Quantum Simulations of Interacting Systems Using Phase Space Methods

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Plan of the talk

1. Truncated Wigner Approach and the Dirac time-dependent MF approximation

2. Cluster TWA.

3. Some applications: diffusion, dynamic structure factor, disordered spin systems

4. Fermion TWA

Variational (saddle point) approach to quantum dynamics

Example: weakly interacting bosons on a lattice (Bose-Hubbard model)

$$\hat{H} = -J\sum_{\langle ij\rangle}(\hat{a}_i^\dagger \hat{a}_j + \hat{a}_i \hat{a}_j^\dagger) + \frac{U}{2}\hat{n}_j(\hat{n}_j + 1), \quad \hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$$

Quench dynamics: interested in some observable:

$$O(t) = \langle \psi_0 | \mathrm{e}^{i\hat{H}t} \hat{O}\, \mathrm{e}^{-i\hat{H}t} | \psi_0 \rangle$$

Operators to numbers: insert a complete set of coherent - classical - states (Schwinger-Keldysh path integral)

$$O(t) = \langle \psi_0 | \alpha_0 \rangle \langle \alpha_0 | \mathrm{e}^{i\hat{H}\delta t} | \alpha_1 \rangle \langle \alpha_1 | \mathrm{e}^{i\hat{H}\delta t} | \alpha_2 \rangle \dots \langle \alpha_N | \hat{O} | \beta_N \rangle$$
$$\langle \beta_N | \mathrm{e}^{-i\hat{H}\delta t} | \beta_{N-1} \rangle \langle \beta_{N-1} | \dots | \beta_0 \rangle \langle \beta_0 | \psi_0 \rangle$$

Take the saddle point (variational) approximation with respect to α_j, β_j. Result: Truncated Wigner Approximation

Standard Truncated Wigner Approximation (TWA)

$$O(t) \approx \int\int D\psi_0^* D\psi_0 W(\psi_{0j}^*, \psi_{0j}) O_W(\psi_j^*(t), \psi_j(t)),$$

$$i\hbar \frac{\partial \psi_j}{\partial t} = \frac{\partial H_W(\psi^*, \psi)}{\partial \psi_j^*}$$

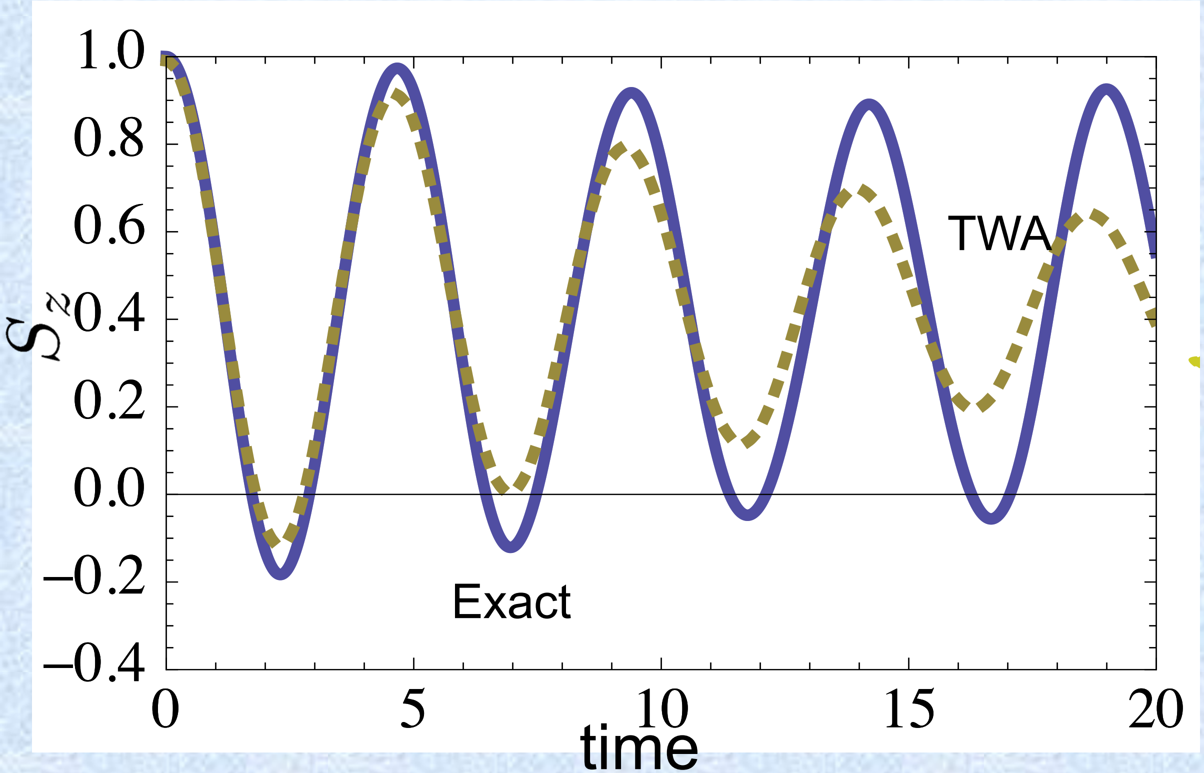
Classical (mean-field) discrete Gross-Pitaevski equation

1. Interpretation: ***many*** mean-field states evolved in parallel, not one like the Dirac time-dependent variational ansatz assumes.
2. TWA is asymptotically exact in the classical limit (large S limit), harmonic limit, or long-range (large N) limit
3. Asymptotically exact at short times
4. Easy to simulate if W is positive. Within accuracy of TWA the Gaussian approximation for W works.
5. Can extend TWA to arbitrary systems with the classical limit (classical Poisson brackets).
6. Many applications: quantum optics, spin systems, cold atoms, quantum chemistry

What if the elementary local degree of freedom (site) has 3 states? E.g. a spin one system.

$$\mathcal{H} = -\mathbf{B}\mathbf{S} + \frac{U}{2} S_z^2$$

TWA fails after a short time unless interactions are weak.



$$\mathcal{H} = -S_x - S_z + \frac{1}{2} S_z^2$$

Prepare the spin initially polarized along *z.*

TWA fails. No small parameter to justify it.

Idea: fix TWA introducing additional (hidden) variables

(S. Davidson and A.P., PRL 2015)

Go to SU(3) group. Any 3x3 Hamiltonian is a linear combination of SU(3) generators.

(Mapping taken from M. Kiselev, et. al. EPL (2013) for LZ problem in a 3 level system)

$$X_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} X_2 = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$X_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

....

$$X_8 = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$X_1 = S_x,\ X_2 = S_y,\ X_3 = S_z,\ X_4 = (S_x)^2 - (S_y)^2,\ X_5 = [S_x, S_y]_+,$$
$$X_6 = [S_x, S_z]_+,\ X_7 = [S_y, S_z]_+,\ X_8 = \frac{1}{\sqrt{3}}\left((S_x)^2 + (S_y)^2 - 2(S_z)^2\right)$$

Schwinger bosons: $X_1 \to a^\dagger_\alpha X_1^{\alpha\beta} a_\beta,\ a^\dagger_1 a_1 + a^\dagger_2 a_2 + a^\dagger_3 a_3 = N = 1$

$$H = \frac{U}{2} S_z^2 - S_z$$

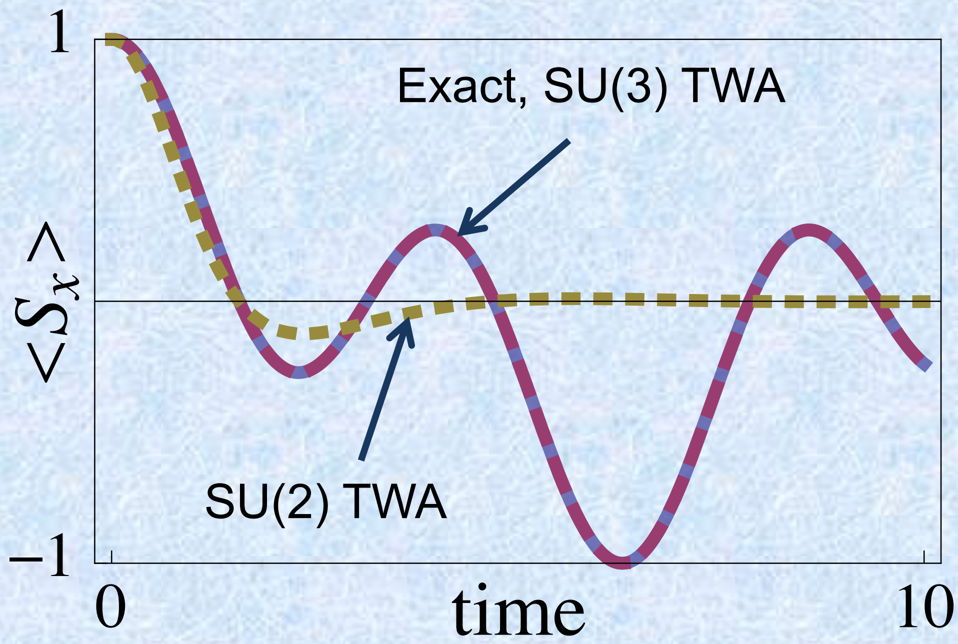
Single site Hamiltonian of Hubbard model: interaction and chemical potential

$$(H_I)_W^{SU(2)} = (U/2) X_3^2 - X_3, \quad (H_I)_W^{SU(3)} = (U/6)(2 - \sqrt{3} X_8) - X_3.$$

Map interacting SU(2) spin to noninteracting (= linear) SU(3)) spin

TWA, solve SU(3) Bloch equation: $\dot{X}_a = f_{abc} \frac{\partial H}{\partial X_\beta} X_\gamma$

Start from a state polarized along x

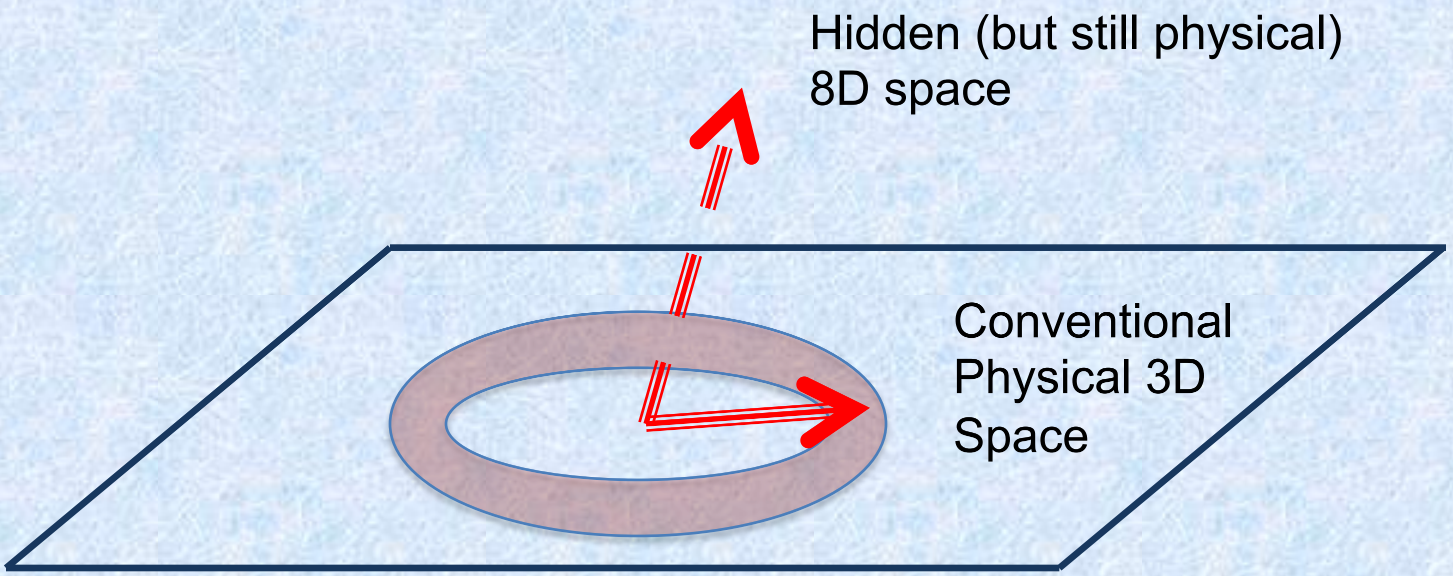


SU(3) TWA – (semi)classical dynamics in 8-dimensional phase space.

Extra variables are like hidden variables.

What did we achieve?

Classical dynamics becomes exact if we go to a higher-dimensional phase space.



If we solve classical equations in 8D space and project to 3D space we are exact (for a single spin one)

Many-body generalization.

