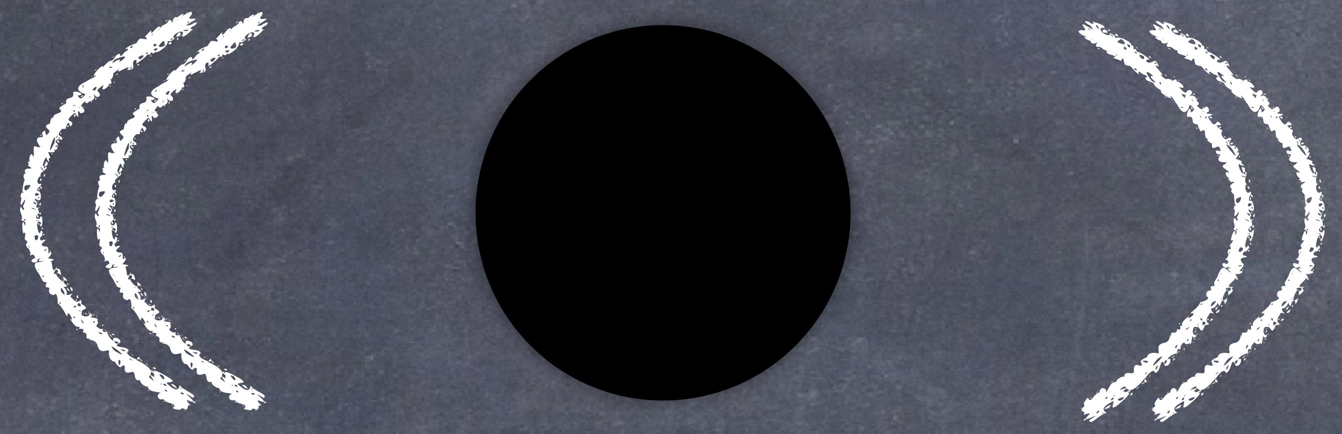
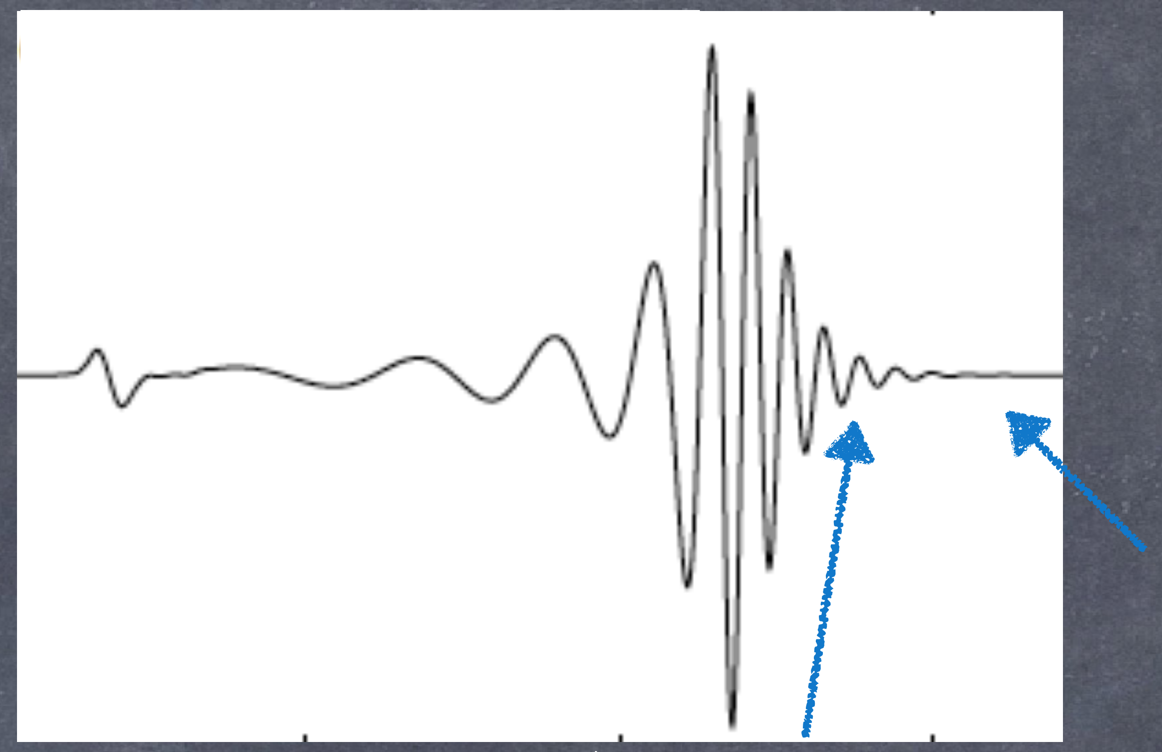
Chapter 2. Black hole perturbation theory

Ringtones of black holes





We would like to know

- Frequencies (quasi-normal modes)
- Amplitudes (as a function of the binary source parameters)

Uniqueness theorems: Kerr(M,J) is the universal final stage of collapse

- Frequencies (M,J) : all is known (1972 Teukolsky; 1985 Leaver)
- Amplitudes : estimates for detectability (1997 Flanagan-Hughes)

