Aspects of near-extremal black holes

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Workshop on Near-extremal black holes
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based on 2010.05932 with Burke, Gair, Edwards
and 2102.08060 with Castro, Godet, Song, Yu

What is this talk about ?

Part I

Deriving Schwarzian dynamics for small BHs

Part II

Identifying the JT mode for near-extremal Kerr

Part III

Comparing the precision in spin measurements, using gravitational waves, between near-extremal and non-extremal Kerr BHs

Part I

Small BHs vs Schwarzian dynamics

Universality of Schwarzian dynamics

Regular extremal BHs with isometry $\mathbb{R} \times \mathrm{U}(1)^{d-3}$ have near horizon geometry [Kunduri, Lucietti, Reall]

$$ds^2 = \Gamma(\theta)[\bar{g}_{\mathrm{AdS}_2} + d\theta^2 + \gamma_{ab}(\theta)(d\varphi^a + e^a\rho dt)(d\varphi^b + e^b\rho dt)]$$

Near-AdS$_2$ perspective

The AdS$_2$ throat can be glued to the full BH, suggesting

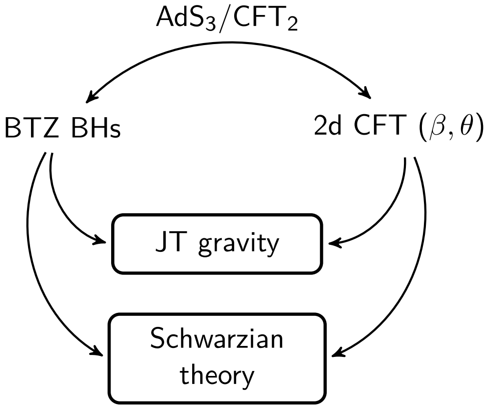
$$r = r_+ + \Phi_{\mathrm{JT}}(t,r)\,, \qquad \Phi_{\mathrm{JT}}(t,r) \ll r_+ \qquad (+\text{ dimensional reduction})$$

In simple and symmetric models $I_{\mathrm{EH}} \to I_{\mathrm{JT}} + \dots$ [Almheiri, Polchinski]

- Low energy $\sim$ JT gravity [Maldacena, Stanford, Yang]

A paradigmatic example

[Ghosh, Maxfield, Turiaci]



Small near-extremal BHs

$\exists$ BHs whose area decreases by shrinking a circle direction along ∂_φ

- Close to the horizon $r \to \epsilon r,\ \epsilon \to 0 \ \Rightarrow\ \gamma_{\varphi\varphi} \sim \gamma(\theta)\, r^2$
- Smoothness requires $\gamma(\theta) = \Gamma(\theta)$
- near horizon geometry

$$ds^2 = \Gamma(\theta)\left(\epsilon^2 r^2(-dt^2 + d\varphi^2) + \frac{dr^2}{r^2}\right) + ds^2_\perp$$

locally AdS_3

Broader perspective

UV field theory

Δ_{ext} $\delta\Delta_{UV}$

Δ_{UV}

AdS | CFT

Gravity

$S_{10d} \sim T$

$\delta\Delta_{UV}$

IR →

Near horizon →

large (singular) gauge transformations

IR 2d CFT

gap $\sim 1/c_{CFT}$

$\delta\Delta_{IR} \sim L_0 - c_{CFT}/24$

$\Delta_{IR} = \Delta_{IR}(\Delta_{UV}, J_i)$

AdS | CFT

Gravity

$S_{3d} \sim S_{Cardy} \sim (c_{CFT}(L_0 - c_{CFT}/24))^{1/2}$

$\sim c_{CFT}\, T$

BTZ intuition

Some thermodynamic formulas

$$M = \frac{r_+^2 + r_-^2}{8G_{\rm N}\ell_3} \qquad J = \frac{r_+ r_-}{4G_{\rm N}\ell_3}$$

$$S_{\rm BTZ} = \frac{\pi}{2G_{\rm N}}\, r_+ \qquad T_{\rm BTZ} = \beta^{-1} = \frac{r_+^2 - r_-^2}{2\pi\ell_3 r_+}$$

Two parameters

- dimensionless *horizon size* in ℓ_3 units : $\frac{r_+}{\ell_3} = \varepsilon$
- *near-extremal* parameters : $r_- = r_+\sqrt{1-\alpha}$