Bose Hubbard model in spin 1 representation (E. Altman 2001)

$$H_{eff} = \frac{U}{2}\sum_i (\hat{S}_z^i)^2 - J\bar{n}\sum_{\langle ij\rangle}(\hat{S}_x^i\hat{S}_x^j + \hat{S}_y^i\hat{S}_y^j) - \mu\sum_i \hat{S}_z^i$$

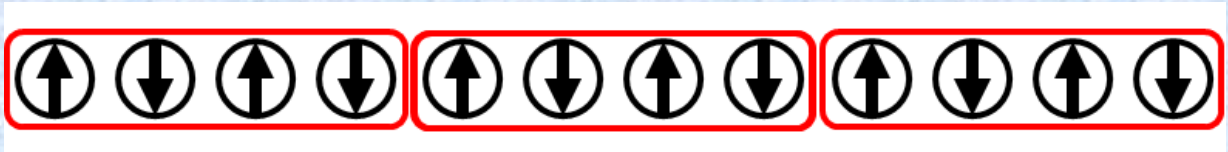
Treat local interactions exactly by mapping to SU(3) spins.
Treat NN interactions semiclassically within TWA.

$$H_{eff} = -\frac{\sqrt{3}U}{6}\sum_i X_8^i - \mu\sum_i X_3^i - J\bar{n}\sum_{\langle ij\rangle}(X_1^iX_1^j + X_2^iX_2^j)$$

Small hopping or large dimensionality (connectivity) – expect SU(3) TWA to work much better than SU(2) TWA.

Similar in spirit to DMFT (asymptotically correct in high dimensions) and DMRG (convert linear Schrodinger equation to nonlinear classical equations). Can treat both spatial and time correlations.

Cluster TWA (CTWA)



Hilbert space of each cluster is spanned by SU(N) group. N – Hilbert Space Dimension. N=16 in the shown example.

$$H_{\text{cluster}} = \sum_i \sum_{\alpha=1}^{N^2-1} h_\alpha^{(i)} \hat{X}_\alpha^{(i)} \qquad H_{\text{cluster}-\text{cluster}} = \sum_{\langle ij \rangle} \sum_{\alpha\beta} J_{\alpha\beta}^{(ij)} \hat{X}_\alpha^{(i)} \hat{X}_\beta^{(j)}$$

Classical equations of motion

$$\dot{X}_\alpha^{(i)} = f_{\alpha\beta\gamma} \frac{\partial H}{\partial X_\beta^i} X_\gamma^{(i)}, \quad i[X_\alpha^{(i)} X_\beta^{(j)}] = \delta_{ij} f_{\alpha\beta\gamma} X_\gamma^{(i)}$$

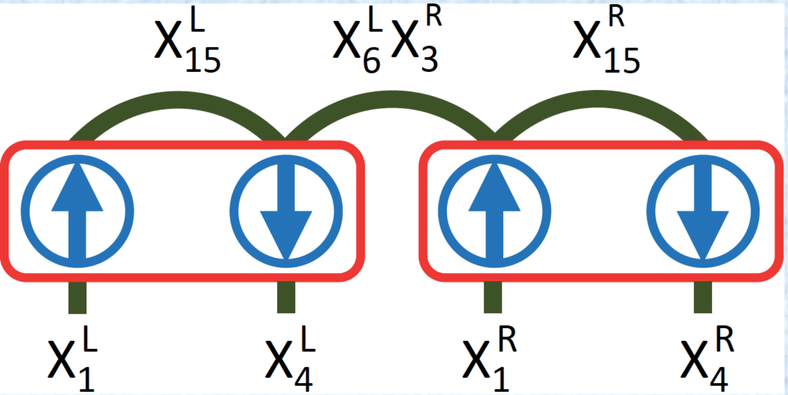
Initial conditions. Choose a Gaussian factorized distribution $W(\vec{X})$

$$\langle \hat{X}_\alpha \rangle = \int D\vec{X} X_\alpha W(\vec{X}), \quad \langle \hat{X}_\alpha \hat{X}_\beta + \hat{X}_\beta \hat{X}_\alpha \rangle \equiv \sum_{\gamma+1}^{N^2} d_{\alpha\beta\gamma} \langle \hat{X}_\gamma \rangle = 2 \int D\vec{X} X_\alpha X_\beta W(\vec{X})$$

This choice can be justified from the short time expansion. Alternative discrete sampling: W. Wooters et. al. 2004; works by A.M. Rey et. al.

Example: four sites

$$\hat{H} = J\sum_{j=1}^{3}\hat{\sigma}_z^{(j)}\hat{\sigma}_z^{(j+1)} + h_x\sum_{j=1}^{4}\hat{\sigma}_x^{(j)}.$$



$$\hat{X}_0 = \hat{I}^{(1)}\otimes\hat{I}^{(2)}$$

$$\hat{X}_1 = \hat{\sigma}_x^{(1)}\otimes\hat{I}^{(2)},\quad \hat{X}_2 = \hat{\sigma}_y^{(1)}\otimes\hat{I}^{(2)},\quad \hat{X}_3 = \hat{\sigma}_z^{(1)}\otimes\hat{I}^{(2)}$$

$$\hat{X}_4 = \hat{I}^{(1)}\otimes\hat{\sigma}_x^{(2)},\quad \hat{X}_5 = \hat{I}^{(1)}\otimes\hat{\sigma}_y^{(2)},\quad \hat{X}_6 = \hat{I}^{(1)}\otimes\hat{\sigma}_z^{(2)}$$

$$\hat{X}_7 = \hat{\sigma}_x^{(1)}\otimes\hat{\sigma}_x^{(2)},\quad \hat{X}_8 = \hat{\sigma}_x^{(1)}\otimes\hat{\sigma}_y^{(2)},\quad \hat{X}_9 = \hat{\sigma}_x^{(1)}\otimes\hat{\sigma}_z^{(2)}$$

$$\hat{X}_{10} = \hat{\sigma}_y^{(1)}\otimes\hat{\sigma}_x^{(2)},\quad \hat{X}_{11} = \hat{\sigma}_y^{(1)}\otimes\hat{\sigma}_y^{(2)},\quad \hat{X}_{12} = \hat{\sigma}_y^{(1)}\otimes\hat{\sigma}_z^{(2)}$$

$$\hat{X}_{13} = \hat{\sigma}_z^{(1)}\otimes\hat{\sigma}_x^{(2)},\quad \hat{X}_{14} = \hat{\sigma}_z^{(1)}\otimes\hat{\sigma}_y^{(2)},\quad \hat{X}_{15} = \hat{\sigma}_z^{(1)}\otimes\hat{\sigma}_z^{(2)}$$

Alternative choice: $\hat{X}_{\alpha\beta} = |\alpha\rangle\langle\beta|$

Treat local correlations (entangled degrees of freedom) as independent variables

$$\hat{H} = J\left(\hat{X}^L_{15} + \hat{X}^R_{15} + \hat{X}^L_6\hat{X}^R_3\right) + h_x\left(\hat{X}^L_1 + \hat{X}^L_4 + \hat{X}^R_1 + \hat{X}^R_4\right).$$

Some operators are correlated

$$\langle\psi_0|\hat{X}_3^{L,R}|\psi_0\rangle = 1,\ \langle\psi_0|\hat{X}_6^{L,R}|\psi_0\rangle = \langle\psi_0|\hat{X}_{15}^{L,R}|\psi_0\rangle = -1$$

$$\left\langle\psi_0\right|\left(\hat{X}^L_\alpha\right)^2\left|\psi_0\right\rangle = 1,\quad \alpha\in\{1,\ldots,15\}$$

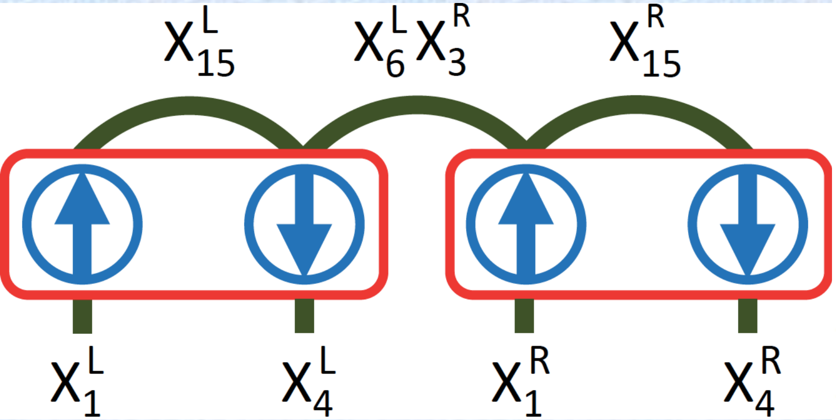
$$\frac{1}{2}\langle\psi_0|\{\hat{X}^L_\alpha,\hat{X}^L_\beta\}_+|\psi_0\rangle = 1\ (\alpha,\beta)\in[(4,13)\quad(5,14)\quad(6,15)\quad(7,11)]$$

$$\frac{1}{2}\langle\psi_0|\{\hat{X}^L_\alpha,\hat{X}^L_\beta\}_+|\psi_0\rangle = -1\ (\alpha,\beta)\in[(1,9)\quad(2,12)\quad(3,6)\quad(3,15)\quad(8,10)],$$

$$\langle\psi_0|\hat{X}_8^2|\psi_0\rangle = \langle\psi_0|\hat{X}_{10}^2|\psi_0\rangle$$

$$= -\frac{1}{2}\langle\psi_0|\{\hat{X}_8,\hat{X}_{10}\}_+|\psi_0\rangle = 1$$

$$\Rightarrow\quad X_8 = -X_{10}$$



$$\hat{X}_0 = \hat{I}^{(1)} \otimes \hat{I}^{(2)}$$

$$\hat{X}_1 = \hat{\sigma}_x^{(1)} \otimes \hat{I}^{(2)}, \quad \hat{X}_2 = \hat{\sigma}_y^{(1)} \otimes \hat{I}^{(2)}, \quad \hat{X}_3 = \hat{\sigma}_z^{(1)} \otimes \hat{I}^{(2)}$$

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$$\hat{X}_7 = \hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_x^{(2)}, \quad \hat{X}_8 = \hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_y^{(2)}, \quad \hat{X}_9 = \hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_z^{(2)}$$

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$$W_L = \frac{1}{Z}\delta(X_3 - 1)\delta(X_6 + 1)\delta(X_{15} + 1)\delta(X_4 - X_{13})\delta(X_5 - X_{14})$$
$$\times\delta(X_1 + X_9)\delta(X_2 + X_{12})\delta(X_7 - X_{11})\delta(X_8 + X_{10}) \exp\left[-\frac{\sum_{\alpha\in\{1,2,4,5,7,8\}} X_\alpha^2}{2}\right],$$

Equations of motion

$$\begin{aligned} \partial_t X_4^L &= J X_3^R X_5^L + J X_{14}^L \\ \partial_t X_3^L &= -h X_2^L \\ \partial_t X_{12}^L &= -h X_{11}^L + h X_{15}^L - J X_1^L \\ \dots &= \dots \end{aligned}$$

Number of independent variables 2^{N+1} (not 4^N). Need one extra ancilla spin.

Schwinger boson TWA

Schwinger bosons: $\hat{X}_\alpha \to a_n^\dagger X_\alpha^{nm} a_m, \; \sum_n a_n^\dagger a_n = 1$

Need to solve D=2^N equations

$$i\dot{a}_n^{(i)} = \frac{\partial H}{\partial a_n^{*\,(i)}}$$

Can almost satisfy initial conditions with the Gaussian state. Works very well.

Reduction from D^2 operators to D Schwinger bosons is like reduction from the density matrix to the wave function.

Make a product ansatz $|\psi\rangle = \prod_j |\psi_j\rangle$

Dirac mean field equations $\langle\psi|i\partial_t|\psi\rangle = \langle\psi|H|\psi\rangle$ are identical to classical equations. TWA is like a statistical mixture of many mean fields. This does make a difference!

Application: diffusion

Model (motivated by discussions with F. Pollmann): XXZ chain

$$\hat{H} = \sum_i^N \hat{S}_x^i \hat{S}_x^{i+1} + \hat{S}_y^i \hat{S}_y^{i+1} + \Delta \hat{S}_z^i \hat{S}_z^{i+1} + \gamma \sum_i^N \hat{S}_x^i \hat{S}_x^{i+2} + \hat{S}_y^i \hat{S}_y^{i+2} + \Delta \hat{S}_z^i \hat{S}_z^{i+2}.$$

Choose $\Delta = 2, \quad \gamma = 1/2, \quad S_\alpha \equiv \sigma_\alpha$

Describes well Yb_2Pt_2Pb, I. Zaliznyak et. al. unpublished

Central object

$$C_{ij}(t,t') = \frac{1}{Z} \sum_n \mathrm{e}^{-\beta E_n} \langle \psi_n | S_\alpha^i(t) S_\alpha^j(t') | \psi_n \rangle, \quad \alpha = \{x, y, z\}$$

Defines the spectral function (dynamic structure factor), spin susceptibilities, diffusion constant, fluctuation-dissipation relation (key indicator of thermalization),...