Plan

2. Black hole Perturbation theory

    2.1. Regge-Wheeler and Zerilli equations

    2.2. Quasi-normal modes of Schwarzschild - Black Hole Spectroscopy

    2.3. Newman-Penrose formalism, Petrov's classification, Teukolsky equation

    2.4. Quasi-normal modes of Kerr

    2.5. Mathisson-Papapetrou-Dixon theory

2.1. Regge-Wheeler and Zerilli equations

board

2.2. Quasi-normal modes of Schwarzschild

Black hole spectroscopy

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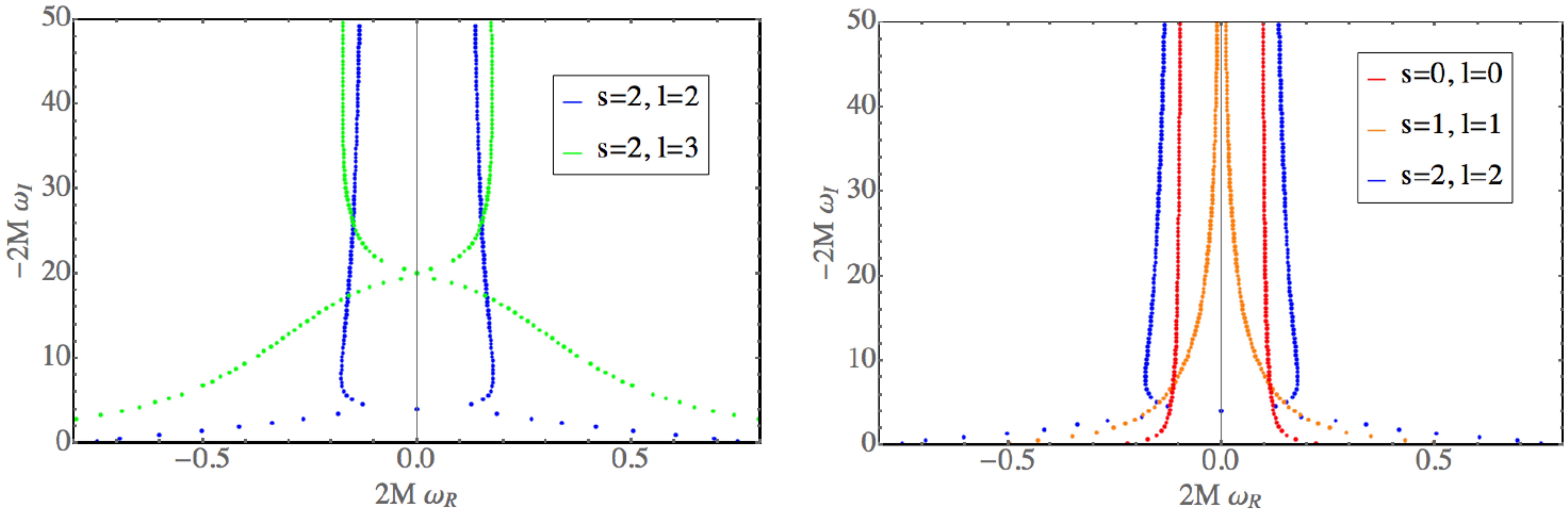
Schwarzschild spectroscopy



(a) *Quasi-normal modes frequencies for gravitational perturbations* ($s = 2$). (b) *Comparison of quasi-normal modes fundamental spectra* $l = |s|$ *for scalar, vector, and gravitational perturbations* ($s = 0, 1, 2$).

Most weakly damped s=2 mode $M\omega = 0.3737 - 0.0890i.$

Adapted from

E. Berti, V. Cardoso, and A. O. Starinets, "Quasinormal modes of black holes and black branes", *Class. Quant. Grav.* **26** (2009) 163001, `arXiv:0905.2975 [gr-qc]`.

"E. Berti's homepage, Ringdown." `https://pages.jh.edu/~eberti2/ringdown/`.

2.3. Newman-Penrose formalism, Petrov's classification, Teukolsky equation

Newman-Penrose formalism

Tetrad : $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$ useful for

- Fermions
- Cartan formalism $e^a = e^a_\mu dx^\mu$

The Newman-Penrose formalism is a tetrad formalism with complex tetrads where the tangent space Minkowski metric η_{ab} is chosen at each point to be

$$\eta_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The tetrad frame is chosen to be a set of 4 *null vectors* $l_\mu, n_\mu, m_\mu, \bar{m}_\mu$ with

$$g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu \,.$$

Null directions suitable to study GW propagation

Reformulation of the connection

4 ∇_μ ⟶

$$D = l^\mu \nabla_\mu \quad ; \quad \Delta = n^\mu \nabla_\mu \quad ; \quad \delta = m^\mu \nabla_\mu \quad ; \quad \bar{\delta} = \bar{m}^\mu \nabla_\mu \,.$$