Small & near-extremal BHs

Triple scaling limit

$$\varepsilon \ll 1\,, \qquad \alpha \ll 1\,, \qquad c \to \infty$$

Some relevant scales

$$S_{\rm BTZ} \sim \varepsilon\, c\,, \qquad M,\, J \sim \varepsilon^2\, c$$
$$T_{\rm BTZ} \sim \varepsilon\, \alpha\,, \qquad M - J \sim (\varepsilon\alpha)^2\, c$$

Sub-AdS scale BHs

- Parameterically large M, J $\Rightarrow \varepsilon \sim \frac{1}{\sqrt{c}}$
- Large entropy $\Rightarrow \varepsilon\, c \gg 1$
- $M - J \sim \alpha^2 \ll 1$

2d CFT analysis

Follow [Ghosh, Maxfield, Turiaci]

$$\beta_{\rm L} = (1+\Omega)\beta \approx 2\beta \gg \beta_{\rm R} = (1-\Omega)\beta \approx \frac{\alpha}{2}\beta \sim \frac{1}{\varepsilon} \sim \frac{c}{S_{\rm BTZ}} \gg 1$$

- Dominant CFT partition function

$$Z_{\rm CFT}(\beta,\theta) \approx (2\pi)^2 \left(\frac{2\pi}{\beta_{\rm L}}\right)^{3/2} \left(\frac{2\pi}{\beta_{\rm R}}\right)^{3/2} \exp\left[\frac{\beta_{\rm L}+\beta_{\rm R}}{24} + (2\pi)^2 \frac{c-1}{24}\left(\frac{1}{\beta_{\rm L}} + \frac{1}{\beta_{\rm R}}\right)\right]$$

- Fixed J-ensemble

$$Z_{\rm J} \propto \int d\theta\, e^{i\theta\, J} \exp\left[\frac{\beta}{12} + \frac{(2\pi)^2(c-1)}{12}\frac{\beta}{\beta^2+\theta^2} - \frac{3}{2}\log(\beta^2+\theta^2)\right]$$

CFT saddle

Saddle equation

$$iJ - (2\pi)^2 \frac{c-1}{6(\beta_{\rm L}\beta_{\rm R})^2}\frac{\beta_{\rm R}^2-\beta_{\rm L}^2}{4i} - \frac{3}{2i}\frac{\beta_{\rm R}-\beta_{\rm L}}{\beta_{\rm R}\beta_{\rm L}} = 0$$

Large entropy guarantees 2nd term $\gg$ 3rd term

$$\frac{c\beta_{\rm R}^{-2}}{\beta_{\rm R}^{-1}} \sim \frac{c}{\beta_{\rm R}} \sim \epsilon\, c \gg 1$$

Same saddle, different regime of parameters :

$$\beta_{\rm R} \approx 2\pi\sqrt{\frac{c}{24J}}$$

Evaluation of the saddle

$$Z_{\rm J} \approx \frac{\pi}{2\sqrt{2}}\left[\left(\frac{\pi}{\beta}\right)^{3/2} \exp\left[\frac{\pi^2 c}{12\beta}\right]\right]\left(\frac{c}{6J^3}\right)^{1/4} \exp\left[-\beta\left(J-\frac{1}{12}\right)+2\pi\sqrt{\frac{cJ}{6}}\right]$$

$$\propto Z_{\rm Schwarzian}\, \exp\left[-\beta E_0 + S_{\rm BTZ} - \frac{3}{2}\log S_{\rm BTZ}\right]$$

Validity of CFT saddle

Validity of vacuum dominance (individual state $h = \bar{h} - J$)

$$R_{\rm h} \equiv \frac{\chi_h(2\pi i/\beta_{\rm L})\,\chi_{\bar{h}}(2\pi i/\beta_{\rm R})}{\chi_{\mathbb{I}}(2\pi i/\beta_{\rm L})\,\chi_{\mathbb{I}}(2\pi i/\beta_{\rm R})}$$
$$\approx \frac{1}{(2\pi)^4}\,\exp\left[-(2\pi)^2\left(-\frac{J}{\beta_{\rm L}} + \bar{h}\left(\frac{1}{\beta_{\rm L}} + \frac{1}{\beta_{\rm R}}\right) - \frac{1}{(2\pi)^2}\log(\beta_{\rm L}\beta_{\rm R})\right)\right]$$

requires

$$\boxed{\frac{\bar{h}_{\rm gap}}{\beta_{\rm R}} - \frac{J}{\beta_{\rm L}} - \frac{1}{(2\pi)^2}\log(\beta_{\rm L}\beta_{\rm R}) \gg 1}$$