This work – focus on infinite temperatures

$$C_{ij}(t,t') = \frac{1}{\mathcal{D}} \mathrm{Tr}\left[S_\alpha^i(t) S_\alpha^j(t') \right] = \frac{1}{2\mathcal{D}} \mathrm{Tr}\left[S_\alpha^i(t) S_\alpha^j(t') + S_\alpha^j(t') S_\alpha^i(t) \right]$$

Expected long time behavior

$$C_{ij}(t,t') \equiv C_{ij}(t-t') = \frac{1}{\mathcal{D}}\mathrm{Tr}\left[S^i_\alpha(t)S^j_\alpha(t')\right] \sim \frac{1}{\sqrt{D|t-t'|}}\exp\left[-\frac{|i-j|^2}{2D|t-t'|}\right]$$

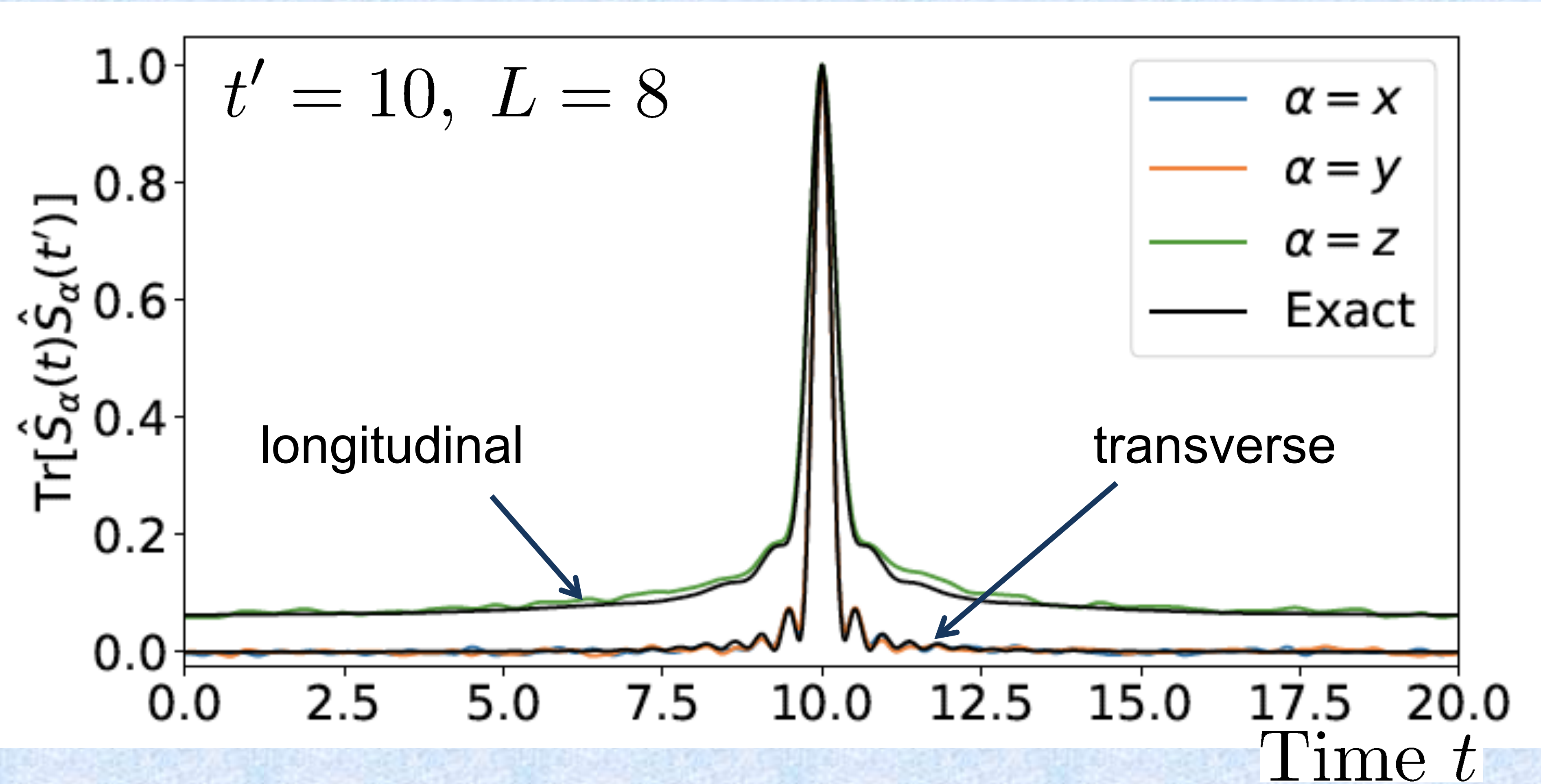
Can be used to extract diffusion constant (D. Luitz and Y. Bar Lev, 2016, 2017)

$$R^2(t) \sim \frac{\sum_{ij=1}^{N}(i-j)^2 C_{ij}(t)}{\sum_{ij} C_{ij}(t)} \sim \frac{N^2}{2\pi^2}\left(1-e^{-4Dt\pi^2/N^2}\right) \sim 2Dt$$

Main challenges: small system sizes amenable to ED can be too small to see asymptotic diffusive behavior.

Approximate methods (DMRG, mean field, TWA, ...) do not preserve time translational invariance, fail at long times.

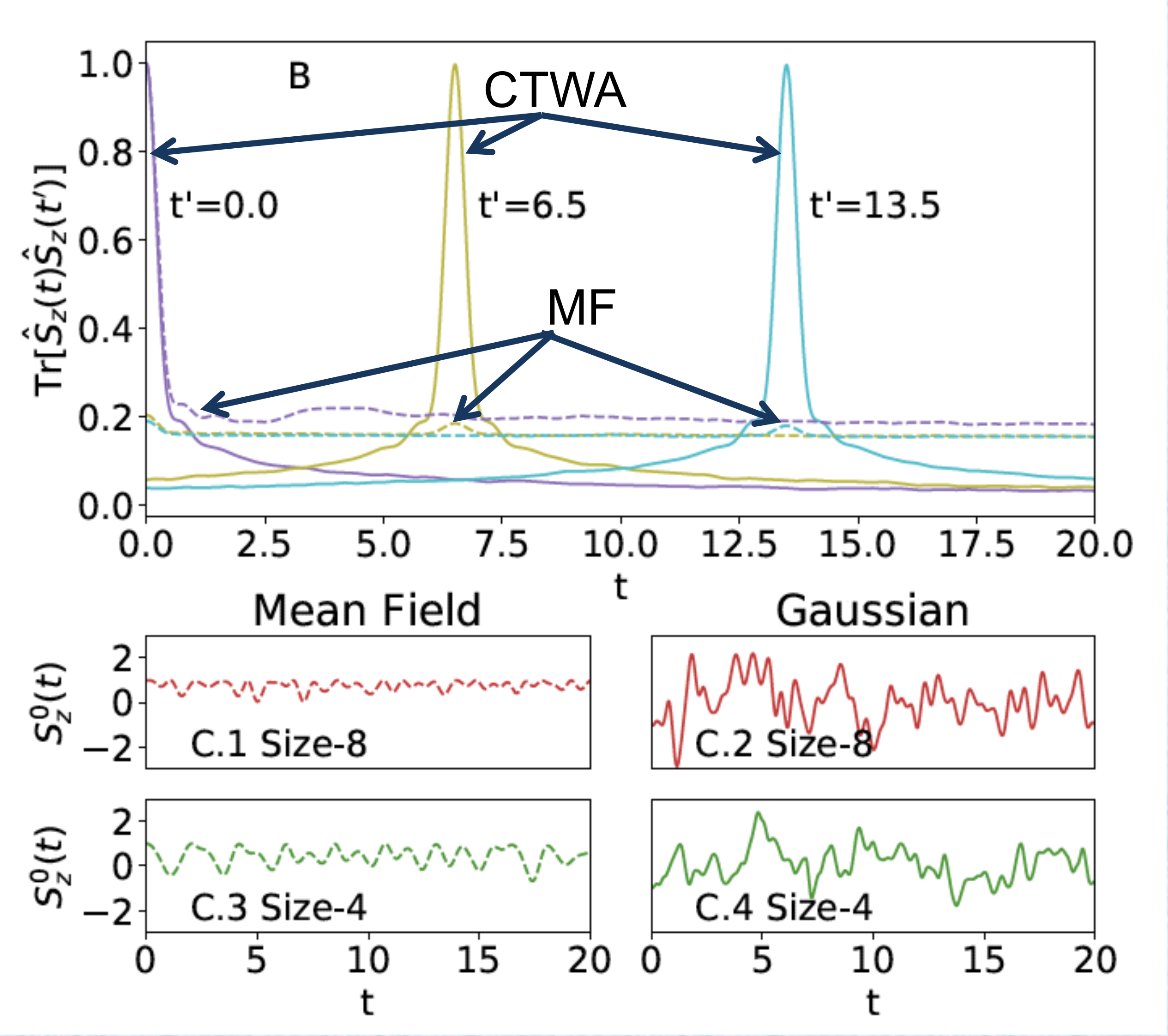
Numerical Results



$$\langle S_z^i(t) S_z^j(t') \rangle \to \frac{\overline{M_z^2}}{N^2} \sim \frac{1}{N}$$

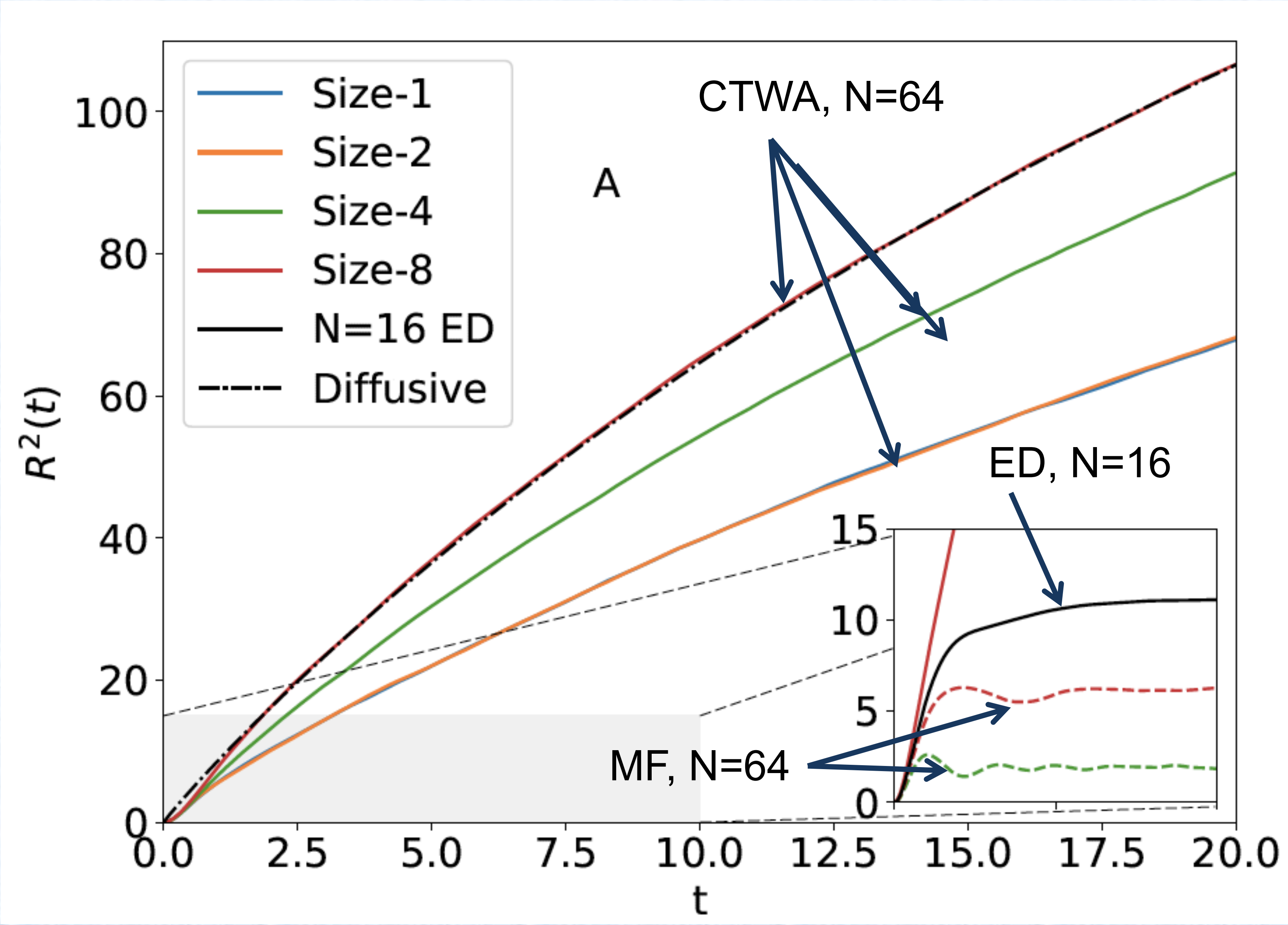
Follows from conservation of Z-magnetization