24=4x6 $\Gamma^\mu_{\nu\rho}$ ⟶

$$\begin{aligned}
\kappa &= -m^\mu l^\nu \nabla_\nu l_\mu \,; \qquad \sigma = -m^\mu m^\nu \nabla_\nu l_\mu \,; \\
\lambda &= -n^\mu \bar{m}^\nu \nabla_\nu \bar{m}_\mu \,; \qquad \nu = -n^\mu n^\nu \nabla_\nu \bar{m}_\mu \,; \\
\rho &= -m^\mu \bar{m}^\nu \nabla_\nu l_\mu \,; \qquad \mu = -n^\mu m^\nu \nabla_\nu \bar{m}_\mu \,; \\
\tau &= -m^\mu n^\nu \nabla_\nu l_\mu \,; \qquad \varpi = -n^\mu l^\nu \nabla_\nu \bar{m}_\mu \,; \\
\epsilon &= -\frac{1}{2}(n^\mu l^\nu \nabla_\nu l_\mu + m^\mu l^\nu \nabla_\nu \bar{m}_\mu) \,; \\
\gamma &= -\frac{1}{2}(n^\mu n^\nu \nabla_\nu l_\mu + m^\mu n^\nu \nabla_\nu \bar{m}_\mu) \,; \\
\alpha &= -\frac{1}{2}(n^\mu \bar{m}^\nu \nabla_\nu l_\mu + m^\mu \bar{m}^\nu \nabla_\nu \bar{m}_\mu) \,; \\
\beta &= -\frac{1}{2}(n^\mu m^\nu \nabla_\nu l_\mu + m^\mu m^\nu \nabla_\nu \bar{m}_\mu).
\end{aligned}$$

Reformulation of the curvature

$R_{\alpha\beta\mu\nu}$ $R_{\mu\nu}$

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu[\rho}R_{\sigma]\nu} + g_{\nu[\rho}R_{\sigma]\mu} + \frac{1}{3}Rg_{\mu[\rho}g_{\sigma]\nu}.$$

20 10 10

10 $W_{\alpha\beta\mu\nu}$ $\longrightarrow$

$$\Psi_0 = W_{\alpha\beta\gamma\delta}l^\alpha m^\beta l^\gamma m^\delta,$$
$$\Psi_1 = W_{\alpha\beta\gamma\delta}l^\alpha n^\beta l^\gamma m^\delta,$$
$$\Psi_2 = W_{\alpha\beta\gamma\delta}l^\alpha m^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_3 = W_{\alpha\beta\gamma\delta}l^\alpha n^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_4 = W_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta.$$

5 Weyl-Newman-Penrose scalars

Reformulation of gauge invariance

$$\Psi_0 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta,$$
$$\Psi_1 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta,$$
$$\Psi_2 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_3 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_4 = W_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta.$$

They are diffeomorphism invariant but dependent upon the choice of tetrad.

One can perform flips $l^\mu \leftrightarrows n^\mu$ and local Lorentz transformation at each spacetime point (6 real functions = 3 complex functions).

- Rotations of type I which leave l^μ unchanged ($a \in \mathbb{C}$);

$$l^\mu \mapsto l^\mu, \qquad n^\mu \mapsto n^\mu + a^* m^\mu + a\bar{m}^\mu + aa^* l^\mu, \qquad m^\mu \mapsto m^\mu + a l^\mu \qquad \bar{m}^\mu \mapsto \bar{m}^\mu + a^* l^\mu.$$

- Rotations of type II which leave n^μ unchanged ($b \in \mathbb{C}$);

$$n^\mu \mapsto n^\mu, \qquad l^\mu \mapsto l^\mu + b^* m^\mu + b\bar{m}^\mu + bb^* n^\mu, \qquad m^\mu \mapsto m^\mu + b n^\mu, \qquad m^\mu \mapsto m^\mu + b^* n^\mu.$$

- Rotations of type III which leave the directions of l^μ and n^μ unchanged and rotate m^μ by an angle in the $m^\mu, \bar{m}^\mu$ plane ($A, \theta \in \mathbb{R}$);

$$l^\mu \mapsto A^{-1} l^\mu, \qquad n^\mu \mapsto A n^\mu, \qquad m^\mu \mapsto e^{i\theta} m^\mu \qquad \bar{m}^\mu \mapsto e^{-i\theta} \bar{m}^\mu.$$

$$\Psi_0 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta,$$
$$\Psi_1 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta,$$
$$\Psi_2 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_3 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_4 = W_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta.$$

Exercise

Prove that under

(i) Rotations of type I which leave l^μ unchanged ($a \in \mathbb{C}$);

$$l^\mu \mapsto l^\mu, \qquad n^\mu \mapsto n^\mu + a^* m^\mu + a\bar{m}^\mu + aa^* l^\mu, \qquad m^\mu \mapsto m^\mu + a l^\mu \qquad \bar{m}^\mu \mapsto \bar{m}^\mu + a^* l^\mu.$$

we have

$$\Psi_0 \mapsto \Psi_0,$$
$$\Psi_1 \mapsto \Psi_1 + a^*\Psi_0,$$
$$\Psi_2 \mapsto \Psi_2 + 2a^*\Psi_1 + (a^*)^2\Psi_0,$$
$$\Psi_3 \mapsto \Psi_3 + 3a^*\Psi_2 + 3(a^*)^2\Psi_1 + (a^*)^3\Psi_0,$$
$$\Psi_4 \mapsto \Psi_4 + 4a^*\Psi_3 + 6(a^*)^2\Psi_2 + 4(a^*)^3\Psi_1 + (a^*)^4\Psi_0;$$

