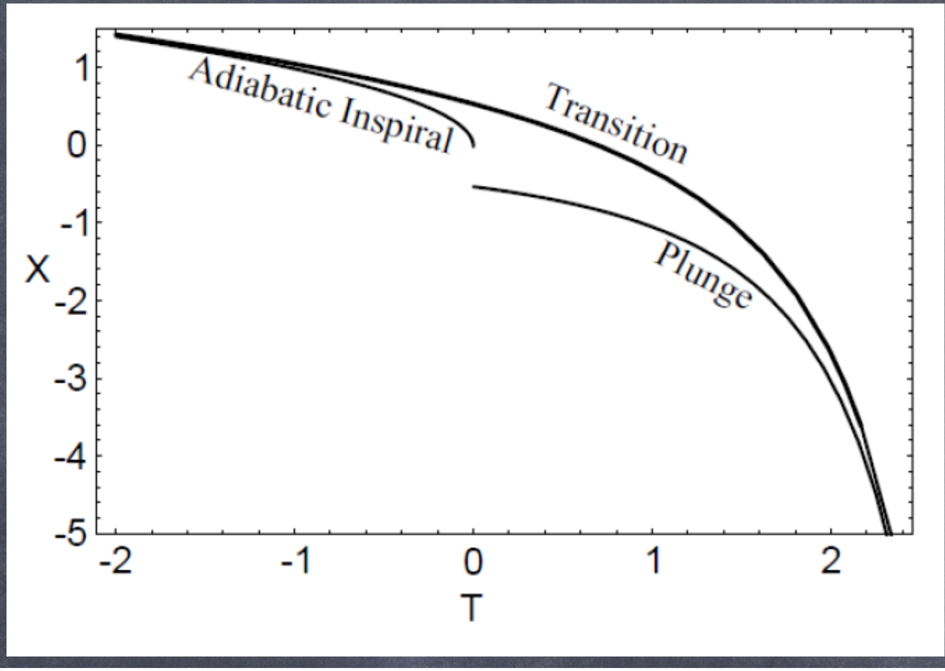
Transition from Inspiral to Plunge



[Ori, Thorne, 2000]
[Buonanno, Damour, 2000]
[Kesden, 2011]

Transition from Inspiral to Plunge:

Standard Spin $\sqrt{1-\frac{a^2}{M^2}} \ll \frac{\mu}{M}$



$$\frac{d^2X}{dt^2} = -X^2 - t + O(\eta^{2/5}), \quad \frac{dY}{dt} = 2X + O(\eta^{2/5}).$$

[Ori, Thorne, 2000]

[Kesden, 2011]

This is the Painlevé transcendent equation of the first kind.

The solution is not valid for Marginally/Extremely High Spins!

Transition from Inspiral to Plunge:

High Spin $\sqrt{1-\frac{a^2}{M^2}} \sim \frac{\mu}{M} \ll 1$

The transition dynamics is now dictated by the equation

$$-\frac{2}{3}\psi - \frac{1}{3}\zeta\frac{d\psi}{d\zeta} + \psi\frac{d\psi}{d\zeta} + \omega^2\frac{d^3\psi}{d\zeta^3} = 0.$$

$$\begin{aligned} t - t_* &= (\frac{2}{9}\zeta + 2^{-2/3})(\frac{\lambda}{\sigma_*\eta})^{2/3}, \\ R_N &= \frac{2}{3}\psi(\zeta) - \frac{2}{3}\zeta, \\ \omega^2 &= \frac{243}{4}(\frac{\sigma_*\eta}{\lambda})^2, \end{aligned}$$

This is the KdV equation with self-similar variables

$$\frac{\partial\Psi}{\partial s} + \Psi\frac{\partial\Psi}{\partial z} + \omega^2\frac{\partial^3\Psi}{\partial z^3} = 0,$$

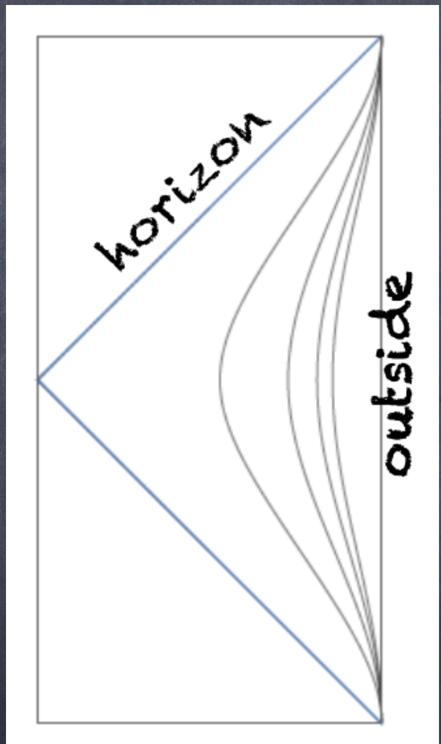
$$\zeta = \frac{z}{s^{1/3}}, \qquad \psi(\zeta) = s^{2/3}\Psi(s,z).$$

The case $\sqrt{1-\frac{a^2}{M^2}} \ll \frac{\mu}{M}$ is ruled out : the cross-section of angular momentum of the primary prevents it.

