# Exploring Scale Invariance in Flatland

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**Solvay meeting, Brussels, Feb. 18-20 2019**

# Scale invariance

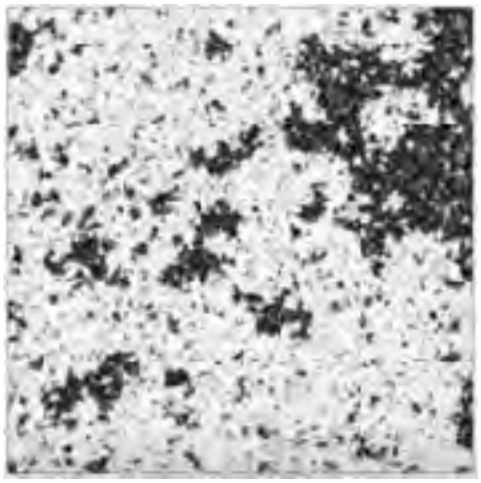
A concept that was introduced in the 70's in high energy physics

Can there be physical systems with no intrinsic energy/length scale?

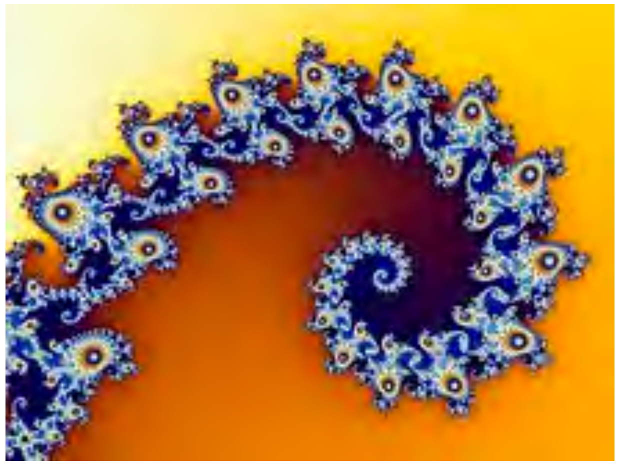
*Need to explain the behavior of $e^-$ - nucleon scattering cross-sections*

This concept later found many applications in physics, maths, biology, etc.

Phase transitions and renormalization group



Fractals



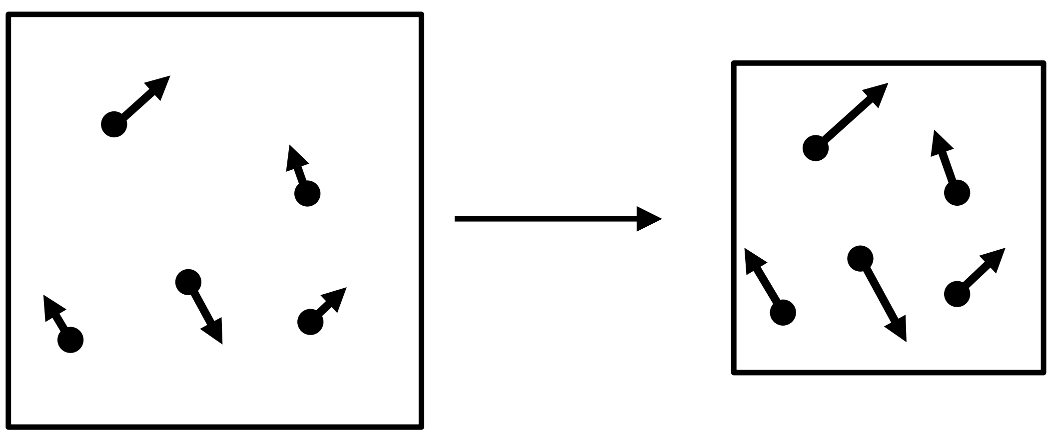
# Scale invariance in a gas of particles

Dilatation operation acting on space and time variables

$$\boldsymbol{r} \to \boldsymbol{r}/\lambda \qquad\qquad t \to t/\lambda^2$$

In such a scaling, the velocity and the kinetic energy become:

$$\boldsymbol{v} \to \lambda \boldsymbol{v} \qquad\qquad E_{\mathrm{kin}} \to \lambda^2 E_{\mathrm{kin}}$$



The action $\propto \int E_{\mathrm{kin}}\, \mathrm{d}t$ is therefore invariant in this transformation

*If there is no interaction, end of the story: the ideal gas is scale-invariant, both in classical and quantum physics*

# Outline of this talk

Explore the expected consequences + find some unexpected ones for the case of an interacting 2D Bose gas

1. Scale/conformal invariance in a cold atomic gas

2. Exploring experimentally dynamical probes of scale invariance

3. Two-dimensional breathers

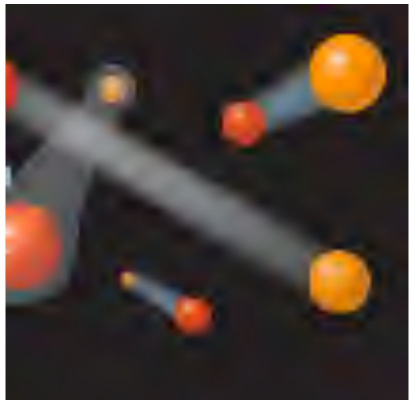
# The interacting case

Since $E_{\text{kin}}$ behaves as $E_{\text{kin}} \to \lambda^2 E_{\text{kin}}$ , an interacting system will be scale-invariant if the interaction energy also satisfies :

$$\boldsymbol{r} \to \boldsymbol{r}/\lambda \qquad\qquad E_{\text{int}} \to \lambda^2 E_{\text{int}}$$

- Interaction potential varying as $V(\boldsymbol{r}_i - \boldsymbol{r}_j) \propto \dfrac{1}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^2}$

- Fermi gas in the unitary regime (not fully obvious, look at Bethe-Peierls boundary conditions)



- Contact interaction in two dimensions (Flatland):