(ii) Rotations of type II which leave n^μ unchanged ($b \in \mathbb{C}$);

$$n^\mu \mapsto n^\mu, \qquad l^\mu \mapsto l^\mu + b^* m^\mu + b\bar{m}^\mu + bb^* n^\mu, \qquad m^\mu \mapsto m^\mu + b n^\mu, \qquad m^\mu \mapsto m^\mu + b^* n^\mu.$$

$$\Psi_0 \mapsto \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4,$$
$$\Psi_1 \mapsto \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4,$$
$$\Psi_2 \mapsto \Psi_2 + 2b\Psi_3 + b^2\Psi_4,$$
$$\Psi_3 \mapsto \Psi_3 + b\Psi_4,$$
$$\Psi_4 \mapsto \Psi_4;$$

(iii) Rotations of type III which leave the directions of l^μ and n^μ unchanged and rotate m^μ by an angle in the $m^\mu, \bar{m}^\mu$ plane ($A, \theta \in \mathbb{R}$);

$$l^\mu \mapsto A^{-1} l^\mu, \qquad n^\mu \mapsto A n^\mu, \qquad m^\mu \mapsto e^{i\theta} m^\mu \qquad \bar{m}^\mu \mapsto e^{-i\theta}\bar{m}^\mu.$$

$$\Psi_0 \mapsto A^2 e^{-2i\theta}\Psi_0,$$
$$\Psi_1 \mapsto A^{-1} e^{i\theta}\Psi_1,$$
$$\Psi_2 \mapsto \Psi_2,$$
$$\Psi_3 \mapsto A e^{-i\theta}\Psi_3,$$
$$\Psi_4 \mapsto A^2 e^{-2i\theta}\Psi_4.$$

Exercise

Remember

$$\Psi_0 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta,$$
$$\Psi_1 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta,$$
$$\Psi_2 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_3 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_4 = W_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta.$$

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$$

$$\eta_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We define the 3 anti-symmetric bivectors

$$X_{\mu\nu} = -2n_{[\mu}\bar{m}_{\nu]} \quad ; \quad Y_{\mu\nu} = 2l_{[\mu}m_{\nu]} \quad ; \quad Z_{\mu\nu} = 2m_{[\mu}\bar{m}_{\nu]} - 2l_{[\mu}n_{\nu]}.$$

Prove that the Weyl tensor is a linear combination of these 3 bivectors

$$\begin{aligned} W_{\alpha\beta\gamma\delta} = &\Psi_0\, X_{\alpha\beta}X_{\gamma\delta} + \Psi_1\left(X_{\alpha\beta}Z_{\gamma\delta} + Z_{\alpha\beta}X_{\gamma\delta}\right) + \Psi_2\left(Y_{\alpha\beta}X_{\gamma\delta} + X_{\alpha\beta}Y_{\gamma\delta} + Z_{\alpha\beta}Z_{\gamma\delta}\right) \\ &+ \Psi_3\left(Y_{\alpha\beta}Z_{\gamma\delta} + Z_{\alpha\beta}Y_{\gamma\delta}\right) + \Psi_4\, Y_{\alpha\beta}Y_{\gamma\delta} + c.c. \end{aligned}$$

Exercise

Remember

$$\Psi_0 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta,$$
$$\Psi_1 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta,$$
$$\Psi_2 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_3 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_4 = W_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta.$$

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$$

$$\eta_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We define the 3 anti-symmetric bivectors

$$X_{\mu\nu} = -2n_{[\mu}\bar{m}_{\nu]} \quad ; \quad Y_{\mu\nu} = 2l_{[\mu}m_{\nu]} \quad ; \quad Z_{\mu\nu} = 2m_{[\mu}\bar{m}_{\nu]} - 2l_{[\mu}n_{\nu]}.$$

Prove that the Weyl tensor is a linear combination of these 3 bivectors

$$W_{\alpha\beta\gamma\delta} = \Psi_0\, X_{\alpha\beta}X_{\gamma\delta} + \Psi_1\left(X_{\alpha\beta}Z_{\gamma\delta} + Z_{\alpha\beta}X_{\gamma\delta}\right) + \Psi_2\left(Y_{\alpha\beta}X_{\gamma\delta} + X_{\alpha\beta}Y_{\gamma\delta} + Z_{\alpha\beta}Z_{\gamma\delta}\right)$$
$$+ \Psi_3\left(Y_{\alpha\beta}Z_{\gamma\delta} + Z_{\alpha\beta}Y_{\gamma\delta}\right) + \Psi_4\, Y_{\alpha\beta}Y_{\gamma\delta} + c.c.$$

Hint: compute

$$X_{\gamma\delta} l^\gamma m^\delta$$
$$Y_{\gamma\delta} l^\gamma m^\delta$$
$$Z_{\gamma\delta} l^\gamma m^\delta$$
$$X_{\gamma\delta} \bar{m}^\gamma n^\delta$$
$$Y_{\gamma\delta} \bar{m}^\gamma n^\delta$$
$$Z_{\gamma\delta} \bar{m}^\gamma n^\delta$$