- $\beta_{\rm L} \sim c$ [Ghosh, Maxfield, Turiaci] $\Rightarrow \alpha \sim \frac{1}{\sqrt{c}}$
- $\bar{h}_{\rm gap} \sim \mathcal{O}(c)$ in 3d pure gravity $\Rightarrow$ $\exists$ consistent regime
- less clear in a generic 2d CFT

Part II

Identifying the JT mode for near-extremal Kerr black holes

JT mode in near-extremal Kerr

Given a near-extremal Kerr metric, whose near horizon limit

$$\tilde{r} = r_+ + \lambda r\,, \qquad \tilde{t} = 2r_+^2 \frac{t}{\lambda}\,, \qquad \tilde{\phi} = \phi + r_+ \frac{t}{\lambda}\,, \quad \lambda \to 0$$

leads to the NHEK geometry

$$ds^2_{\mathrm{NHEK}} = J(1+\cos^2\theta)\left[-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2} + \mathrm{d}\theta^2\right] + J\frac{4\,\sin^2\theta}{1+\cos^2\theta}\left[\mathrm{d}\phi + r\,\mathrm{d}t\right]^2$$

Questions

- how do we identify Φ_{JT} when spherical symmetry is broken ?
- how is Φ_{JT} compatible with Wald's theorem ?

How do we identify Φ_{JT} ?

Strategy

1. Expand NHEK & work at linear order in perturbation h

$$g = g_{\text{NHEK}} + h$$

   in some specific gauge

2. Study sphere harmonics in detail & compare with gauge invariant quantities, such as Weyl scalars Ψ_0, Ψ_4 used in gauge invariant Teukolsky formalism

Axisymmetric NHEK perturbations

$$ds^2 = J\left(1+\cos^2\theta+\epsilon\chi(x,\theta)\right)\left[g_{ab}\mathrm{d}x^a\mathrm{d}x^b+\mathrm{d}\theta^2\right]$$
$$+4J\,\frac{\sin^2\theta}{1+\cos^2\theta+\epsilon\chi(x,\theta)}\left(\mathrm{d}\phi+A_a\mathrm{d}x^a+\epsilon\mathcal{A}\right)^2+O(\epsilon^2)$$

At linear order in ϵ, h is determined by

$$\Box_2\chi+\frac{\sin^3\theta}{\cos\theta}\partial_\theta\left(\frac{\cos^2\theta}{\sin^3\theta}\partial_\theta\left(\frac{\chi}{\cos\theta}\right)\right)=0$$

Eigen-mode expansion

$$\chi(x,\theta)=\sin^2\theta\sum_\ell S_\ell(\theta)\chi_\ell(x)\ ,$$

- $S_\ell \sim$ associated Legendre polynomials with $\ell \geq 2$
- $\chi_\ell(x)$ satisfy the AdS_2 wave equation

$$\Box_2\chi_\ell=\ell(\ell+1)\chi_\ell$$

Tower of AdS_2 modes with $\Delta=\ell+1\geq 3$

$\ell \geq 2$ modes

Matching ingoing/outgoing modes in Teukolsky formalism

δg in IRG [ingoing radiation gauge] $\Leftrightarrow$ Hertz potential Ψ_{H_0}

Using $(\boldsymbol{l}^a, \boldsymbol{n}^a)$ Newman-Penrose tetrads

- Relating $\chi(\mathbf{x}, \theta)$ and Ψ_{H_0} for $\ell \geq 2$

$$\chi(\mathbf{x}, \theta) = -\sin^2\theta\, \boldsymbol{l}^a \boldsymbol{l}^b \nabla_a \nabla_b \Psi_{\mathrm{H}_0}(\mathbf{x}, \theta)$$

- Inversely, if $\Psi_{\mathrm{H}_0}(\mathbf{x}, \theta) = \sum_{\ell \geq 2} U_\ell(\mathbf{x}) S_\ell(\theta)$

$$U_\ell(\mathbf{x}) = -\frac{4}{(\ell-1)\ell(\ell+1)(\ell+2)} \boldsymbol{n}^a \boldsymbol{n}^b \nabla_a \nabla_b \chi_\ell(\mathbf{x})$$

Conclusion : Normalizable $\ell \geq 2$ modes in one–to–one correspondence with ingoing/outgoing modes in Teukolsky formalism

Where is Φ_{JT} ?