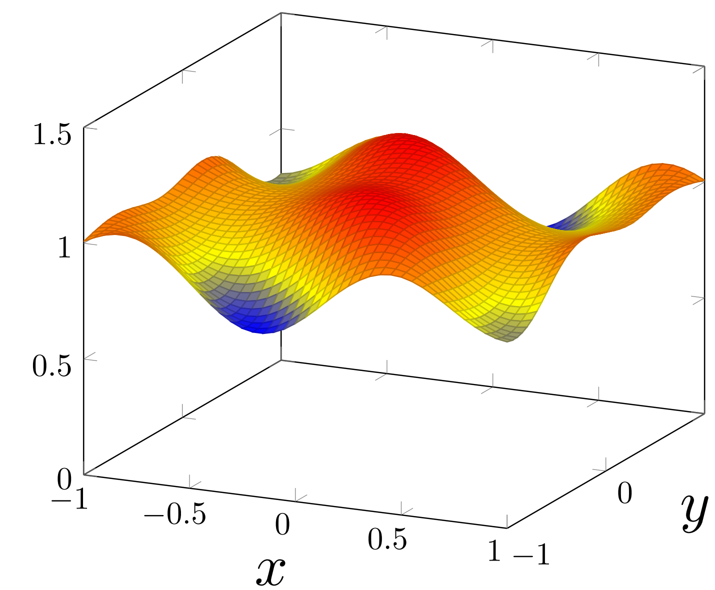
$$\boldsymbol{r} \to \boldsymbol{r}/\lambda \qquad\qquad g\,\delta(\boldsymbol{r}) \to g\,\delta(\boldsymbol{r}/\lambda) = \lambda^2\, g\,\delta(\boldsymbol{r})$$

*BUT… 2D contact interaction is singular when treated in quantum mechanics*

# Classical field approach to the 2D Bose gas

Describe the gas by a classical field $\psi(\boldsymbol{r},t)$ obeying the Gross-Pitaevskii equation

Energy of the gas: $E(\psi) = E_{\rm kin}(\psi) + E_{\rm int}(\psi)$




$$E_{\rm kin}(\psi) = \frac{\hbar^2}{2m}\int |\boldsymbol{\nabla}\psi|^2$$

$$E_{\rm int}(\psi) = \frac{\hbar^2}{2m}\tilde{g}\int |\psi|^4$$

$\tilde{g}$ : interaction strength

*No singularity at the classical field level*

In 2D, the interaction strength $\tilde{g}$ is dimensionless: no length scale, nor energy scale associated with the interactions, as required for scale invariance

# Enriching the scale invariance

Pitaevskii & Rosch, 1997

Conformal invariance $\longleftrightarrow$ Dynamical symmetry [SO(2,1)]

In addition to the standard Galilean transformations (translations, rotations), there exist three types of transformations that leave the 2D Gross-Pitaevskii invariant:

Dilatations: $\boldsymbol{r} \to \boldsymbol{r}/\lambda \qquad t \to t/\lambda^2$

Time translations: $\boldsymbol{r} \to \boldsymbol{r} \qquad t \to t + t_0$

"Expansions": $\boldsymbol{r} \to \dfrac{\boldsymbol{r}}{\gamma t + 1} \qquad t \to \dfrac{t}{\gamma t + 1}$

The ensemble forms a 3-parameter group ("time-dependent" dilatations):

$$\boldsymbol{r} \to \frac{\boldsymbol{r}}{\gamma t + \delta} \qquad t \to \frac{\alpha t + \beta}{\gamma t + \delta} \qquad \alpha\delta - \beta\gamma = 1$$

**Group SL(2,R) [real 2x2 matrices of determinant 1], which is isomorphous to SO(2,1) (Lorentz group in two spatial dimensions)**

Pitaevskii & Rosch, 1997

# "Revealing" the SO(2,1) symmetry

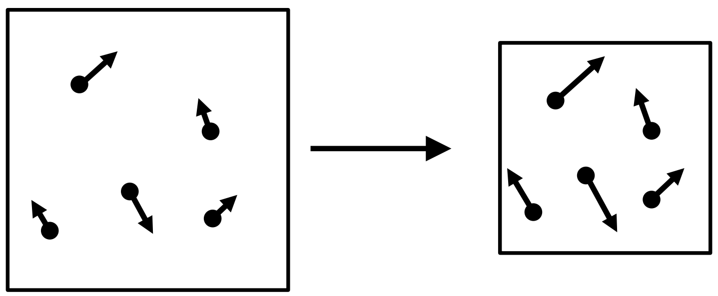
Add a harmonic confinement $\frac{1}{2}m\omega^2 r^2$ leading to $E_{\rm pot} = \frac{1}{2}m\omega^2 \int r^2\,|\psi|^2$

Naive reaction: this breaks the scale/conformal invariance!

$$\boldsymbol{r} \to \frac{1}{\lambda}\,\boldsymbol{r}$$

$$E_{\rm kin} + E_{\rm int} \to \lambda^2\ (E_{\rm kin} + E_{\rm int})$$

$$E_{\rm pot} \to \frac{1}{\lambda^2}\,E_{\rm pot}$$



However one can still exhibits a 3-parameter group of transformations leaving the GP equation invariant:

$$\tan(\omega t) \to \frac{\alpha\tan(\omega t)+\beta}{\gamma\tan(\omega t)+\delta} \qquad \alpha\delta-\beta\gamma=1 \qquad \boldsymbol{r} \to \frac{\boldsymbol{r}}{\lambda(t)}$$

λ = dynamical parameter: oscillatory exchange between $E_{\rm kin} + E_{\rm int}$ and $E_{\rm pot}$

**Undamped "breathing mode" at frequency 2ω**

# The SO(2,1) symmetry in a nutshell

The three contributions to the Hamiltonian

$$\hat{H}_{\rm kin} = \sum_j \frac{\hat{\boldsymbol{p}}_j^2}{2m} \qquad \hat{H}_{\rm int} = \frac{1}{2}\sum_{i\neq j} V(\hat{\boldsymbol{r}}_i - \hat{\boldsymbol{r}}_j) \qquad \hat{H}_{\rm pot} = \sum_j \frac{1}{2} m\omega^2 \hat{\boldsymbol{r}_j}^2$$

Define the three operators:

$$\begin{cases} \hat{L}_1 = \dfrac{1}{2\hbar\omega}\left(\hat{H}_{\rm kin} + \hat{H}_{\rm int} - \hat{H}_{\rm pot}\right) \\ \hat{L}_2 = \dfrac{1}{4}\sum_j \left(\hat{\boldsymbol{r}_j}\cdot\hat{\boldsymbol{p}_j} + \hat{\boldsymbol{p}_j}\cdot\hat{\boldsymbol{r}_j}\right) \\ \hat{L}_3 = \dfrac{1}{2\hbar\omega}\left(\hat{H}_{\rm kin} + \hat{H}_{\rm int} + \hat{H}_{\rm pot}\right) \end{cases}$$

Commutation relations: $[\hat{L}_1, \hat{L}_2] = -\mathrm{i}\hbar\hat{L}_3 \qquad [\hat{L}_2, \hat{L}_3] = \mathrm{i}\hbar\hat{L}_1 \qquad [\hat{L}_3, \hat{L}_1] = \mathrm{i}\hbar\hat{L}_2$