Longitudinal correlations, comparison with mean-field dynamics



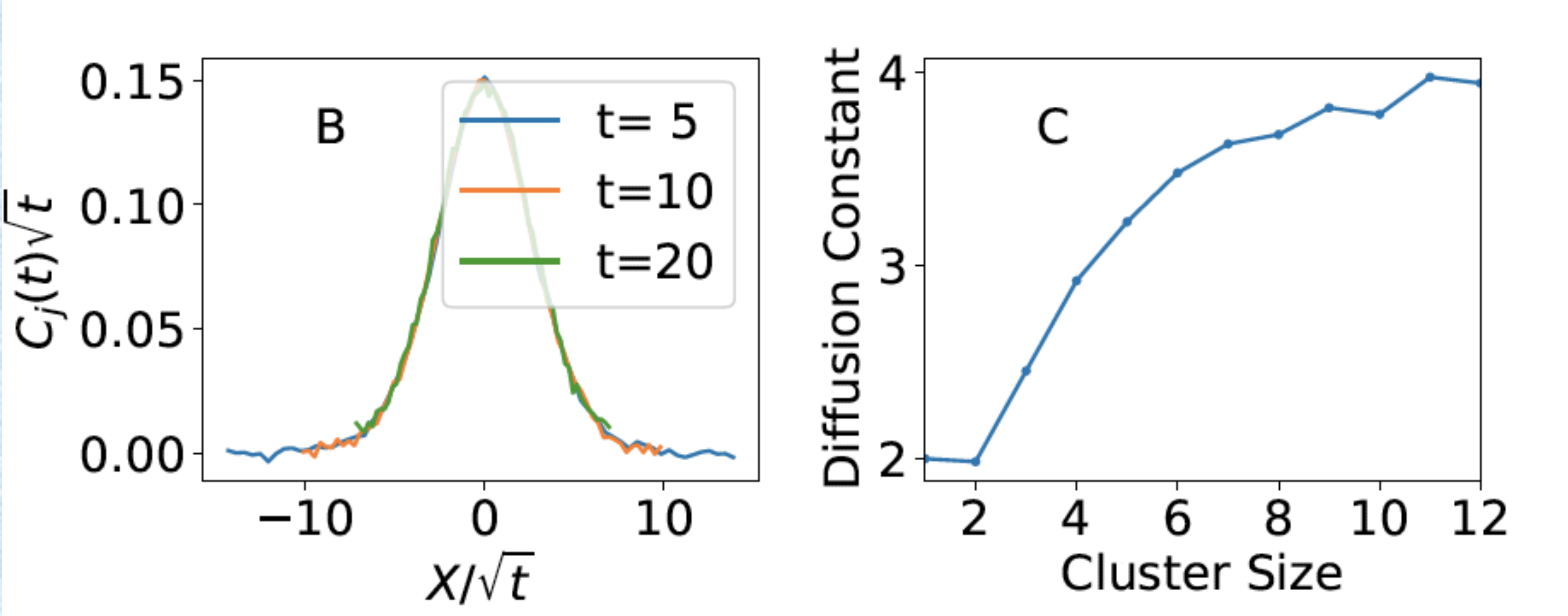
CTWA respects time-translation invariance: correct noise. MF fails, increasing cluster size makes things even worse due to ETH. Non-equilibrium initial state: MF is expected to fail completely.

Extracting diffusion constant



MF fails, ED gives a wrong diffusion constant

Excellent convergence to diffusive profile for all cluster sizes

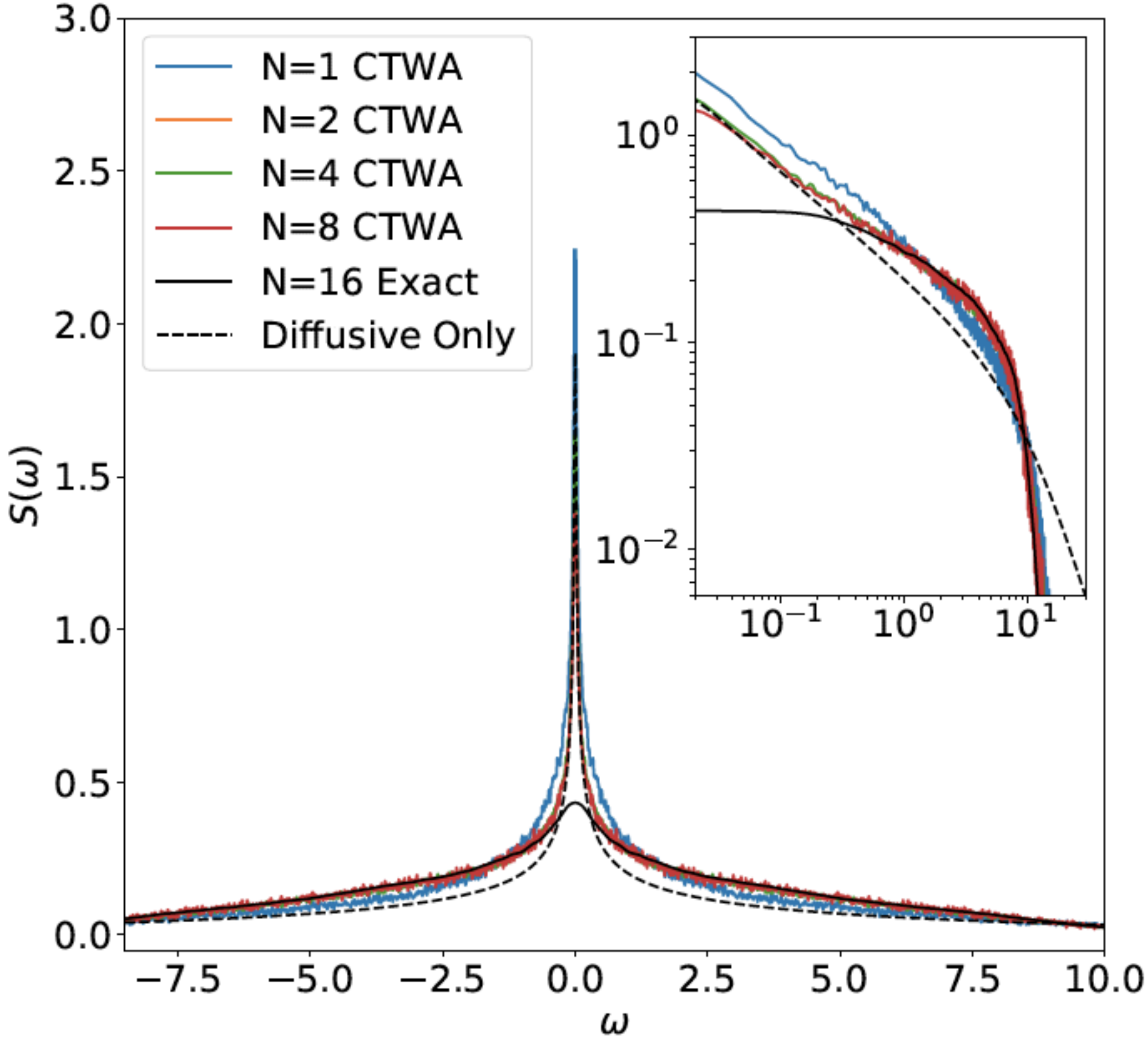


Very slow saturation of the diffusive constant with the cluster size (strong quantum renormalization).

Much faster saturation if we remove Z-conservation law. MF (classical) dynamics gives very accurate diffusion constant.

Can reproduce well the whole dynamical structure factor

$$S(k,\omega) = \sum_{ij} \int_{-\infty}^{\infty} dt e^{i\omega t + ik(i-j)} C_{ij}(t), \quad S(\omega) = \frac{1}{N} \sum_k S(k,\omega) = \frac{2\pi}{N} \sum_i \int_{-\infty}^{\infty} dt e^{i\omega t} C_{ii}(t)$$



Small frequency tail

$$S(\omega) \propto \frac{1}{\sqrt{D\omega}}$$

indicates asymptotic diffusive behavior. Only visible for N>32.

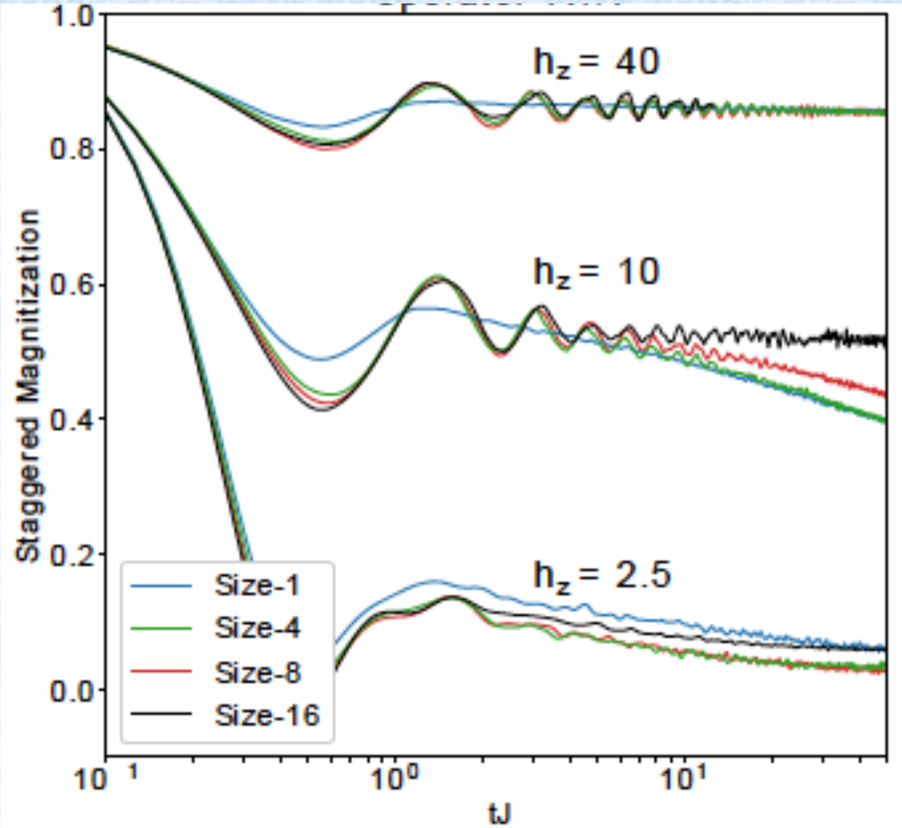
High frequency (exponential) asymptotes are quantum and can not be recovered from hydrodynamic approaches.

CTWA captures both!

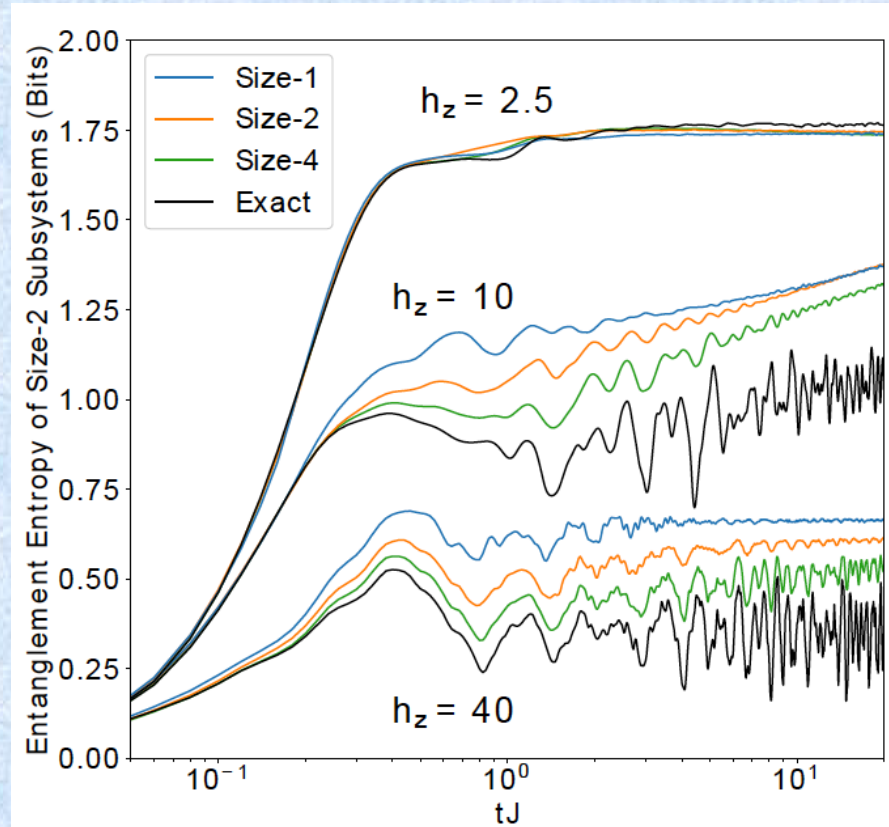
Less favorable example: MBL in a disordered Heisenberg spin chain

$$H = \sum \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + h_z \sum_j \delta_j \sigma_j^z, \quad \delta_j \in [-1, 1]$$