Hints

- There are no associated Legendre polynomials with $\ell = 0, 1$. However, these values are allowed by the AdS_2 Breitenlohner-Freedman bound
- More precisely, $\exists\, S_\ell(\theta)$ for $\ell = 1, 0$
  - non-normalizable on the 2-sphere
  - have conical-like singularities at either north/south poles
- Both Ψ_0 and Ψ_4 diverge at these singularities $\Rightarrow$ infinite energy flux

Observation

Require the energy flux to vanish, while keeping the $\ell = 1, 0$ modes

- $\ell = 1$: $\Psi_0 = \Psi_4 = 0$ + AdS_2 wave equation implies

$$\nabla_a \nabla_b \chi - g_{ab} \Box_2 \chi + g_{ab} \chi = 0 \qquad \text{JT equations !!}$$

- $\ell = 0$: $\Psi_0 = \Psi_4 = 0$ + AdS_2 wave equation $\Rightarrow$ zero mode

What about the singularity ?

$\ell = 1$ modes with vanishing $\Psi_0 = \Psi_4 = 0 \Rightarrow \chi_1(x)$ satisfies JT eqs

- our NHEK perturbation remains singular

Hint & Suggestion

- *h* was written in a particular gauge
- can we apply a diffeo to describe *h* as an smooth perturbation ?
  - can we find a *singular* diffeomorphism that allows to remove the singularity in our gauge ?

Balancing the conical-like singularity

1. Since ϕ is periodic, any diffeomorphism of the form

$$\xi^{\mu}(x,\theta,\phi) = \frac{\epsilon}{2}\phi\,\zeta^{\mu}(x,\theta)$$

   will be *non-single valued* (generating a conical-like singularity)

2. Requiring the action of the diffeo to be axisymmetric

$$\partial_{\phi}(\mathcal{L}_{\xi} g_{\mathrm{NHEK}}) = 0 \quad \Rightarrow \quad \zeta(x,\theta) \text{ Killing}$$

3. Killing vector fields of metrics (including g_{NHEK})

$$ds^2 = \Lambda(\theta)(g_{\mathrm{AdS}_2} + \mathrm{d}\theta^2) + \Gamma(\theta)(\mathrm{d}\phi + A_a \mathrm{d}x^a)^2$$

   are of the form

$$\zeta = \varepsilon^{ba}\nabla_b\Phi_{\zeta}\partial_a + (\Phi_{\zeta} + \varepsilon^{ab}A_a\nabla_b\Phi_{\zeta})\partial_{\phi} \quad \text{with} \quad \Phi_{\zeta} \equiv c_{(\phi)} + \Phi_{\mathrm{JT}}$$

Balancing the conical-like singularity

1. Killing vectors ζ are in one–to–one correspondence with a scalar field $\Phi_{\rm JT}(x)$ solving the JT equations of motion ($\ell = 1$ mode) and a constant zero mode $c_{(\phi)}$ ($\ell = 0$ mode).
   - $\Phi_{\rm JT}(x)$ is non-dynamical (it is a Killing vector field !!)
   - however, the diffeomorphism ξ is *physical*, it generates a conical-like singularity carrying energy !!
2. Tuning the diffeomorphism, i.e. choosing

$$\boxed{\chi(x,\theta) = \Phi_{\rm JT}(x) + \frac{1}{2}(1+\cos^2\theta)\, c_{(\phi)}}$$

gives rise to smooth $\ell = 1, 0$ perturbations

$$g = g_{\rm NHEK} + h + \mathcal{L}_\zeta g_{\rm NHEK}$$

where the scalar controlling the perturbation satisfies the JT eqs

Consistency with Wald's theorem

Wald's theorem

Smooth $\delta\tilde{g}$ in full Kerr with vanishing Weyl scalars must satisfy

$$\delta\tilde{g} = \delta_M\tilde{g} + \delta_J\tilde{g} + \epsilon\,\mathcal{L}_{\tilde{\xi}}\tilde{g}$$