Close to an angular momentum (SO(3)), but not quite

The invariant is here: $\hat{L}_1^2 + \hat{L}_2^2 - \hat{L}_3^2$

# A few consequences of scale invariance for cold gases

Dynamics in a harmonic trap: Pitaevskii & Rosch (1997)

Paris (2001), Grimm group (2004), Köhl group (2012)
+ Vale and Jochim groups (2018)

Universal thermodynamics for 2D Bose gas Chin group (2011), Paris (2011-14)

+ second sound measurements, Paris (2018)

Universal thermodynamics for Fermi gas at unitarity:

Salomon group (2010), Zwierlein group (2012)

Universal viscosity in a unitary Fermi gas:

predictions by Son (2007), Zwerger (2011) Thomas group (2011)

# Outline

1. Scale/conformal invariance in a cold atomic gas

2. Exploring experimentally dynamical probes of scale invariance

3. Two-dimensional breathers

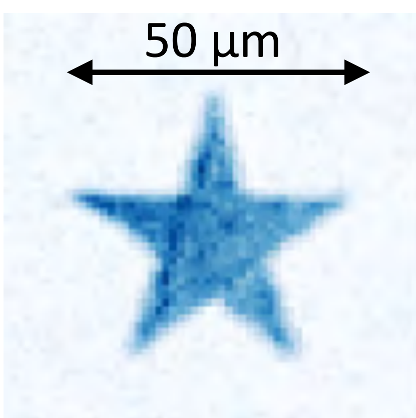
# Experimental setup (rubidium)

Frozen motion along the vertical direction $z$

$$\omega_z/2\pi = 4\,\mathrm{kHz}$$

Initial confinement in the $xy$ plane:

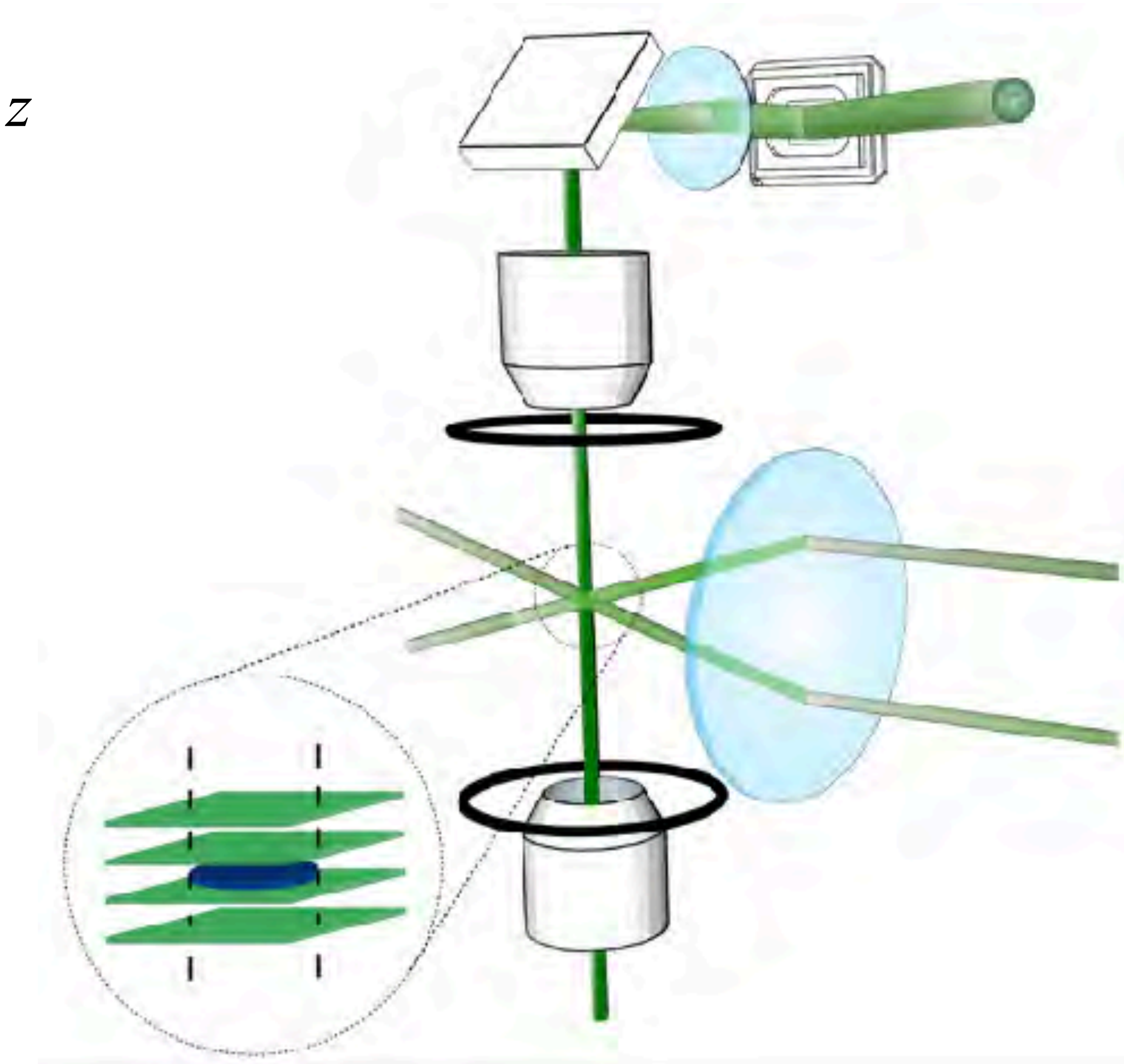
Box-like potential with arbitrary shape




internal state

$$|F=1, m=0\rangle$$

**Uniform gas in the Thomas-Fermi regime with a few $10^4$ atoms**



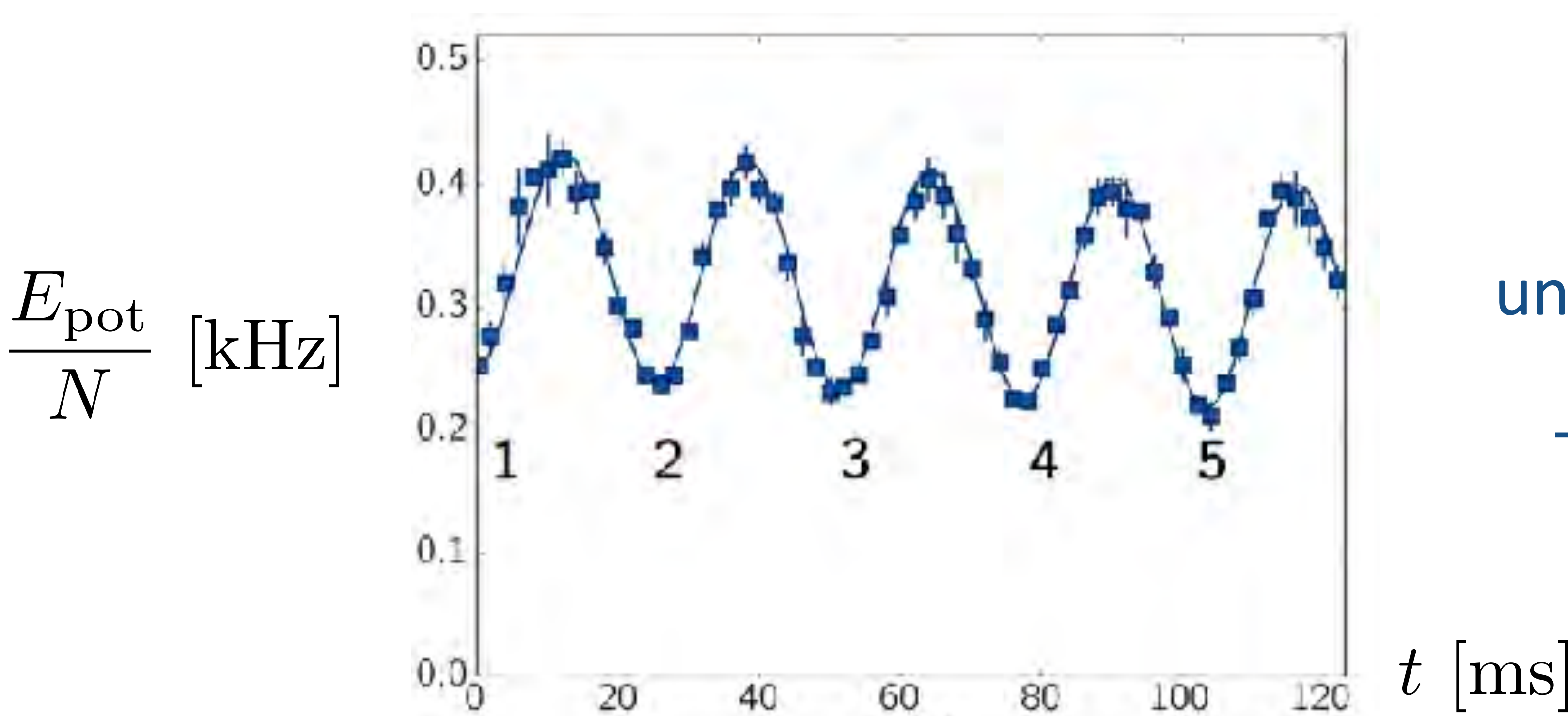
At time $t$ = 0, switch off the box potential and transfer the atoms to $|F=1, m=1\rangle$

Harmonic magnetic potential in the $xy$ plane with $\omega/2\pi = 20\,\mathrm{Hz}$

# Periodic evolution of $\langle r^2 \rangle$

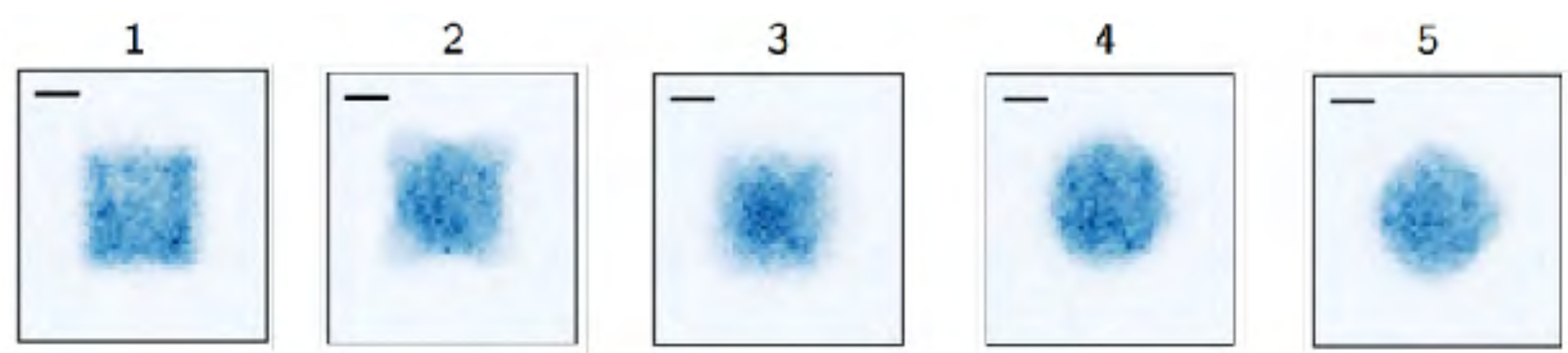
Consequence of the SO(2,1) symmetry:
Periodic exchange of energy between $E_{\rm pot}$ and $E_{\rm kin} + E_{\rm int}$

We measure $E_{\rm pot}$ using in-situ pictures: $E_{\rm pot}(t) = \int \frac{1}{2} m\omega^2 r^2 \, n(\boldsymbol{r}, t) \, {\rm d}^2 r$




Results for an initially uniform square distribution

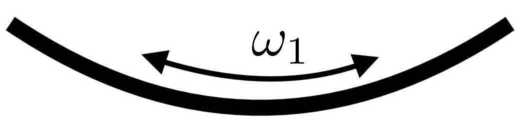
⟶ Oscillation at $2\omega$



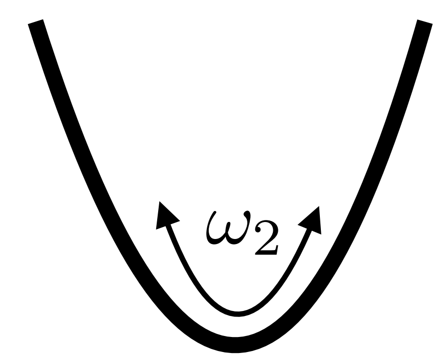
# Linking different solutions of the GP equation

Start with a solution: $$i\hbar\frac{\partial\psi_1}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi_1 + \frac{\hbar^2}{m}N\tilde{g}\,|\psi_1|^2\psi_1 + \frac{1}{2}m\omega_1^2 r^2\psi_1$$

Assuming Thomas-Fermi regime for the initial state, one can link for a given initial shape (square, disk, star, triangle, etc…) the evolutions of:




$$\begin{pmatrix} N_1 \\ \tilde{g}_1 \\ L_1 \\ \omega_1 \end{pmatrix} \xleftrightarrow{\textit{correspondance law}} \begin{pmatrix} N_2 \\ \tilde{g}_2 \\ L_2 \\ \omega_2 \end{pmatrix}$$