Staggered magnetization

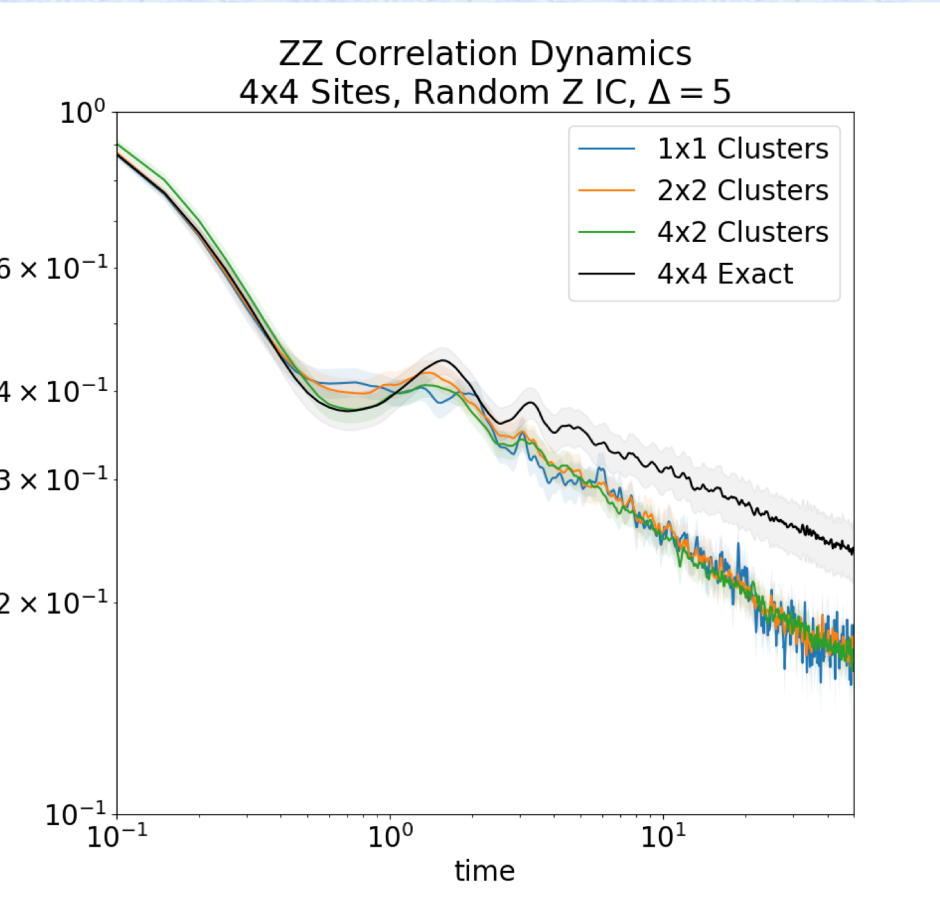


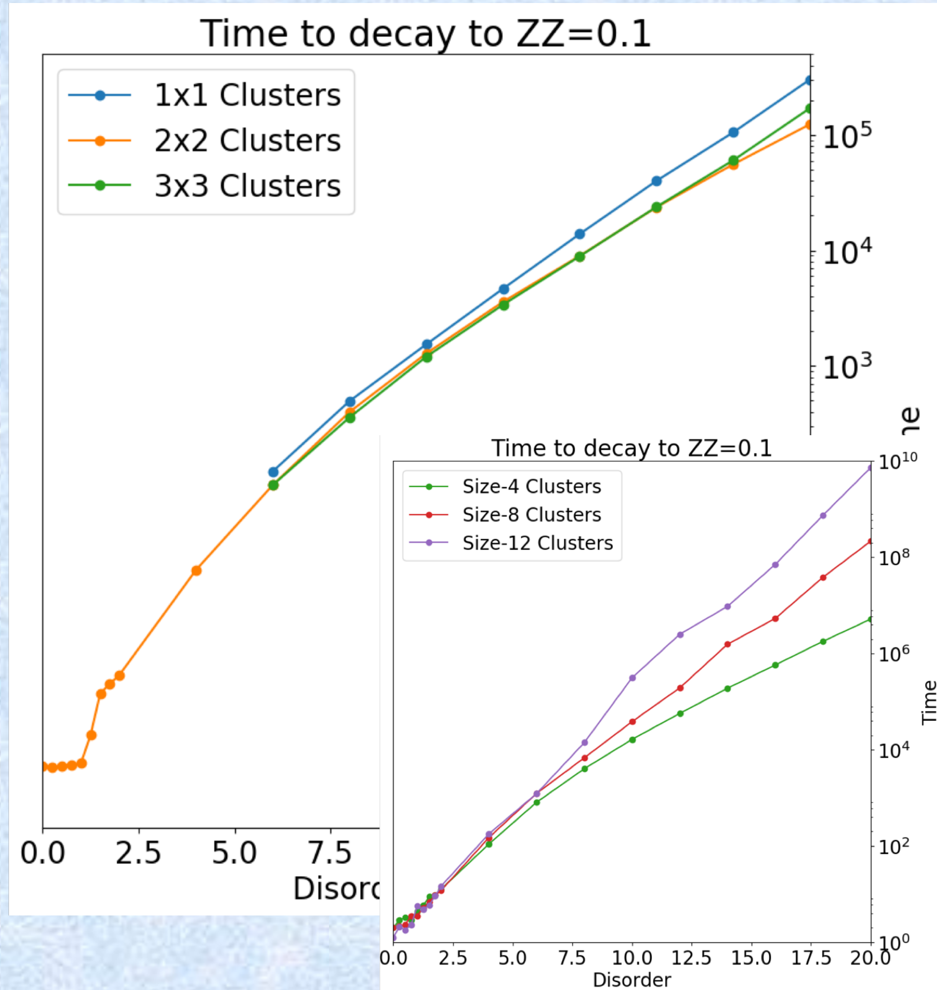
Entanglement entropy



Long time-diffusion, but can see the evidence of localization.
Higher entanglement in the classical limit

Disordered 2D XY chain (=hard core bosons). Preliminary results





Very small dependence on the cluster size. Evidence for a subdiffusive behavior, very strong (exponential) scaling of decay time with disorder.

Fermions. No obvious classical limit.

Main idea: use bilinear strings as dynamical variables. Non-locality is crucial

$$\hat{E}^{\alpha}_{\beta} = \hat{c}^{\dagger}_{\alpha}\hat{c}_{\beta} - \frac{1}{2}\delta_{\alpha\beta}, \quad \hat{E}_{\alpha\beta} = \hat{c}_{\alpha}\hat{c}_{\beta}, \quad \hat{E}^{\alpha\beta} = \hat{c}^{\dagger}_{\alpha}\hat{c}^{\dagger}_{\beta}.$$

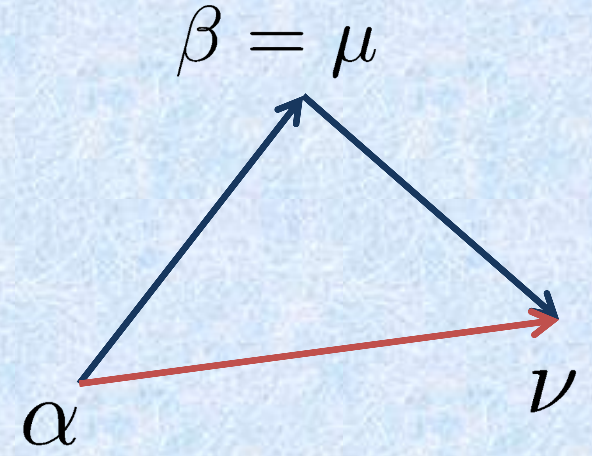
Group structure $U(N) = \{\hat{E}^{\alpha}_{\beta}\}$

$$SO(2N) = \{\hat{E}^{\alpha}_{\beta}, \hat{E}_{\alpha\beta}, \hat{E}^{\alpha\beta}\}.$$

Treat string variables as SO(2N) nonlocal spin degrees of freedom. Phase space dimensionality ~ $2N^2$ (instead of 2N).

Non-interacting system. Hamiltonian is linear. TWA is exact.

Poisson brackets (commutation relations). Encode locality



$$
\begin{aligned}
[\hat{E}^{\alpha}_{\beta}, \hat{E}^{\mu}_{\nu}]_- &= \delta_{\beta\mu}\hat{E}^{\alpha}_{\nu} - \delta_{\alpha\nu}\hat{E}^{\mu}_{\beta}, \\
[\hat{E}^{\alpha}_{\beta}, \hat{E}_{\mu\nu}]_- &= \delta_{\alpha\nu}\hat{E}_{\beta\mu} - \delta_{\alpha\mu}\hat{E}_{\beta\nu}, \\
[\hat{E}^{\alpha\beta}, \hat{E}_{\mu\nu}]_- &= \delta_{\alpha\nu}\hat{E}^{\beta}_{\mu} + \delta_{\beta\mu}\hat{E}^{\alpha}_{\delta} - \delta_{\alpha\mu}\hat{E}^{\beta}_{\nu} - \delta_{\beta\nu}\hat{E}^{\alpha}_{\mu}, \\
[\hat{E}_{\alpha\beta}, \hat{E}_{\mu\nu}]_- &= 0, \quad [\hat{E}^{\alpha\beta}, \hat{E}^{\mu\nu}]_- = 0.
\end{aligned}
$$

New non-local phase space variables

$$\rho_{\alpha\beta} = \left(\hat{E}^{\alpha}_{\beta}\right)_W, \; \tau_{\alpha\beta} = \left(\hat{E}_{\alpha\beta}\right)_W, \; \tau^*_{\alpha\beta} = -\left(\hat{E}^{\alpha\beta}\right)_W,$$

$$\rho_{\alpha\beta} = \rho^*_{\beta\alpha}, \quad \tau_{\alpha\beta} = -\tau_{\beta\alpha}$$

These variables satisfy canonical Poisson bracket relations, e.g.

$$\{\rho_{\alpha\beta}, \rho_{\mu\nu}\} = \delta_{\beta\mu}\rho_{\alpha\nu} - \delta_{\alpha\nu}\rho_{\mu\beta}$$

Equations of motion

$$\dot{\rho}_{\alpha\beta} = \{\rho_{\alpha\beta}, H_W\}, \quad \dot{\tau}_{\alpha\beta} = \{\tau_{\alpha\beta}, H_W\}$$

Initial conditions: exact Wigner function is too complicated. Use the best Gaussian (alternatively discrete sampling A. M. Rey group).

$$\langle\hat{\rho}_{\alpha\beta}\rangle = \int D\rho D\tau \rho_{\alpha\beta} W(\rho,\tau), \quad \langle\rho_{\alpha\beta}\rho_{\mu\nu}+\rho_{\mu\nu}\rho_{\alpha\beta}\rangle = 2\int D\rho D\tau \rho_{\alpha\beta}\rho_{\mu\nu} W(\rho,\tau),$$

Alternatively exact discrete sampling (Wooters, A.M. Rey in progress)

Example: initial free Fermi sea, indexes –momentum modes

$$\langle\rho_{\alpha\beta}\rangle = \delta_{\alpha\beta}(n_\alpha - 1/2), \ \langle\tau_{\alpha\beta}\rangle = 0,$$

$$\rho_{\alpha\beta} \leftrightarrow c_\alpha^\dagger c_\beta - 1/2, \quad \tau_{\alpha\beta} \leftrightarrow c_\alpha c_\beta$$

$$\left\langle\rho^*_{\alpha\beta}\rho_{\mu\nu}\right\rangle_c = \frac{1}{2}\delta_{\alpha\mu}\delta_{\beta\nu}\left(n_\alpha + n_\beta - 2n_\alpha n_\beta\right), \qquad (6)$$