In 2102.08060, we proved

- all such finite perturbations correspond in the $\lambda \to 0$ to our smooth $\ell = 0, 1$ NHEK perturbations (up to diffeos)

$$\chi(\mathbf{x}, \theta) = \Phi_{\mathrm{JT}}(\mathbf{x}) + \frac{1}{2}(1 + \cos^2\theta)\, c_{(\phi)}$$

- the identification crucially depends on $\delta M \sim \lambda^n\,\epsilon$
- singular $\ell = 0$ NHEK perturbations give rise to Taub-Nut and/or C-metric deformations
- $\exists\ \delta_M\tilde{g} = \delta_J\tilde{g} = 0$ and $\mathcal{L}_{\tilde{\xi}}\tilde{g} \neq 0$ Kerr diffeos, but not a NHEK one

Part III

Precision in spin measurements using gravitational waves

Precision in spin measurement

Before the lockdown, I was asked

1. Can the spin of a near-extremal Kerr black hole (BH) be measured with higher accuracy than for non-extremal BHs using gravitational waves ?
2. Either way, can you (ideally quantitatively) explain why ?

Context

Binary black hole (BH) system with masses $M \gg \mu$ $(\eta = \mu/M \ll 1)$

- Approximate the motion of the secondary BH (μ) as a point particle inspiraling towards the primary BH (M) with outer horizon (EMRI)

$$r_+ = M\left(1 + \sqrt{1 - a^2}\right) \equiv M(1+\epsilon) \qquad a = \hat{a}/M$$

- Consider circular orbits (absence of radiation)

$$E(\tilde{r}, a) = \mu \, \frac{1 - 2/\tilde{r} + a/\tilde{r}^{3/2}}{\sqrt{1 - 3/\tilde{r} + 2a/\tilde{r}^{3/2}}} \quad \text{with} \quad \tilde{r} = r/M$$

- Due to energy conservation $(E + E_{\rm GW} = \text{const})$

$$\frac{dE}{dt} = \partial_r E \, \frac{dr}{dt} = -\dot{E}_{\rm GW}$$

Perturbative expansion

Define $\tilde{E} = E/\mu$, $\tilde{t} = t/M$, the inwards spiral trajectory

$$\frac{d\tilde{E}}{d\tilde{t}} = \partial_{\tilde{r}}\tilde{E}\,\frac{d\tilde{r}}{d\tilde{t}} = -\frac{1}{\eta}\dot{E}_{\mathrm{GW}}(\tilde{r}, a) = -P_{\mathrm{GW}}\,,$$

- $\dot{E}_{\mathrm{GW}} \sim \mathcal{O}(\eta^2)$ is energy rate carried away by gravitational waves
  - Computed by first principles solving Teukolsky's equation in the presence of the source (μ)
- $\Rightarrow\ P_{\mathrm{GW}} \sim \mathcal{O}(\eta)$
- Spiral trajectory contains two pieces of information
  1. kinematic : geodesic information $(\partial_{\tilde{r}}\tilde{E})$
  2. dynamic : Teukolsky's equation $[\dot{E}_{\mathrm{GW}}(\tilde{r}, a)]$

Fisher matrix for gravitational waves

The spin precision Δa is given by

$$\Delta a = \sqrt{(\Gamma^{-1})_{aa}} \quad \text{with} \quad \Gamma_{aa} = 4 \int df \, \frac{|\partial_a \tilde{h}(f)|^2}{S_n(f)} \, ,$$

- $\tilde{h}(f)$ is the Fourier transformed of the amplitude in the gravitational wave

$$h(t) = \sum_m h_m(t) \approx \sum_m \frac{2\sqrt{\dot{E}_m^\infty}}{m \tilde{\Omega} \tilde{D}} \sin(m \tilde{\Omega} \tilde{t}) \, ,$$

- $S_n(f)$, power spectral density (PSD) : describes the LISA noise.