3 parameters needed: $$\mu^2 = \frac{\tilde{g}_2 N_2}{\tilde{g}_1 N_1} \qquad \rho = \frac{L_2}{L_1} \qquad \zeta = \frac{\omega_2}{\omega_1}$$

Rescaling of positions and non-linear rescaling of time: $n_2(\boldsymbol{r},t) = \lambda^2\mu^2\, n_1(\lambda\boldsymbol{r},\tau)$

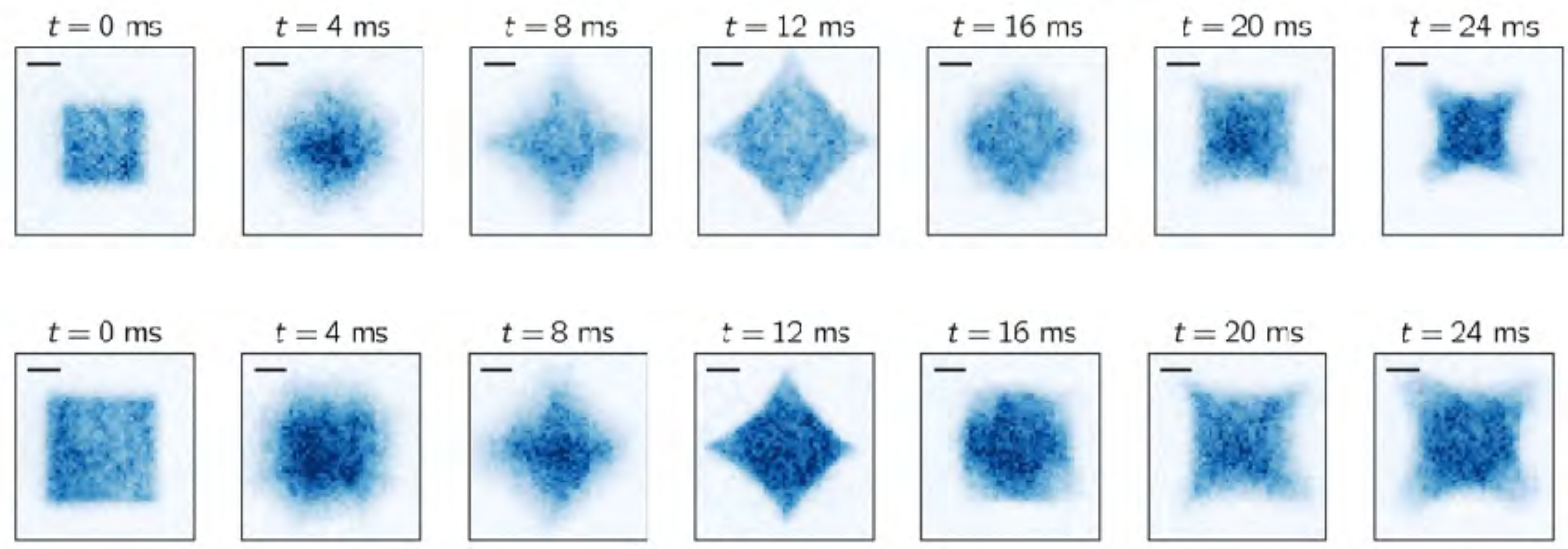
$$\lambda(t) = \left[\rho^2\cos^2(\omega_2 t) + \left(\frac{\mu}{\rho\zeta}\right)^2 \sin^2(\omega_2 t)\right]^{-1/2} \qquad \tan(\omega_1\tau) = \frac{\mu}{\zeta\rho^2}\tan(\omega_2 t)$$

# Experimental check of the correspondance law

Compare $(\tilde{g}_1 N_1,\, L_1)$ and $(\tilde{g}_2 N_2,\, L_2)$, keeping $\omega_1 = \omega_2$ for simplicity