Exercise

Use

$$W_{\alpha\beta\gamma\delta} = \Psi_0\, X_{\alpha\beta}X_{\gamma\delta} + \Psi_1\left(X_{\alpha\beta}Z_{\gamma\delta} + Z_{\alpha\beta}X_{\gamma\delta}\right) + \Psi_2\left(Y_{\alpha\beta}X_{\gamma\delta} + X_{\alpha\beta}Y_{\gamma\delta} + Z_{\alpha\beta}Z_{\gamma\delta}\right) + \Psi_3\left(Y_{\alpha\beta}Z_{\gamma\delta} + Z_{\alpha\beta}Y_{\gamma\delta}\right) + \Psi_4\, Y_{\alpha\beta}Y_{\gamma\delta} + c.c.$$

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$$

$$\eta_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

to prove that

$$l_{[\alpha}W_{\beta]\gamma\delta[\rho}l_{\sigma]}l^\gamma l^\delta = \Psi_0 l_{[\alpha}\bar{m}_{\beta]}l_{[\rho}\bar{m}_{\sigma]} + \Psi_0^* l_{[\alpha}m_{\beta]}l_{[\rho}m_{\sigma]}.$$

Petrov's Classification

[Petrov, 1954][Géhéniau, 1957]

Higher dimensions: 2004

Petrov classified the Weyl tensor by the number of degenerate local eigenvalues and (antisymmetric) eigenbivectors of the Weyl tensor. The eigenvalue equation reads as

$$W^{\mu\nu}{}_{\alpha\beta} X^{\alpha\beta} = \lambda \, X^{\mu\nu}.$$

Distinction: $G_{\mu\nu} \sim T_{\mu\nu}, \qquad W_{\mu\nu\alpha\beta}$

Classification of the Weyl tensor categorizes both vacuum and non-vacuum solutions to Einstein's equations

$W_{\mu\nu\alpha\beta}$ is conformally invariant $\longrightarrow$ $W_{\mu\nu\alpha\beta}$ identical for AdS and Minkowski

Three equivalent classifications

(1)

Petrov classified the Weyl tensor by the number of degenerate local eigenvalues and (antisymmetric) eigenbivectors of the Weyl tensor. The eigenvalue equation reads as

$$W^{\mu\nu}{}_{\alpha\beta}X^{\alpha\beta} = \lambda\, X^{\mu\nu}.$$

(2)

A non-trivial result due to Penrose in 1960 shows that solving this eigenvalue problem is equivalent to classify spacetimes according to the degeneracy of *principal null directions* of the Weyl tensor. Such directions are spanned by null vectors k^μ obeying

$$k_{[\alpha}W_{\beta]\gamma\delta[\rho}k_{\sigma]}k^\gamma k^\delta = 0.$$

(3)

Yet another equivalent formulation of the classification is the following. We have just seen that with respect to a chosen tetrad, the Weyl tensor is completely determined by the five Weyl-Newman-Penrose scalars. The third formulation of the classification consists in determining how many of these scalars can be made to vanish for a given spacetime by choosing a suitable orientation of the tetrad frame.

For the proof (1) <-> (2) : see [Stephani, Kramer, MacCallum, Hoenselaers, Herlt, 2004]

Petrov classification using formulation (3)

Type I

$$\begin{aligned}\Psi_0 &\mapsto \Psi_0,\\ \Psi_1 &\mapsto \Psi_1 + a^*\Psi_0,\\ \Psi_2 &\mapsto \Psi_2 + 2a^*\Psi_1 + (a^*)^2\Psi_0,\\ \Psi_3 &\mapsto \Psi_3 + 3a^*\Psi_2 + 3(a^*)^2\Psi_1 + (a^*)^3\Psi_0,\\ \Psi_4 &\mapsto \Psi_4 + 4a^*\Psi_3 + 6(a^*)^2\Psi_2 + 4(a^*)^3\Psi_1 + (a^*)^4\Psi_0;\end{aligned}$$

Type II

$$\begin{aligned}\Psi_0 &\mapsto \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4,\\ \Psi_1 &\mapsto \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4,\\ \Psi_2 &\mapsto \Psi_2 + 2b\Psi_3 + b^2\Psi_4,\\ \Psi_3 &\mapsto \Psi_3 + b\Psi_4,\\ \Psi_4 &\mapsto \Psi_4;\end{aligned}$$

Type III

$$\begin{aligned}\Psi_0 &\mapsto A^2e^{-2i\theta}\Psi_0,\\ \Psi_1 &\mapsto A^{-1}e^{i\theta}\Psi_1,\\ \Psi_2 &\mapsto \Psi_2,\\ \Psi_3 &\mapsto Ae^{-i\theta}\Psi_3,\\ \Psi_4 &\mapsto A^2e^{-2i\theta}\Psi_4.\end{aligned}$$

Given a NP tetrad $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$ we compute the scalars $\{\Psi_i\}_{i=1,\dots,5}$

If Weyl vanishes, we are done. Let us assume it is not vanishing.

We assume $\Psi_4 \neq 0$. Otherwise do a Type I rotation.

We consider a Type II rotation with complex parameter b.