Estimating the Fisher matrix

- Assume $S_n(f) \approx S_n(f_\circ)$ (standard in relevant literature) and use Parseval
$$|\partial_a h(t)|^2 = \sum_m (\partial_a h_m)^2 + 2 \sum_{n<m} \partial_a h_n \, \partial_a h_m \, ,$$
- Consider "diagonal terms"

$$\partial_a h_m(t) = |h_m(t)| \left\{ \sin(m\tilde{\Omega}\tilde{t}) \, \mathcal{B}_m + (m\tilde{t}\partial_a\tilde{\Omega}) \, \cos(m\tilde{\Omega}\tilde{t}) \right\}$$

$$\mathcal{B}_m \equiv \frac{\partial_a \dot{E}_m^\infty}{2\dot{E}_m^\infty} - \frac{\partial_a \tilde{\Omega}}{\tilde{\Omega}}$$

$$(\partial_a h_m)^2 = \frac{|h_m|^2}{2} \left\{ (m\tilde{t} \, \partial_a\tilde{\Omega})^2 + (\mathcal{B}_m)^2 + \cos(2m\tilde{\Omega}\tilde{t}) \left[(m\tilde{t} \, \partial_a\tilde{\Omega})^2 - (\mathcal{B}_m)^2 \right] + 2\sin(2m\tilde{\Omega}\tilde{t}) \, \mathcal{B}_m \, (m\tilde{t} \, \partial_a\tilde{\Omega}) \right\} .$$

Approximation

- EMRI physics $\Rightarrow P_{\rm GW} \sim \eta \ll 1 \Rightarrow \tilde{t} \sim \eta^{-1}$
  Dominant contribution from the $\mathcal{O}(\tilde{t}^2)$
- More precisely, we assume (numerically checked)

$$\left|\frac{\partial_a \dot{E}_{\infty 2}}{\tilde{t}\, \dot{E}_{\infty 2}}\right| \ll \partial_a \tilde{\Omega}$$

- the time integral is over long periods $\Rightarrow$ oscillatory terms would be subleading (anyway)
- similar arguments hold for off-diagonal terms

Fisher matrix estimation

Since $\tilde{\Omega}^{-1} = \tilde{r}^{3/2} + a$, it follows

$$\partial_a \tilde{\Omega} = -\tilde{\Omega}^2 \left(1 + \frac{3}{2}\sqrt{\tilde{r}}\,\partial_a \tilde{r}\right)$$

Hence,

$$\boxed{\Gamma_{aa} \approx \frac{16M}{\tilde{D}^2\, S_n(f_\text{o})} \int_{\tilde{t}_0}^{\tilde{t}_\text{cut}} d\tilde{t}\; \dot{E}_{\infty 2}\, (\tilde{\Omega}\tilde{t})^2 \left(1 + \frac{3}{2}\sqrt{\tilde{r}}\,\partial_a \tilde{r}\right)^2}$$

or, in radial coordinate (using the spiral equation)

$$\boxed{\Gamma_{aa} \approx \frac{16\mu}{(\eta\tilde{D})^2\, S_n(f_\text{o})} \int_{\tilde{r}_\text{cut}}^{\tilde{r}_0} d\tilde{r}\, (\partial_{\tilde{r}} \tilde{E})\, (\eta\tilde{t}\,\tilde{\Omega})^2 \left(1 + \frac{3}{2}\sqrt{\tilde{r}}\,\partial_a \tilde{r}\right)^2 .}$$

Spin dependence on the trajectory

Define $u = \partial_a \tilde{r}$

- Remember the inwards spiral equation

$$\partial_{\tilde{r}} \tilde{E}(\tilde{r}, a)\, \frac{d\tilde{r}}{d\tilde{t}} = -P_{\rm GW}(\tilde{r}, a)$$

- Apply $\frac{d}{da}$, taking into account explicit and implicit dependence

$$\frac{d}{da}\partial_{\tilde{r}}\tilde{E} = (\partial^2_{\tilde{r}}\tilde{E})\, u + \partial^2_{a\tilde{r}}\tilde{E}\,,$$
$$\frac{dP_{\rm GW}}{da} = \left(\partial_{\tilde{r}} P_{\rm GW}\right)\, u + \partial_a P_{\rm GW}\,.$$