**Set 1 of param.**

**Set 2 of param.**

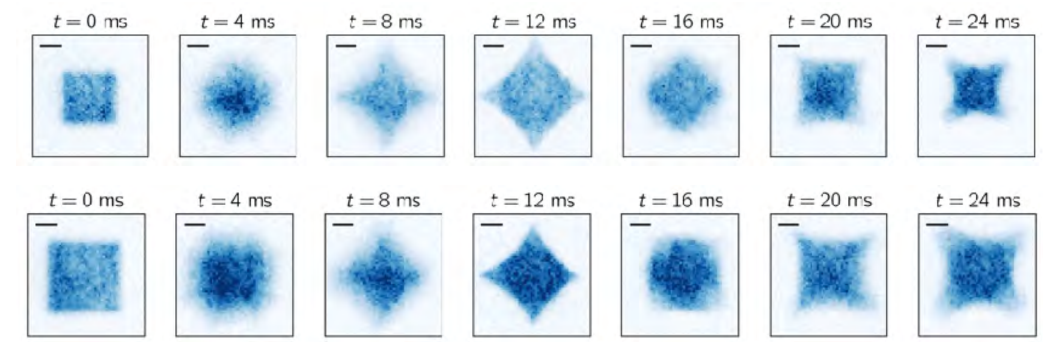


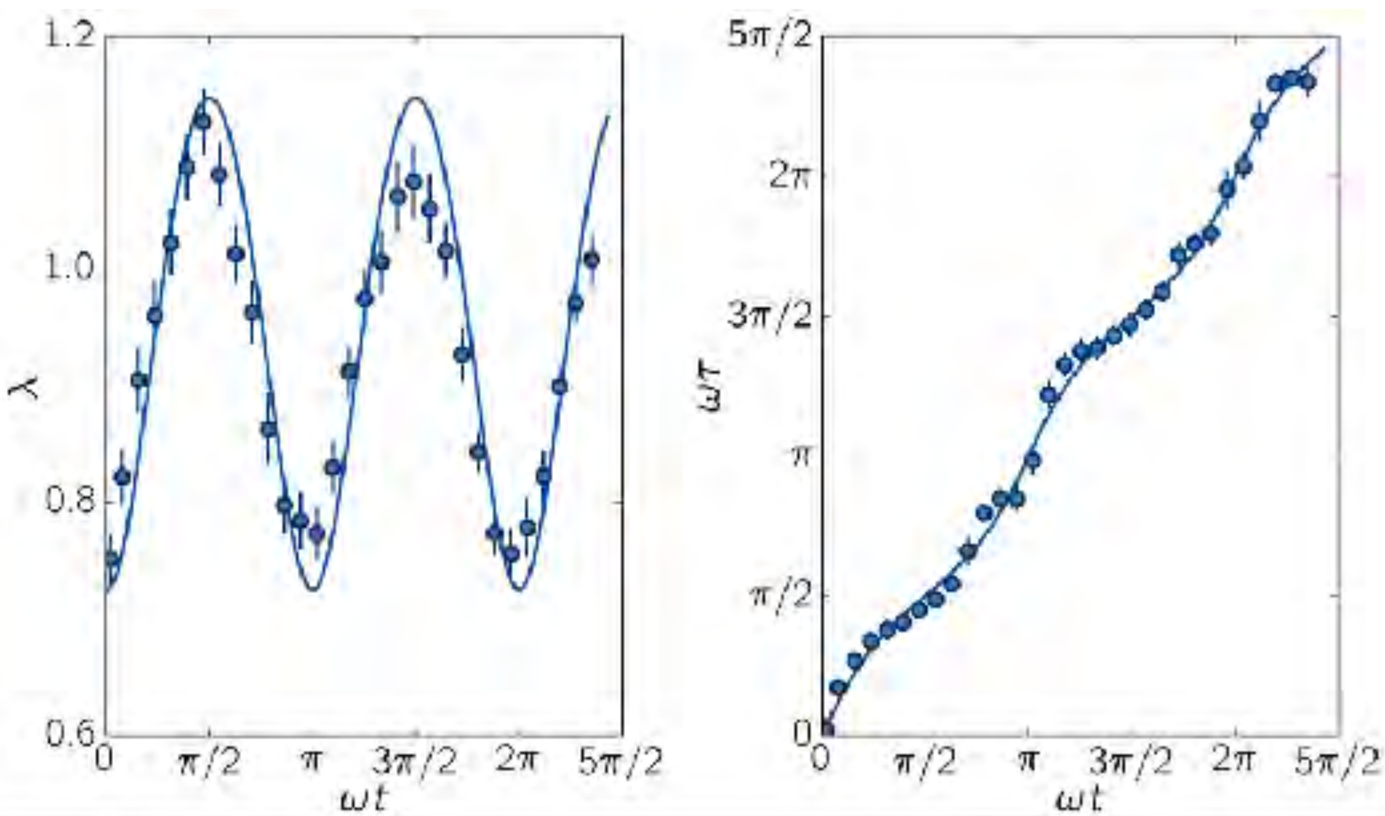

For each image of run 2, find the best match in run 1 after rescaling:

$$\mathcal{O}(n_1, n_2) = \max_{\lambda} \frac{\lambda \int n_1(\lambda \boldsymbol{r})\, n_2(\boldsymbol{r})\, \mathrm{d}^2 r}{||n_1||\ ||n_2||}$$

This best match provides $\lambda(t)$ and $\tau(t)$

# Experimental check of the correspondance law (2)



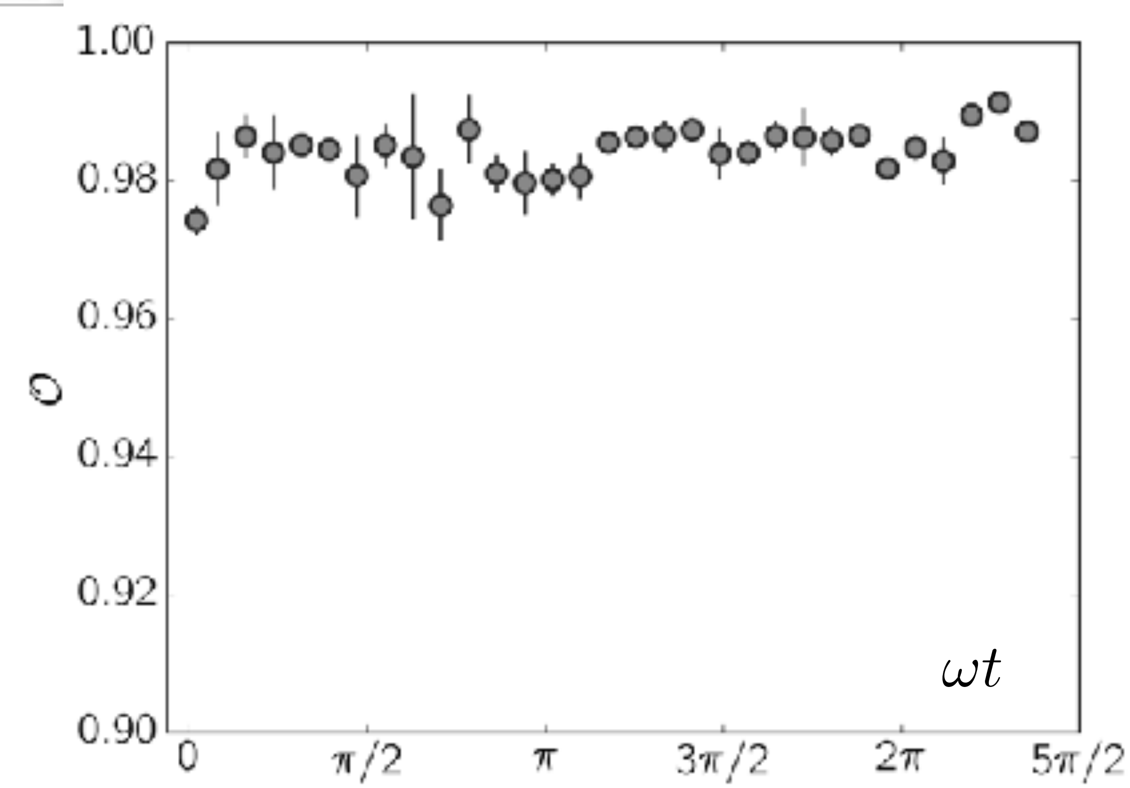




$$\frac{1}{\lambda^2(t)} = \rho^2 \cos^2(\omega_2 t) + \left(\frac{\mu}{\rho}\right)^2 \sin^2(\omega_2 t)$$