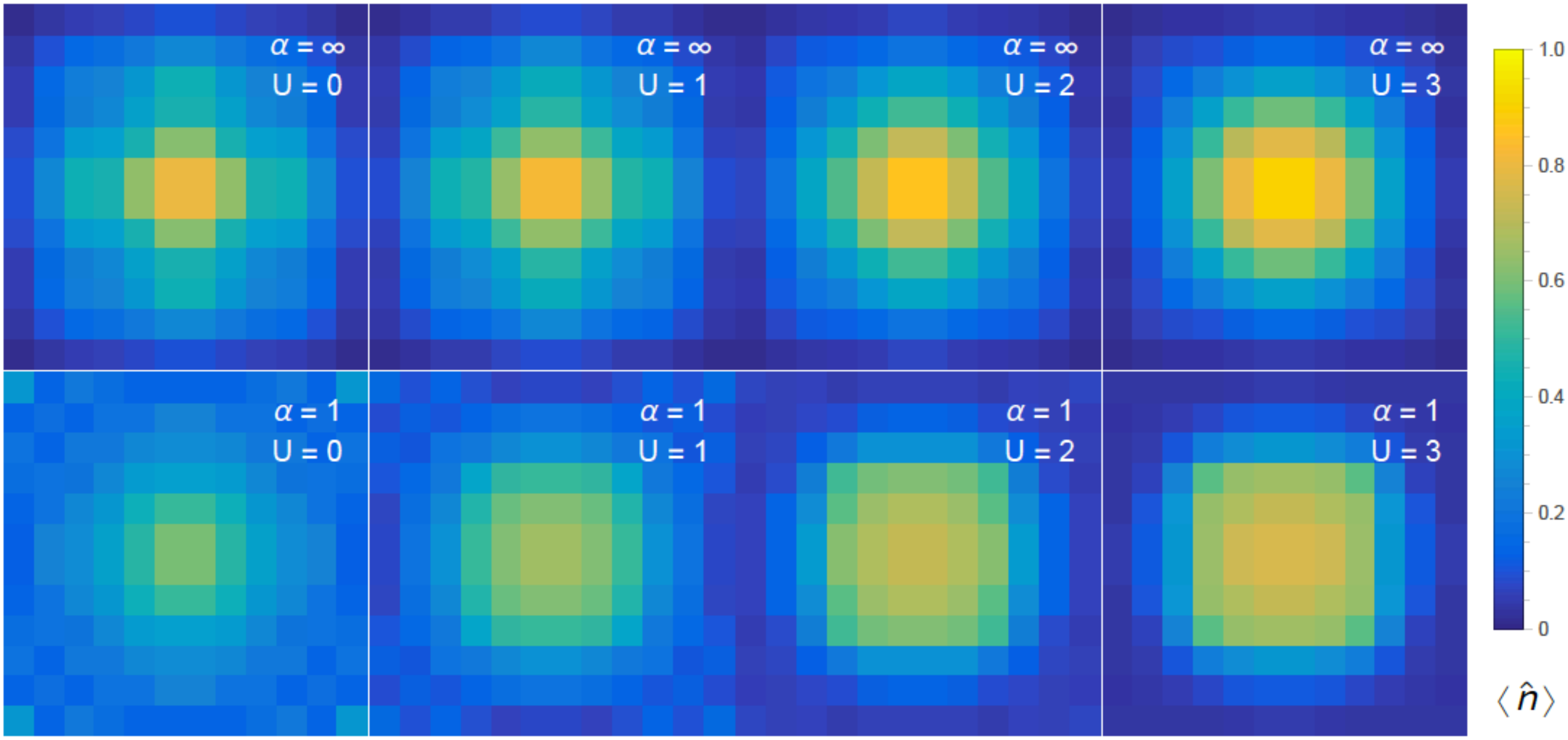
$$\langle\tau^*_{\alpha\beta}\tau_{\mu\nu}\rangle_c = \frac{1}{2}\left(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\beta\mu}\delta_{\alpha\nu}\right)\left(1 + 2n_\alpha n_\beta - n_\alpha - n_\beta\right)$$

Normal variables: no fluctuations at zero or unit filling.
Superconducting variables – always fluctuate.

Interaction blockade. Fermion expansion with NN and long-range hopping

$$\hat{H}_{\mathrm{Hub}} = -\sum_{ij\sigma} J_{ij}\left(\hat{c}^{\dagger}_{i\sigma}\hat{c}_{j\sigma} + \mathrm{h.c}\right) + U\sum_{i}\hat{n}_{i\uparrow}\hat{n}_{i\downarrow},$$

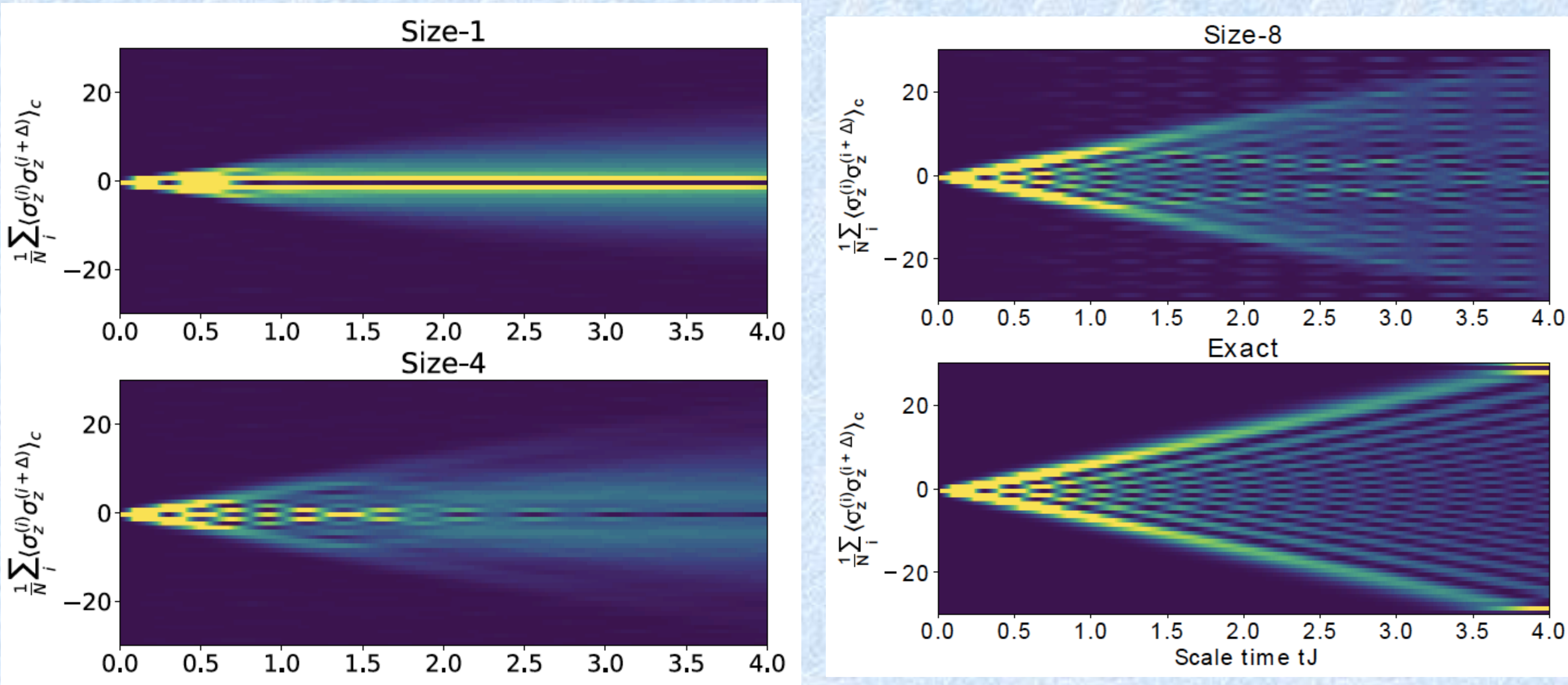
$$J_{ij} = \frac{t}{|i-j|^{\alpha}}$$



Long range hopping + interactions leads to stronger localization

Non-local correlations: cluster vs. fermion TWA for XY chain

$$\hat{H} = \sum_{i=0}^{64} \hat{\sigma}_x^{(i)}\hat{\sigma}_x^{(i+1)} + \hat{\sigma}_y^{(i)}\hat{\sigma}_y^{(i+1)}.$$



Accuracy of TWA depends on the choice of basis operators!

Integrability is seeing as emerging asymptotically from CTWA with increasing cluster size.

Conclusions

Can incorporate (short-distance) quantum fluctuations into TWA by adding more degrees of freedom.

CTWA - cluster degrees of freedom; fTWA – fermionic bilinears as degrees of freedom. In general need a closed set of commutation relations to define Poisson brackets..

TWA goes beyond mean field. Fluctuations in initial conditions are crucial for recovering non-equal time correlation functions and correct hydrodynamic behavior.

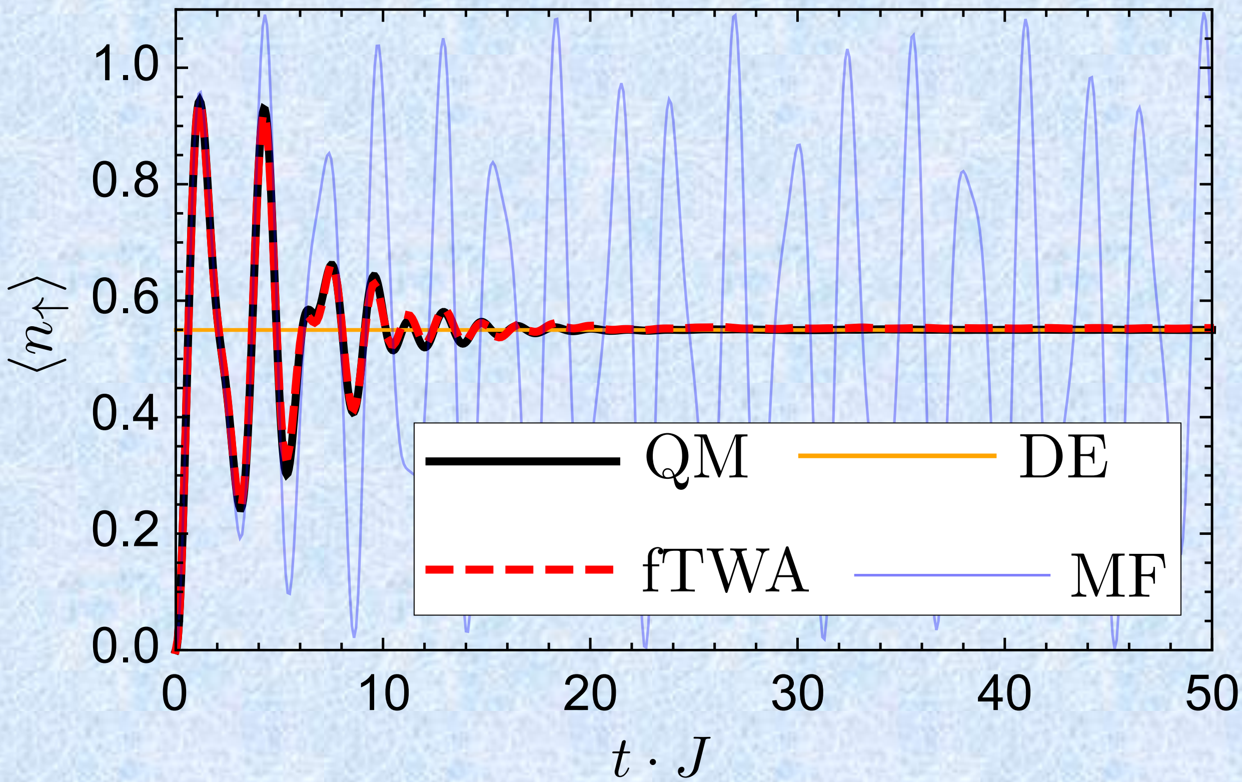
Can dramatically improve accuracy of TWA by using better degrees of freedom.

Application: two channel model (cartoon for gauge theories)

$$H = \mu_B \sum_j b_j^\dagger b_j - J \sum_{\sigma,\langle ij\rangle} \left(c_{\sigma i}^\dagger c_{\sigma j} + h.c.\right) + g \sum_j \left(b_j c_{\uparrow j}^\dagger c_{\downarrow j}^\dagger + h.c.\right)$$

Large positive (negative) μ_B – attractive (repulsive) Hubbard model

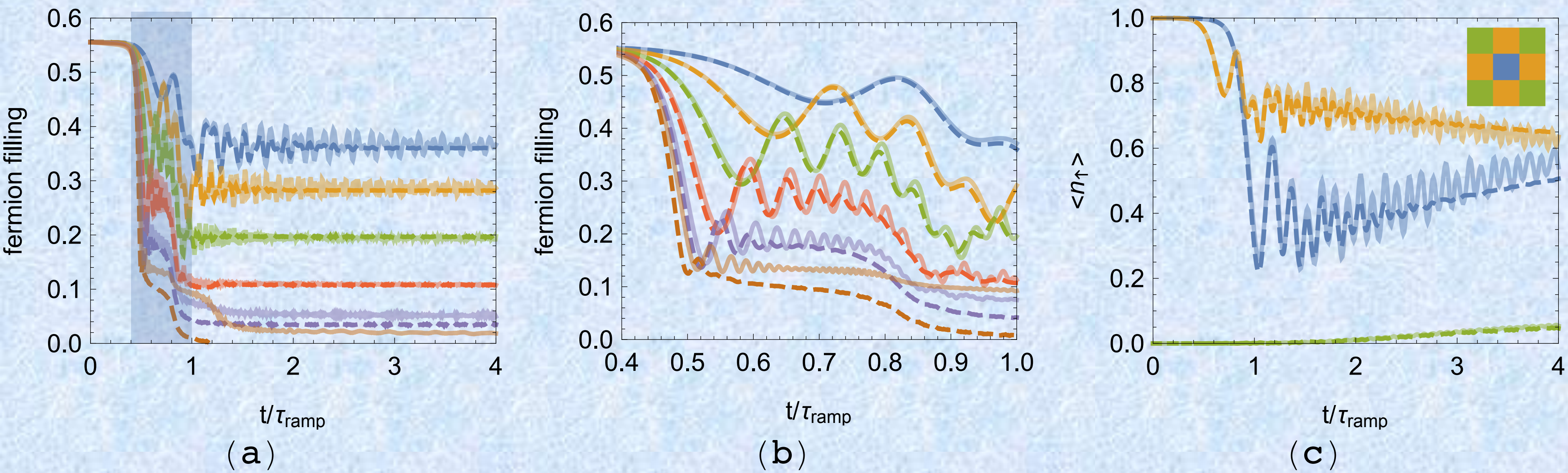
Two-site model, near mean-field regime. Fermion vacuum, coherent state for bosons with N=9 per site. Quench to $\mu_B = 1,\ g = 1/3$



- MF – only short times
- fTWA nearly exact including long time limit (but no revivals)
- Hilbert space is sufficiently large to thermalize.