Ψ_0 can be made to vanish if b is a solution to

$$\Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 = 0.$$

There are exactly 4 roots. The corresponding Type II rotations lead to

$$n^\mu \mapsto n^\mu, \quad l^\mu \mapsto l^\mu + b^*m^\mu + b\bar{m}^\mu + bb^*n^\mu, \quad m^\mu \mapsto m^\mu + bn^\mu, \quad m^\mu \mapsto m^\mu + b^*n^\mu.$$

They define the 4 principal null directions of the Weyl tensor

Using $$l_{[\alpha}W_{\beta]\gamma\delta[\rho}l_{\sigma]}l^\gamma l^\delta = \Psi_0 l_{[\alpha}\bar{m}_{\beta]}l_{[\rho}\bar{m}_{\sigma]} + \Psi_0^* l_{[\alpha}m_{\beta]}l_{[\rho}m_{\sigma]}.$$

the principal null directions obey $l_{[\alpha}W_{\beta]\gamma\delta[\rho}l_{\sigma]}l^\gamma l^\delta = 0.$ This proves (3) -> (2)

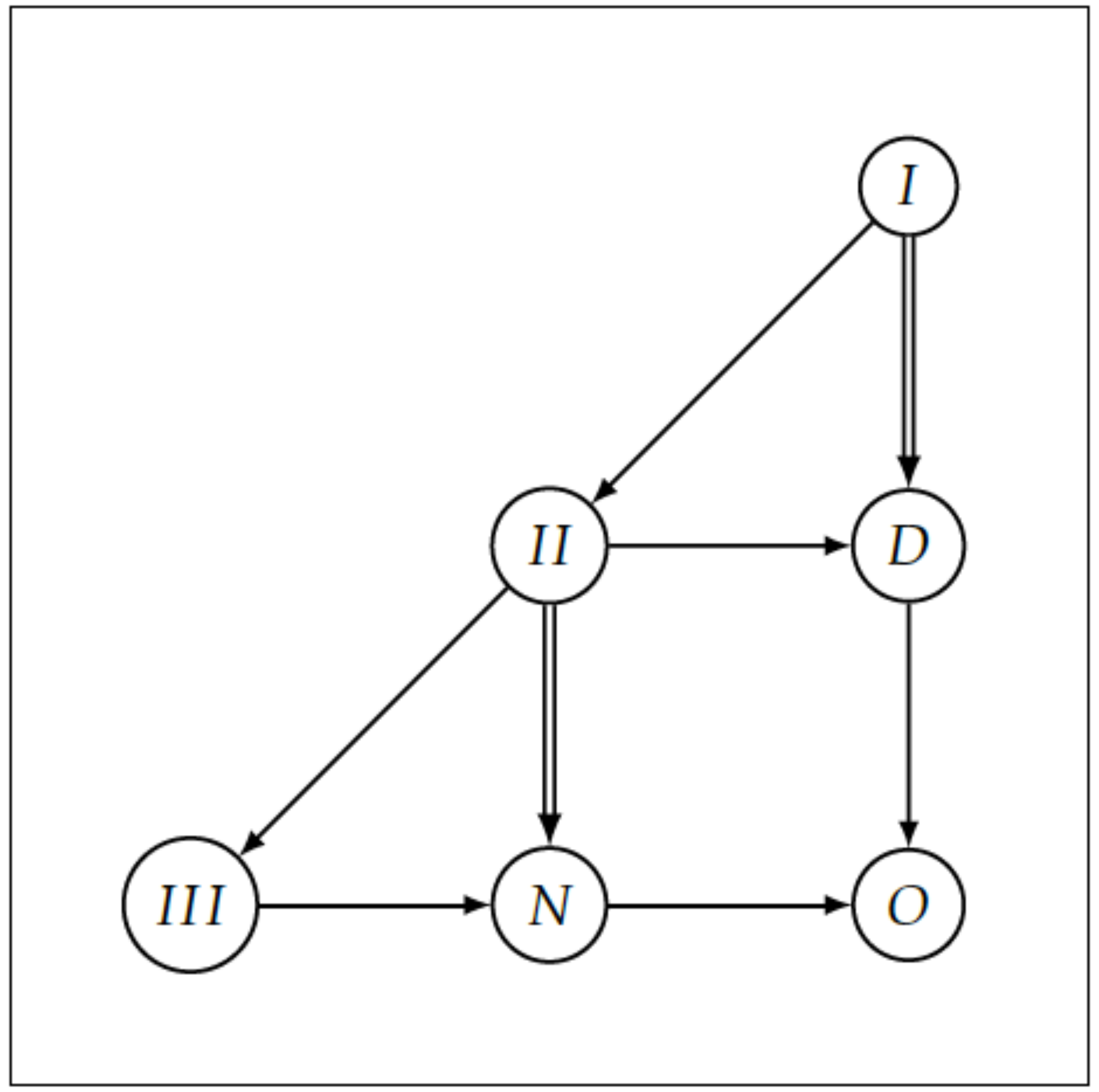
The degeneracy structure of the roots of a quartic polynomial lead to the classification.