$$\tan(\omega_1 \tau) = \frac{\mu}{\rho^2} \tan(\omega_2 t)$$

No adjustable parameter

Overlap: excellent correspondance between the two series



# Outline

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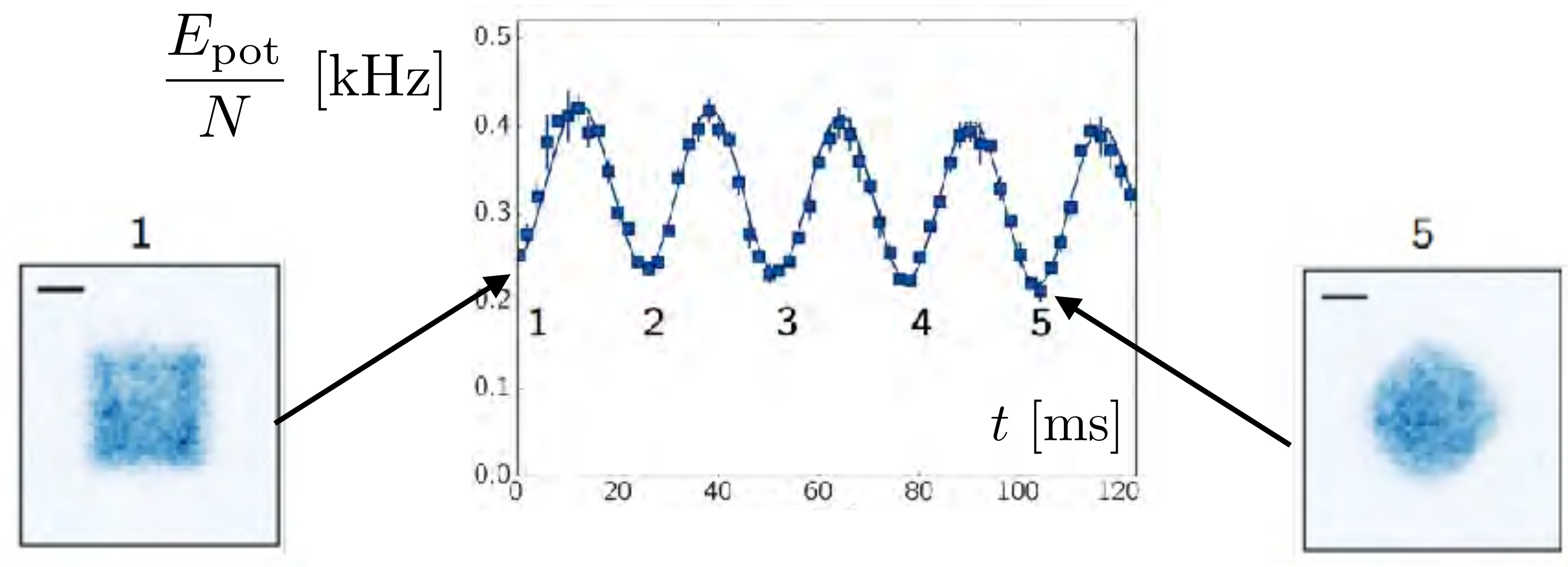
# Periodic evolution of shapes

The breathing mode of Pitaevskii-Rosch deals with the periodicity of average quantities:




Are there shapes that evolve periodically?

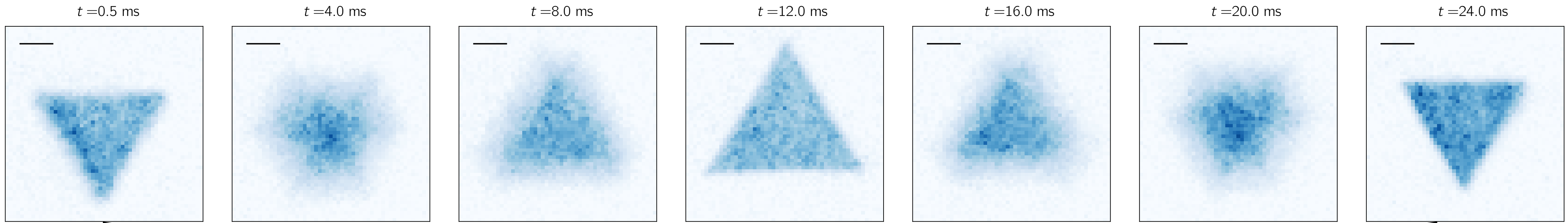
⟶ Much stronger requirement than simply the periodicity of $\langle r^2 \rangle$

⟶ The existence of such shapes is not guaranteed by the SO(2,1) symmetry

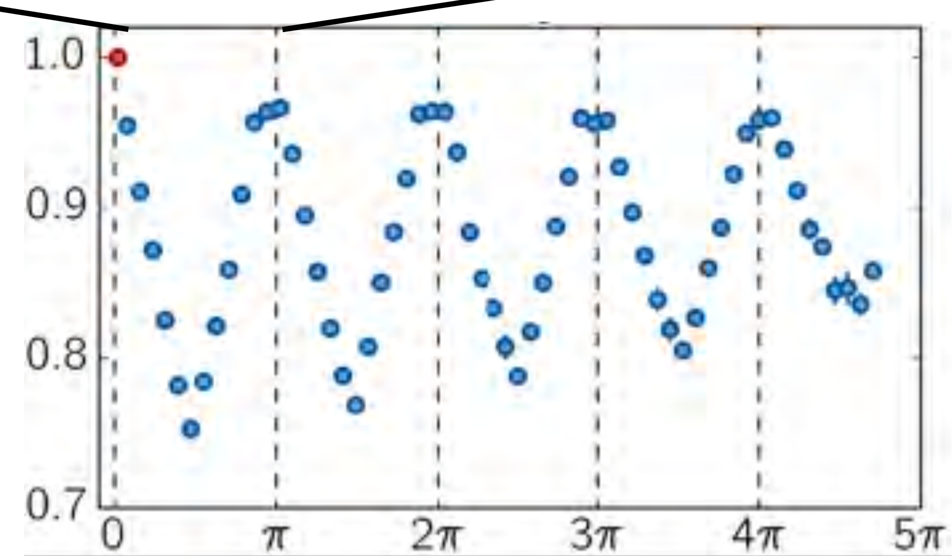
However...