- Using

$$\frac{du}{d\tilde{t}} = \frac{du}{d\tilde{r}}\frac{d\tilde{r}}{d\tilde{t}}\,,$$

one derives a linear ODE

$$\boxed{\frac{du}{d\tilde{r}} + \left(\frac{\partial^2_{\tilde{r}}\tilde{E}}{\partial_{\tilde{r}}\tilde{E}} - \frac{\partial_{\tilde{r}} P_{\rm GW}}{P_{\rm GW}}\right)\, u = -\frac{\partial^2_{a\tilde{r}}\tilde{E}}{\partial_{\tilde{r}}\tilde{E}} + \frac{\partial_a P_{\rm GW}}{P_{\rm GW}}\,.}$$

Two cases to keep in mind

1. Near-extremal and close to the extremal horizon $x \equiv \tilde{r} - 1 \ll 1$

$$P_{\rm GW} = \eta\, \tilde{C}\, x$$ [Gralla, Porfyriadis, Warburton]

2. Non-extremal (generic Finn-Thorne parameterisation)

$$P_{\rm GW} = \frac{32}{5}\eta\, \tilde{\Omega}^{10/3}\, \dot{\mathcal{E}}$$

$\dot{\mathcal{E}}$ relativistic corrections (computed numerically)

$$\partial_a \tilde{r} = \frac{1}{\mathcal{Q}}\left(k_0 - \int \mathcal{Q}\, \partial_a \log \mathcal{Q}\, d\tilde{r}\right), \quad \mathcal{Q} = \frac{\partial_{\tilde{r}} \tilde{E}}{\tilde{\Omega}^{10/3} \dot{\mathcal{E}}}.$$

with a source term allowing the decomposition

$$\mathcal{Q}\, \partial_a \log \mathcal{Q} = \frac{\partial_{\tilde{r}} \tilde{E}}{\tilde{\Omega}^{10/3} \dot{\mathcal{E}}}\left(\frac{\partial^2_{a\tilde{r}} \tilde{E}}{\partial_{\tilde{r}} \tilde{E}} - \frac{\partial_a \dot{\mathcal{E}}}{\dot{\mathcal{E}}} + \frac{10}{3}\tilde{\Omega}\right)$$

- First and third terms are *kinematic*, i.e., driven by geodesic physics
- Second term is *dynamical*, i.e., driven by the energy flux

Brief comparison

Let

- ϵ be near-extremal parameter
- $\tilde{r} - \tilde{r}_{\rm isco} \sim \delta$ coordinate distance to ISCO

Analytic estimates

$$\partial_a \tilde{r} \propto \begin{cases} \frac{1}{\delta}, & \text{moderate spins} \\ \frac{\epsilon^{2/3}}{\delta(\delta+\epsilon^{2/3})^2}, & \text{near-extremal spins} \end{cases}$$

suggest the spin dependence in near-extremal Kerr is larger

Ratio of Fisher matrices

Ignoring angular velocity variation and including all modes

$$\Gamma_{aa} \approx 18 \frac{\mu}{(\eta \tilde{D})^2 \, S_n(f_\circ)} \, \tilde{r}_{\rm ext} \, \tilde{\Omega}^2_{\rm ext} \sum_m \int_0^{\tilde{t}_{\rm cut}} d(\eta \tilde{t}) \frac{d\tilde{E}_m^\infty}{\eta d\tilde{t}} \, (\eta \tilde{t})^2 \, (\partial_a \tilde{r})^2 .$$

Ratio of spin precisions

- Numerical evaluation (single Fisher parameter)

$$\frac{\Gamma_{aa}^{\rm ext}}{\Gamma_{aa}^{\rm mod}} \sim 500$$

confirms our analytic estimates

Conclusions

Part I

Small near-extremal BHs with sub-AdS scale local AdS_3 geometries may still be controlled by Schwarzian dynamics

Part II

- $\exists\ \ell = 1$ smooth irrelevant AdS_2 perturbations satisfying JT equations of motion
- When glued to asymptotically flat Kerr, it corresponds to a mass perturbation, in agreement with Wald's theorem
- Similar statements hold for $\ell = 0$ marginal AdS_2 deformations

Part III

- Analytic techniques to estimate Fisher matrices in EMRI set-ups
- Near-extremal Kerr BHs expected to have 2 orders of magnitude increase in the precision of spin using gravitational waves within an EMRI set-up compare to moderate spin ones