Petrov's classification

Table and diagram adapted from
[Stephani, Kramer, MacCallum, Hoenselaers, Herlt, 2004]

Petrov type	Multiplicity of p.n.d.	Vanishing Weyl components	Criterion on $W_{\alpha\rho\sigma\beta}$
I	$(1,1,1,1)$	$\Psi_0 = 0$	$k_{[\gamma}W_{\alpha]\rho\sigma[\beta}k_{\delta]}k^\rho k^\sigma = 0$
II	$(2,1,1)$	$\Psi_0 = \Psi_1 = 0$	$k_{[\gamma}W_{\alpha]\rho\sigma\beta}k^\rho k^\sigma = 0$
D	$(2,2)$	$\Psi_0 = \Psi_1 = 0$	$k_{[\gamma}W_{\alpha]\rho\sigma\beta}k^\rho k^\sigma = 0$
III	$(3,1)$	$\Psi_0 = \Psi_1 = \Psi_2 = 0$	$k_{[\gamma}W_{\alpha]\rho\sigma\beta}k^\rho = 0$
N	(4)	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$	$W_{\alpha\rho\sigma\beta}k^\alpha = 0$
O	$\varnothing$	$\Psi_i = 0, \forall i$	$W_{\alpha\rho\sigma\beta} = 0$

(a) *Characterisation of Petrov types. k^μ is always the most degenerate principal null direction (p.n.d.).*



(b) *The Penrose graph summarizing the degeneracy growth in Petrov's classification. Each arrow indicates one additional degeneracy.*

Kerr is Type D: it admits 2 distinct principal null directions.

A Newman-Penrose tetrad adapted to these null directions and such that Ψ_3, Ψ_4 vanish is called a Kinnersley tetrad. Only Ψ_2 is non-vanishing in a Kinnersley tetrad.

The Goldman-Sachs theorem implies that for a Type D spacetime, the principal null directions are shear-free geodesic congruences. The Newman-Penrose formalism is therefore well-adapted for the study of GW in Kerr!

Quasi-normal modes: definition

We consider linear perturbations of matter and the metric around the Kerr geometry.
All linear perturbations are collectively denoted as $\Phi^i(t,r,\theta,\phi)$

Thanks to the 2 Killing vectors of Kerr, there is no explicit time or angular dependence in the field equations. Therefore, the linear solution can be decomposed in isolated Fourier modes:

$$\Phi^i(t,r,\theta,\phi) = \frac{1}{2\pi}\int d\omega e^{-i\omega t}\sum_{m\in\mathbb{Z}} e^{im\phi}F^i(r,\theta)$$

The linear equations are partial differential equations in r,θ, which dependence upon (M,a=J/M) and m,ω

The boundary conditions are

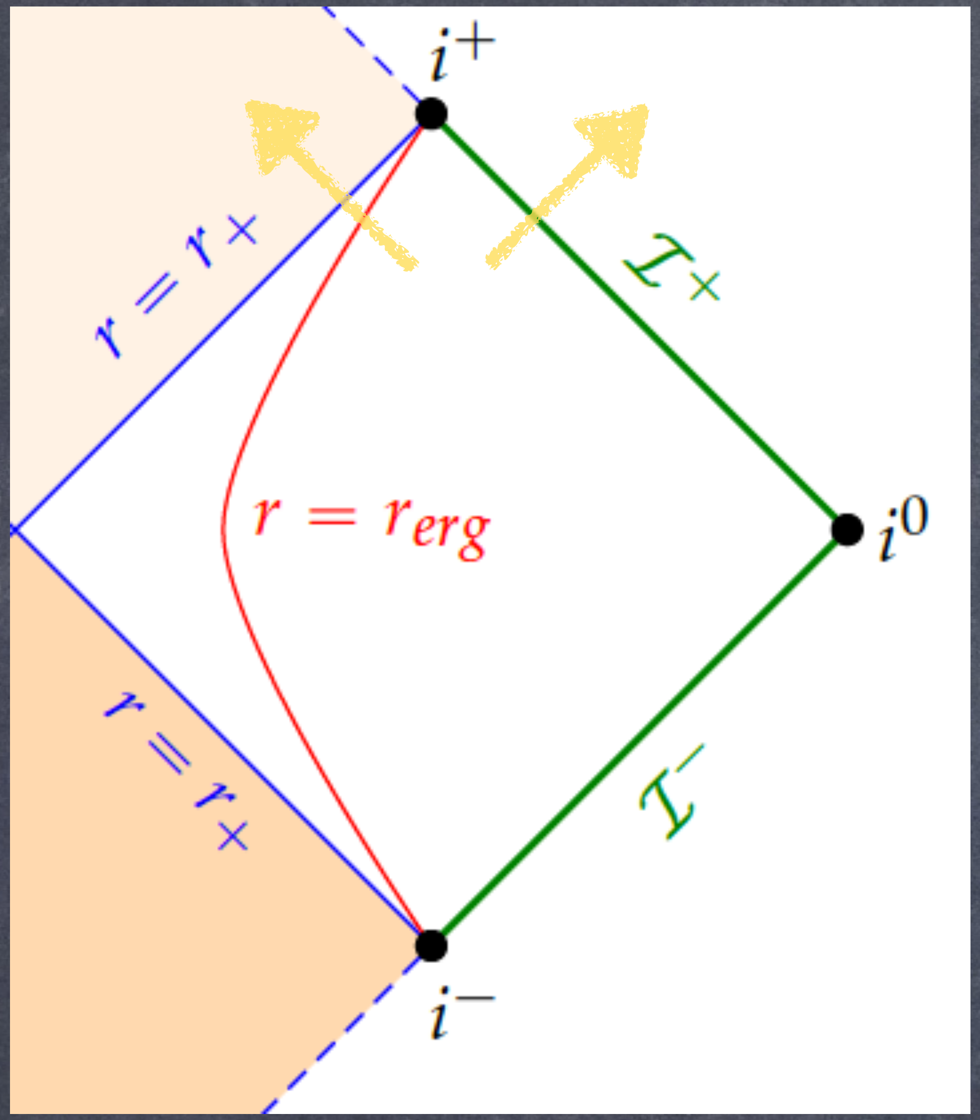
- "In" : Ingoing at the horizon $e^{-i\omega t+im\phi}F(r,\theta) \xrightarrow{r\to r_+} e^{-i\omega v_\star+im\phi_\star}F(\theta)$

where $r_\star$ is the *tortoise coordinate*, $v_\star = t + r_\star$ the advanced time and $\phi_\star$ the angular coordinate which define the regular ingoing Eddington-Finkelstein coordinates $v_\star, r_\star, \theta, \phi_\star$ (or in other words, which resolves the geometry near the horizon).