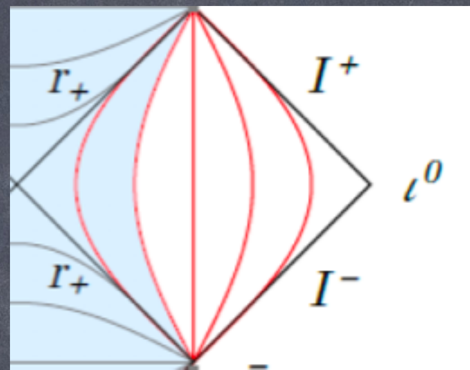
[Compère, Fransen, Jonas, 1909.12848]

Feature #6: Non-decoupling between NHEK and the exterior region

NHEK



Exterior



Travelling waves

"AdS2 with finite cap"

[Amsel, Marolf, Horowitz, Roberts, 2009]

Neat example: scalar self-force

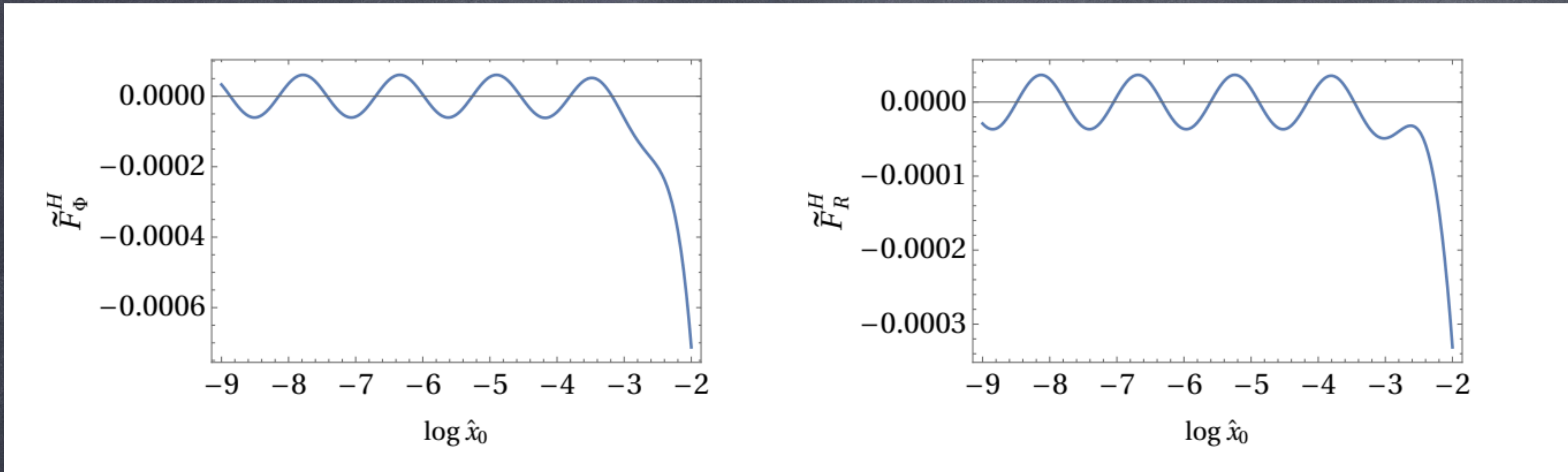
- Complex scalar in Kerr sourced by a point charge on a worldline

$$\nabla_\mu \nabla^\mu \Psi = -4\pi\rho, \qquad \rho = q\int_\gamma \frac{\delta^{(4)}(x^\mu - z^\mu(\tau))}{\sqrt{-g}} \mathrm{d}\tau.$$

- The self-force acting on the particle is $F_\mu(\tau) = \frac{q}{2}(\partial_\mu \Psi)\,|_{z(\tau)} + \text{c.c.}$