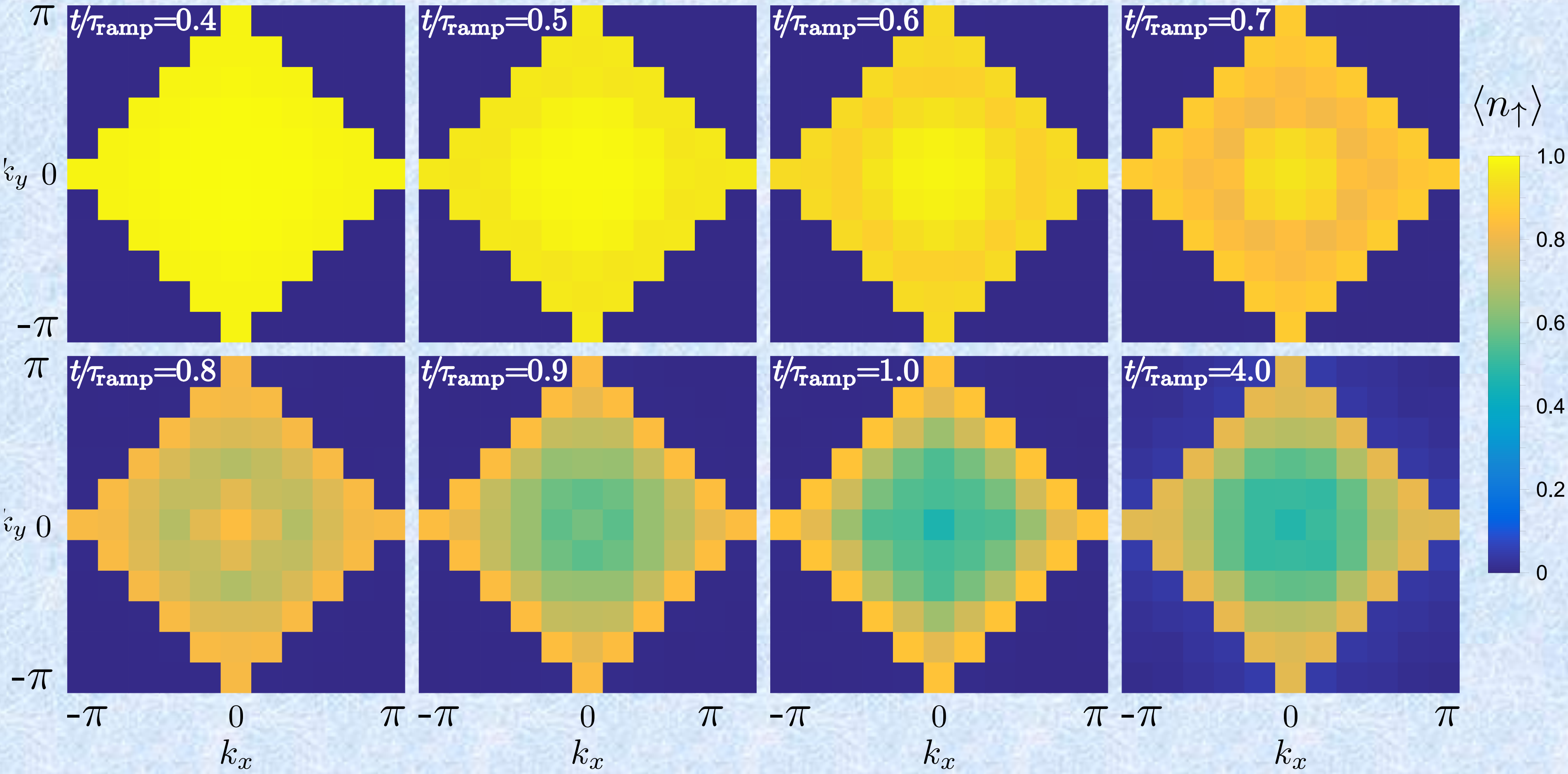
Same model. Initially no bosons, half filling of fermions. No obvious small parameter 3x3 system.

$$\mu_B(t) = -10(1 - e^{-(t/\tau_{\mathrm{ramp}})^2}) \text{ and } g(t) = 1 - e^{-(t/\tau_{\mathrm{ramp}})^2}$$



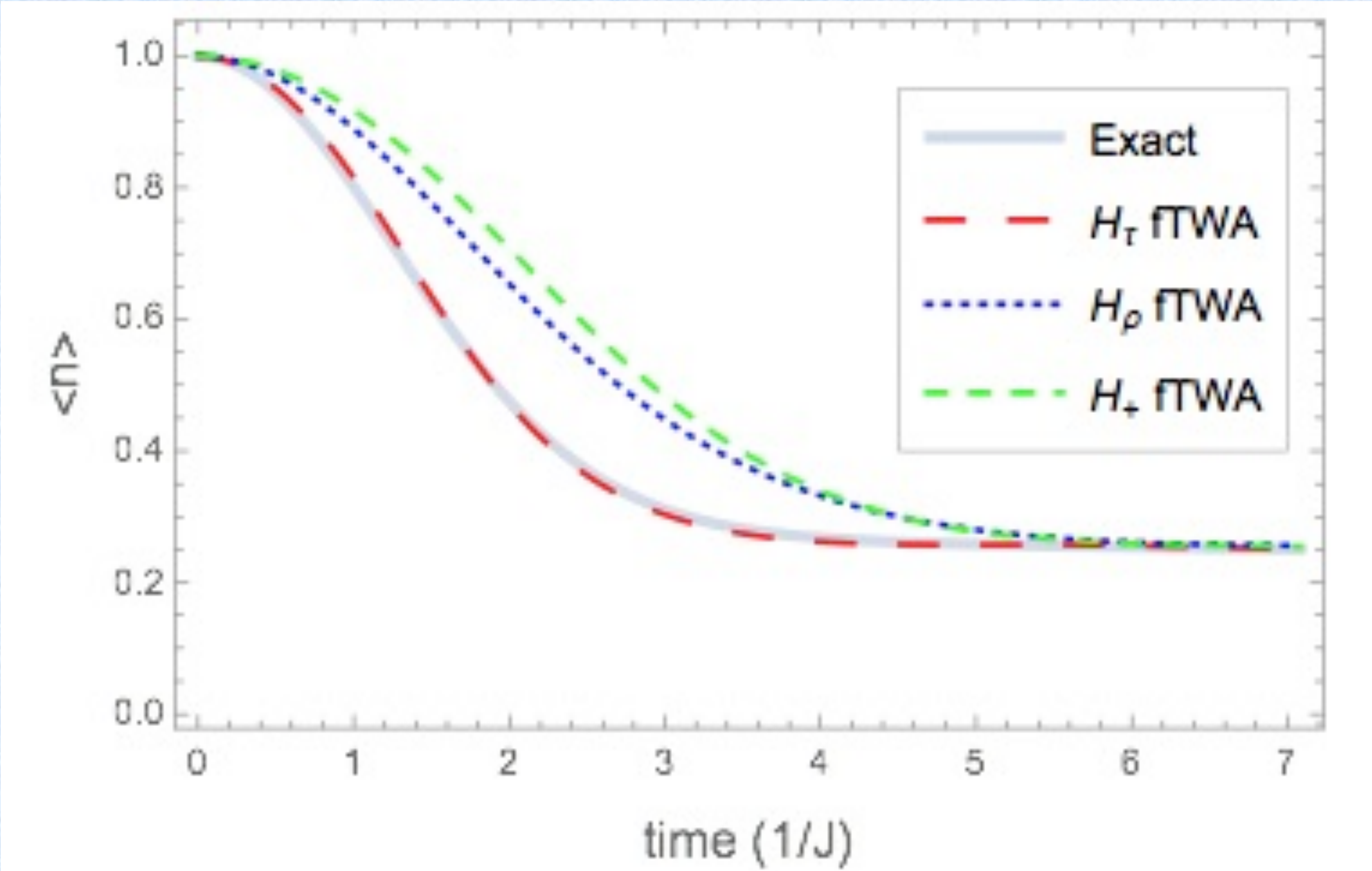
fTWA works very well except for very slow ramps. Can not predict correctly strongly-correlated GS. Works very well for short and intermediate time ramps.

Same as in the previous slide but for 10x10 lattice

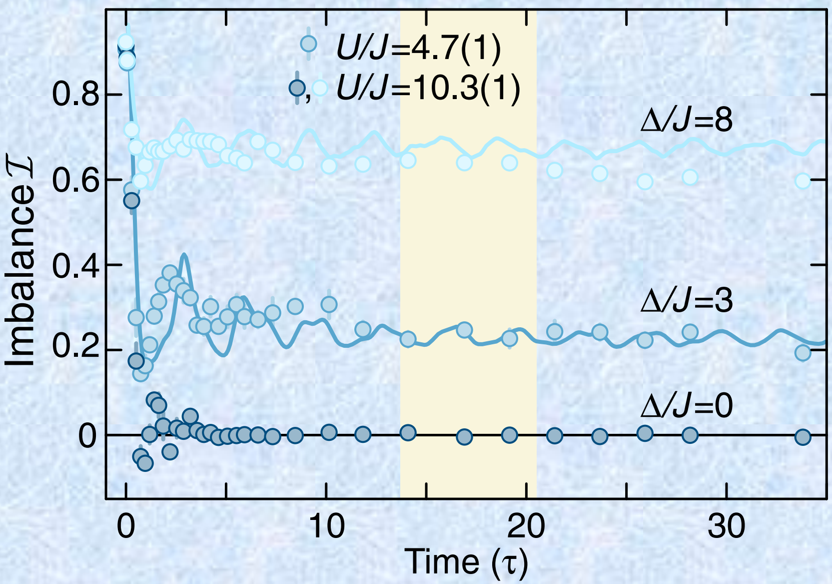


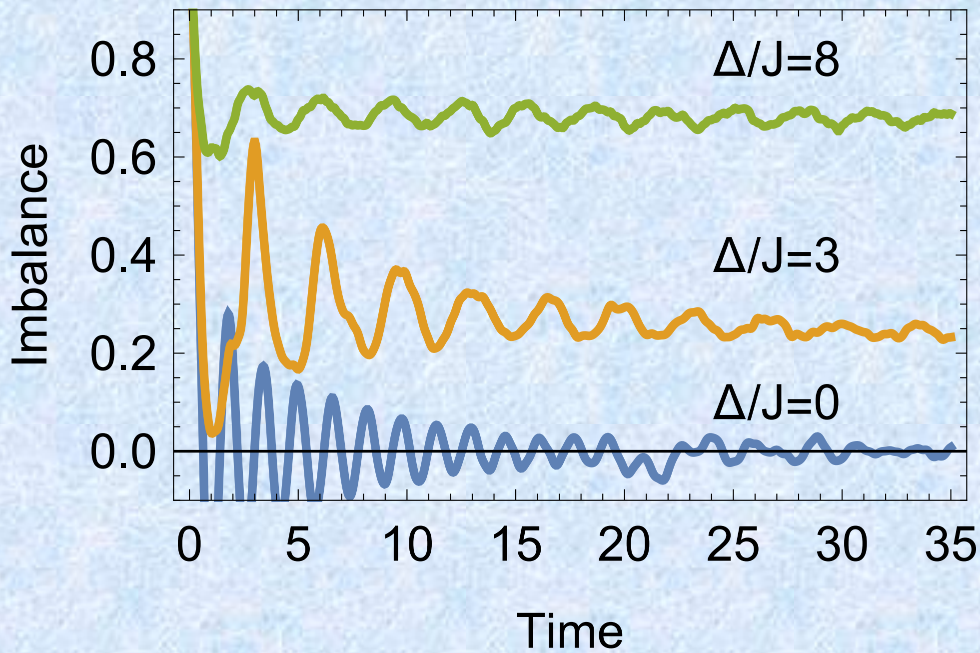
Emergence of a very unusual (ring-type) state of fermions.