# The equilateral triangle

Period $T/2$ with $T = 2\pi/\omega$

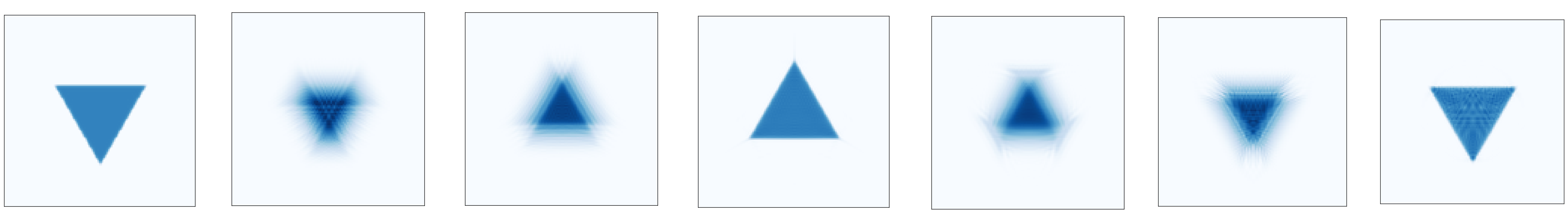



“Scalar product” of images with the initial one




$\omega t$

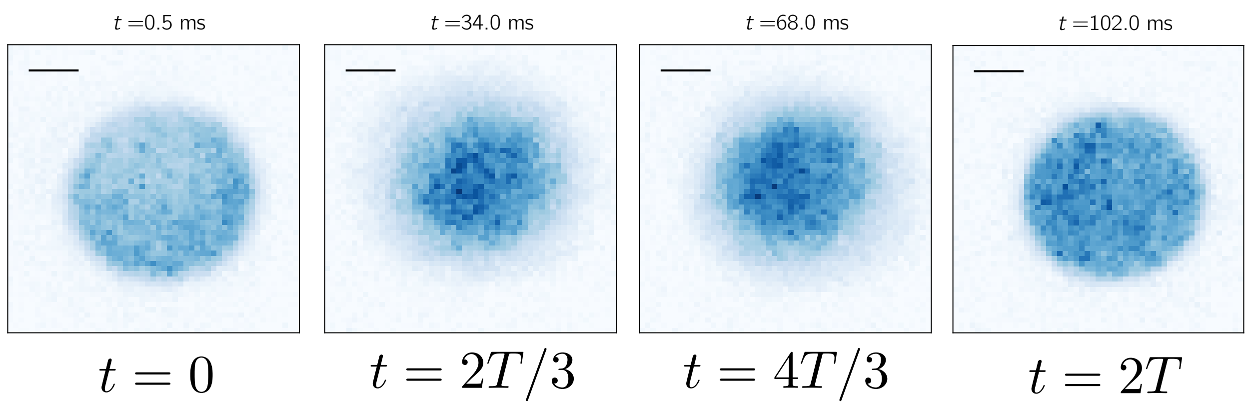
Numerical simulation starting from the GP ground state in the triangular box:



Overlap of wave functions: $|\langle\psi_i|\psi_f\rangle| > 0.995$ (calculation grid 1024 x 1024)

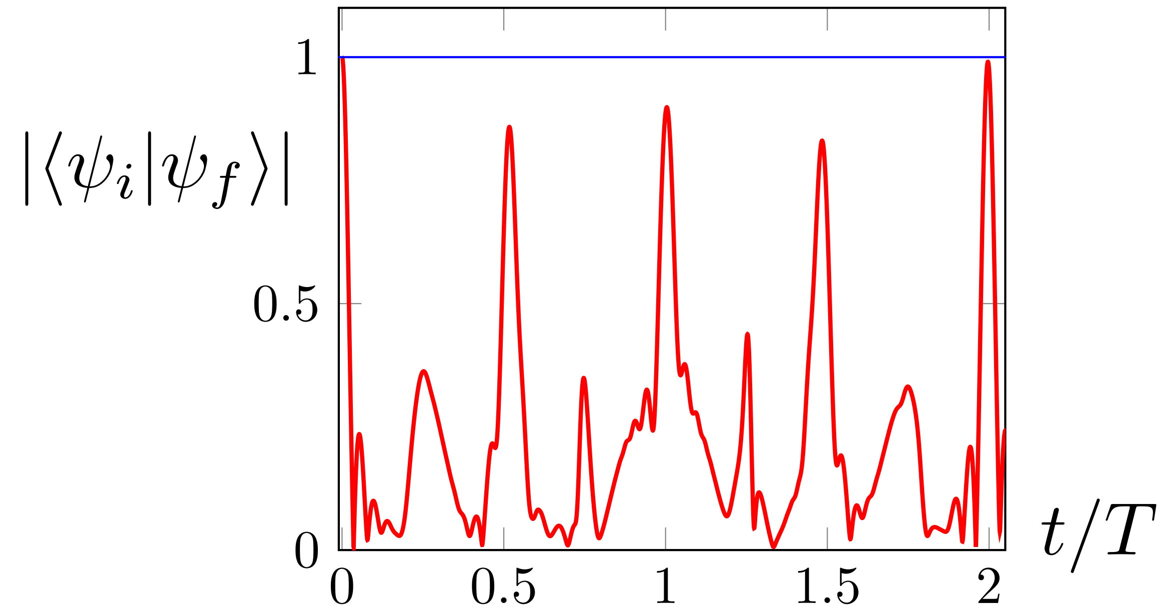
# Other examples of breathers? Only one so far: Disk

Experimentally:




Period $2T$ with $T = 2\pi/\omega$

Numerical simulation:



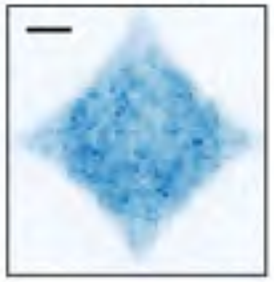

Overlap: $|\langle\psi_i|\psi_f\rangle| > 0.998$
on a grid 1024 x 1024

Genuine non-linear effect, which can (probably) not be captured by a linearization of the motion around an equilibrium position

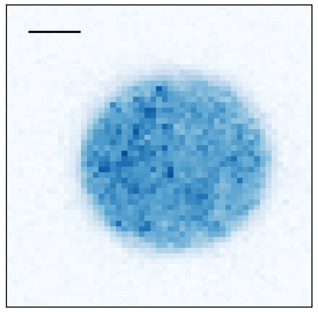
**Possible approach: Multi-mode treatment + mode-locking via non linear effects?**

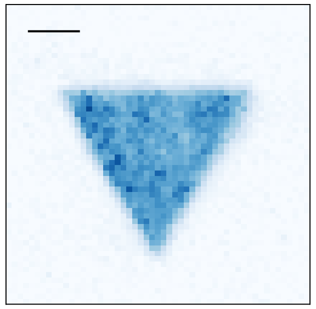
*No breathers found for squares, pentagons, hexagons, 6-branch stars,…*

# The rubidium team at Collège de France



# Conclusion and outlook





Quantum gases constitute an excellent platform to study scale/conformal invariance

*3D unitary Fermi gas, 2D (weakly interacting) Bose gas*

Here we explored with 2D Bose gases some predicted effects (connection between evolutions in various settings) as well as unexpected phenomena

*Breathers (triangle and disks)*

**Open questions:**

- Do such breathers also show up for other systems with SO(2,1) symmetry?

  *3D unitary Fermi gas, gas with $1/r^2$ interaction potential*

- Are the breathers robust against quantum effects when $\tilde{g} \gtrsim 1$ ?

  *Quantum anomaly explored recently by the Vale and Jochim groups*

- Robustness with respect to thermal effects?