- We compute the self-force on a circular orbit in the NHEK geometry

Result



$$\hat{x}_0 \equiv \frac{\hat{r}_0 - \hat{r}_+}{M} = R_0 \lambda^{2/3}$$

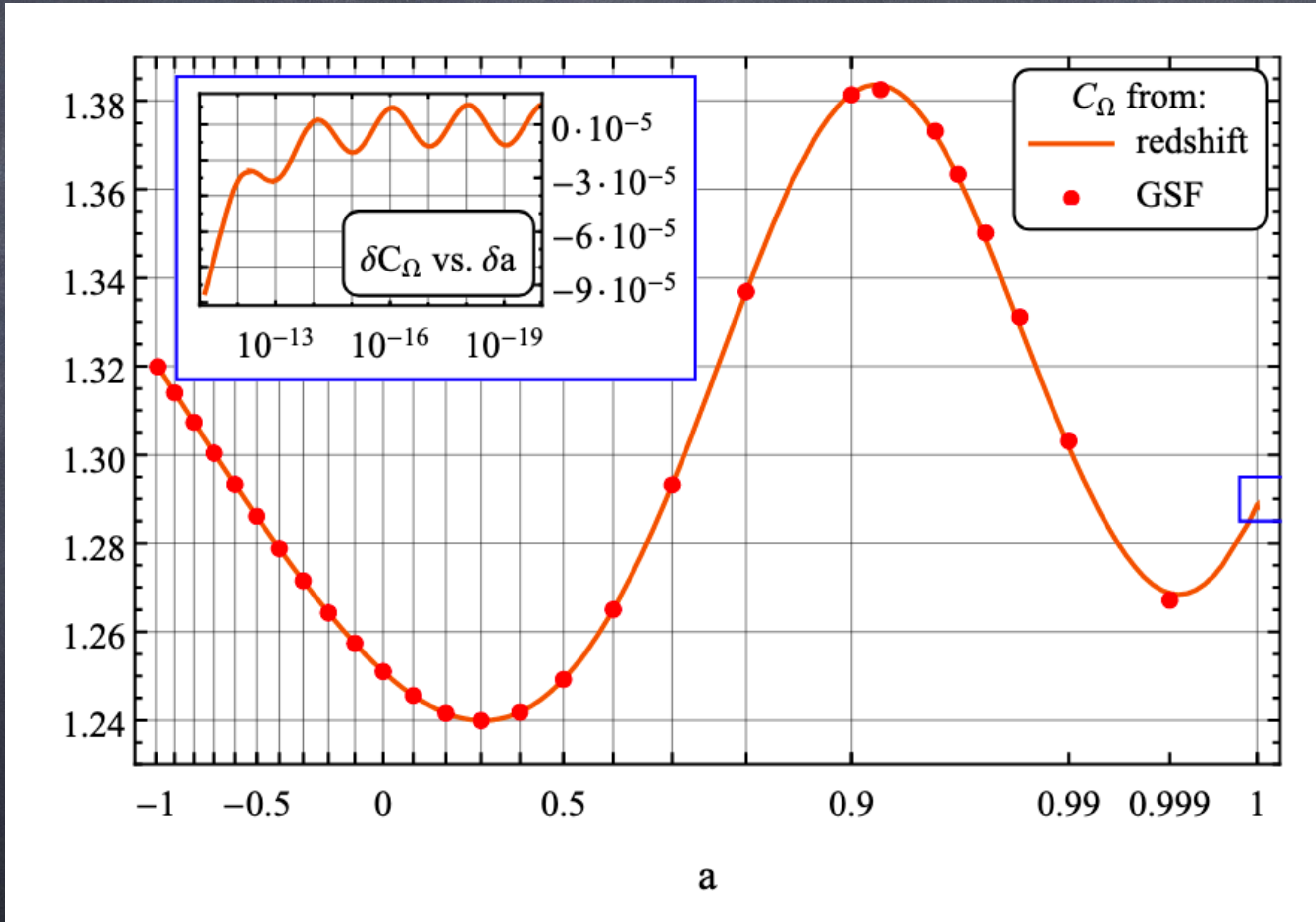
The travelling wave $\ell = m = 2$ is dominant.

The self-force scales as $F_\mu \sim \cos(2\sqrt{-\eta_{22}}\log \hat{x}_0)$ where $h_{22} = \frac{1}{2}(1+\sqrt{\eta_{22}})$ is the conformal weight corresponding to the $\ell = m = 2$ mode.

Conformal invariance is broken to discrete logarithmic periodicity

[Compère, Fransen, Herzog, Liu, 2019]

This effect is only seen in the extremely high spin limit



[van de Meent, 2016]

Conclusion

(Near-)extremal black holes

1) Make us think deeply about quantum gravity (effective field theory, singularities, quantum corrections, classical features, ...)
2) Have mysterious features (Kerr/CFT entropy match, zero entropy?, ...)
3) Lead to specific predictions for observation. Caveat: very high spin is necessary, which is astrophysically very marginal!