Comparison with the normal variable representation



Application to MBL experiment (M. Schreiber et. al.). Same parameters, same number of dublons. L=40



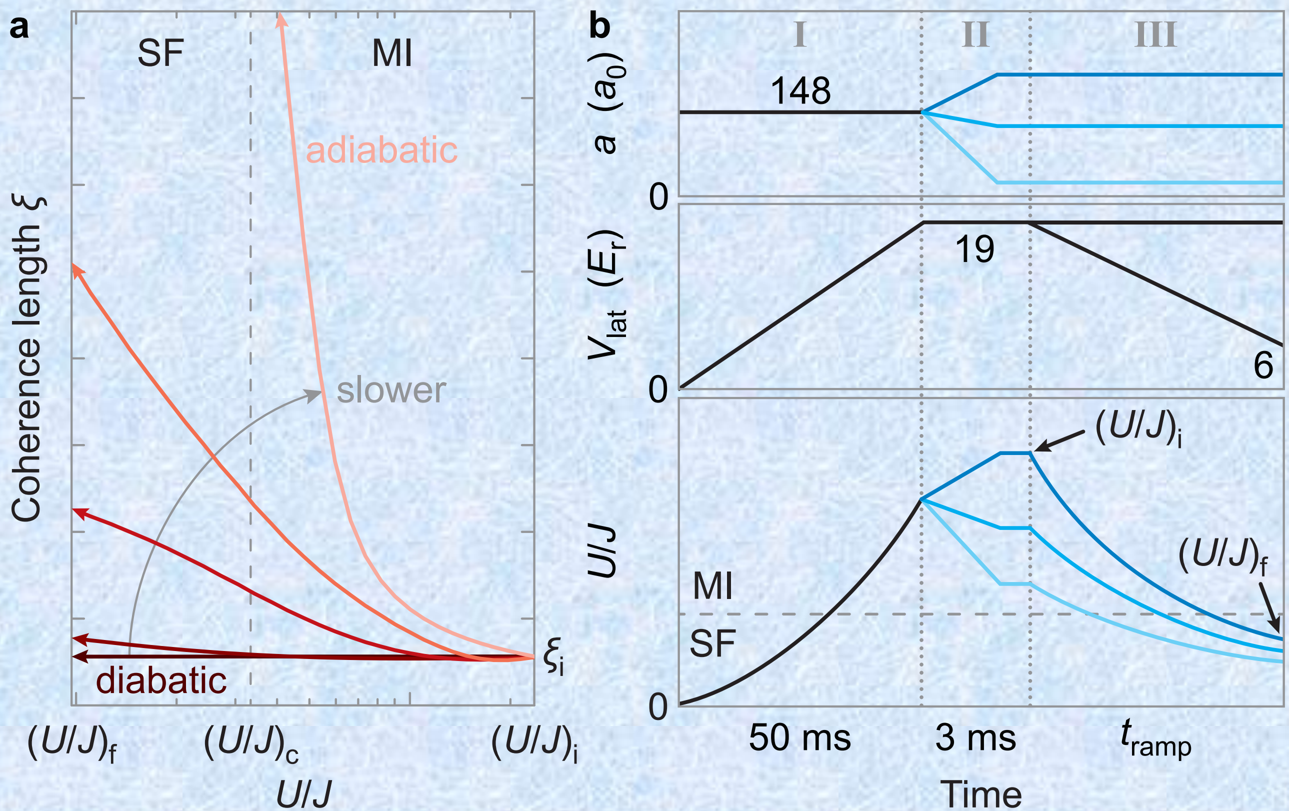


$$\hat{H} = -J\sum_{<ij>\sigma}\left(\hat{c}^\dagger_{i\sigma}\hat{c}_{j\sigma} + \text{h.c}\right) + \sum_i \mu_B \hat{b}^\dagger_i \hat{b}_i$$
$$+ g\sum_i\left(\hat{b}_i\hat{c}^\dagger_{\uparrow i}\hat{c}^\dagger_{\downarrow i} + \text{h.c.}\right) + \Delta\sum_{i\sigma}\cos\left(2\pi\beta i + \phi\right)\hat{c}^\dagger_{i\sigma}\hat{c}_{i\sigma}$$

$$g = 2,\ \mu_b = 10 \Rightarrow U \approx 0.4$$

fTWA works qualitatively well for at least intermediate times and better than CTWA. Long times – tendency to decay.

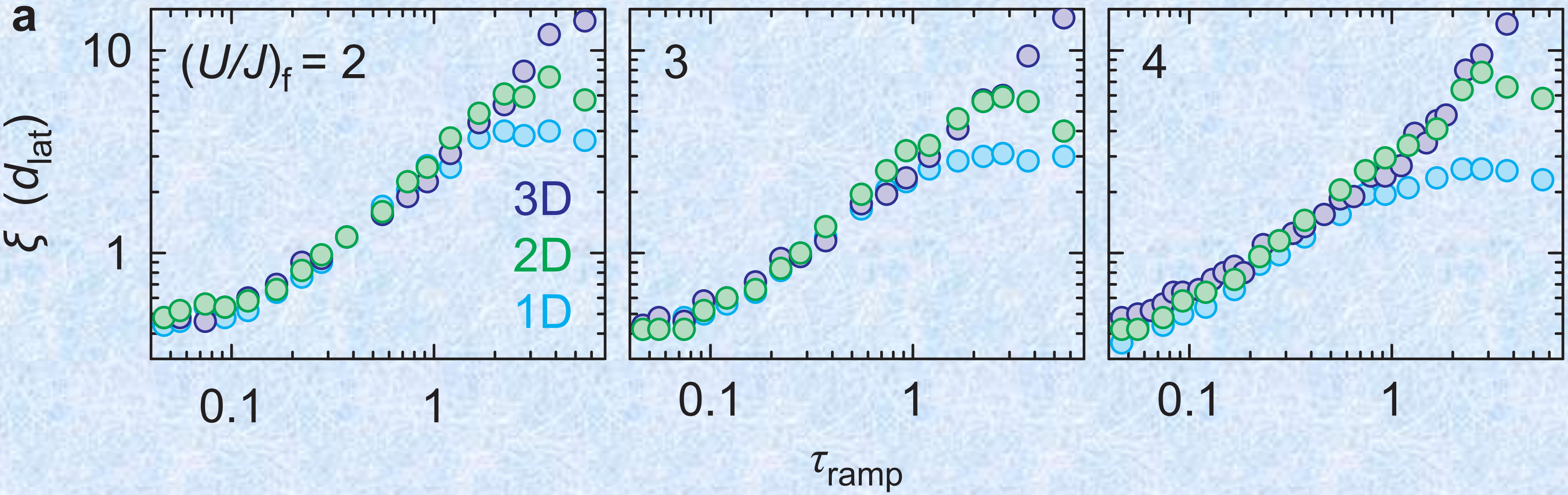
Slow Ramps from IN to SF

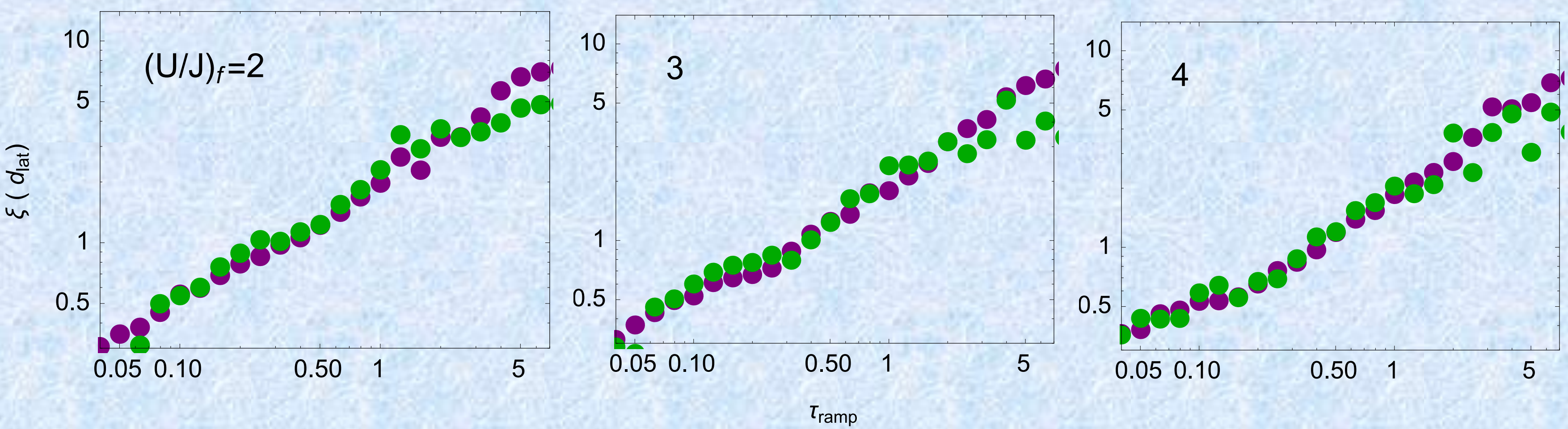


S. Braun, … I. Bloch, U. Schneider, J. Eisert, PNAS 2015

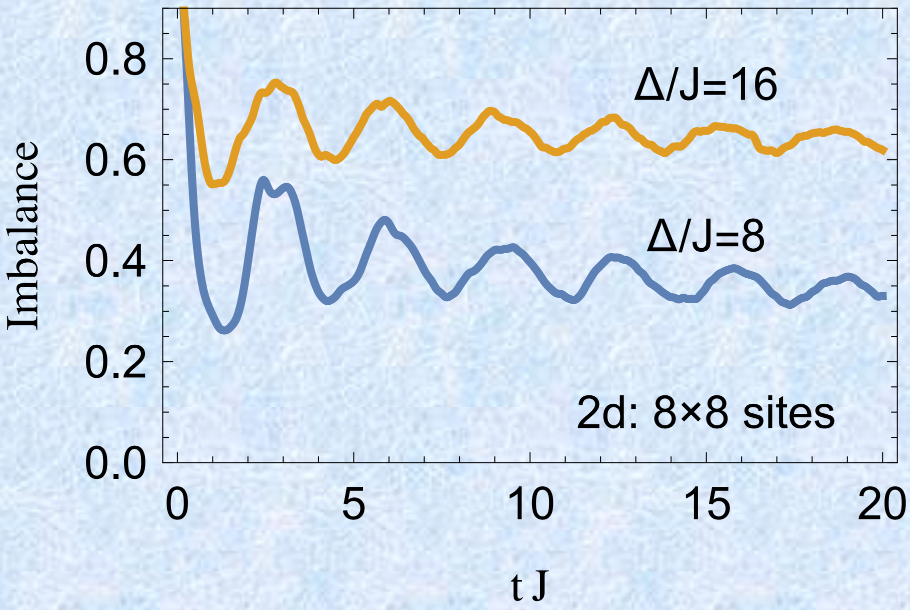
Check correlation length in the SF state as a function of ramp rate

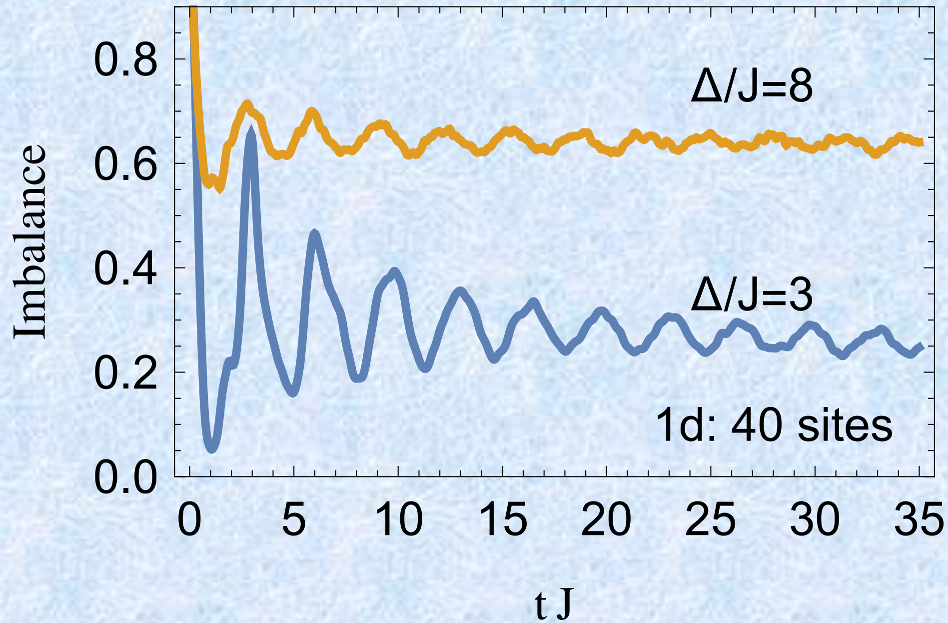
Experiment vs. SU(3) TWA





2D simulation (uncorrelated disorder), 8x8 lattice (quick run)





Reliable for the time scales shown.