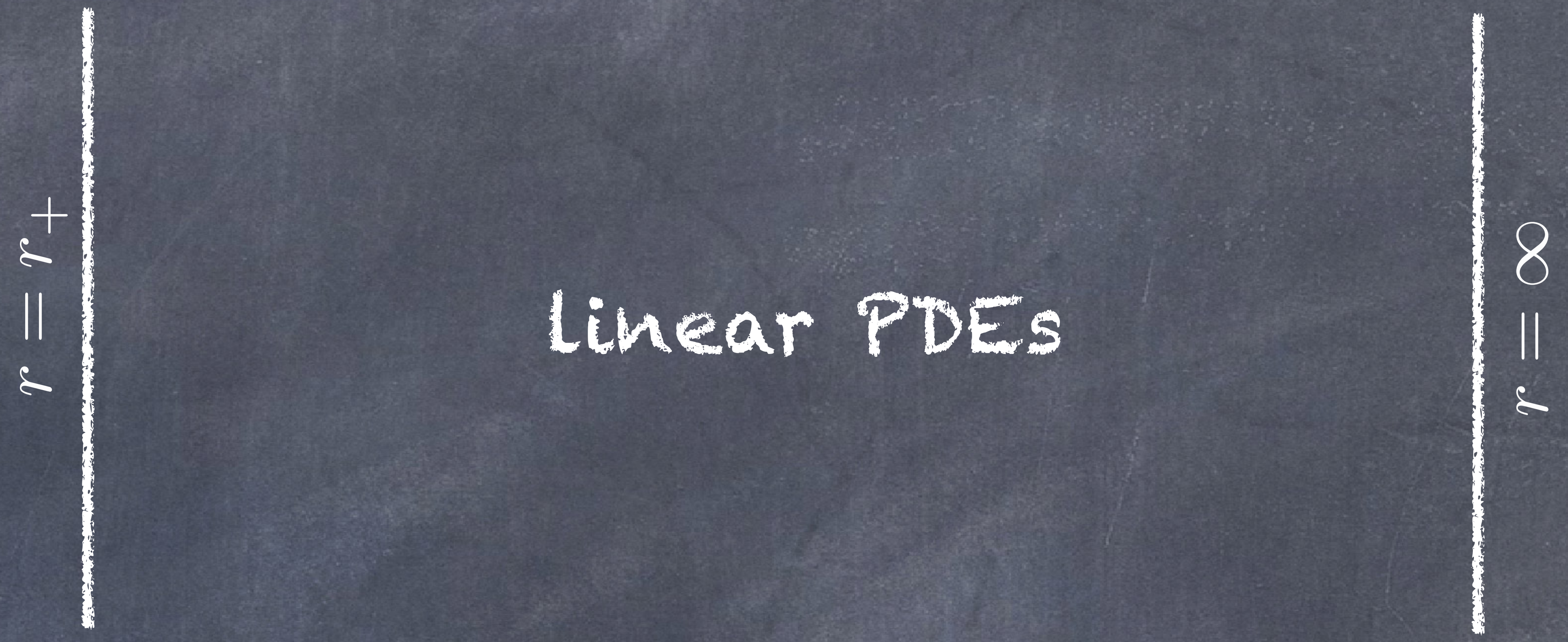
- "Up" : Outgoing at infinity $e^{-i\omega t+im\phi}F(r,\theta) \xrightarrow{r\to\infty,\, u \text{ fixed}} e^{-i\omega u+im\phi}\tilde{F}(\theta)$

where $u = t - r$ is the asymptotically flat retarded time.

Quasi-normal modes: qualitative solution



Penrose diagram of the domain of outer communication of Kerr



Radial interval at fixed Boyer-Linquist time

This is a boundary value problem. It admits an infinite set of solutions labelled by

– Spheroidal harmonic numbers l, m

– Overtone $N = 0, 1, 2, \ldots$

The frequencies are $\omega_{lmN} = \mathrm{Re}(\omega_{lmN}) + i\,\mathrm{Im}(\omega_{lmN})$ Linear stability is equivalent to $\mathrm{Im}(\omega_{lmN}) < 0$

Teukolsky equation

Teukolsky found during his PhD thesis with Kip Thorne in 1972 how to separate the radial and polar equations for the Weyl components.

He started to write down the linear perturbation equations in terms of the 5 Weyl-Newman-Penrose complex scalars for a Kinnersley tetrad. In Kerr (in Boyer-Linquist coordinates), only

$$\Psi_2 = -\frac{M}{(r - ia\cos\theta)^3}.$$

is non-vanishing. It turns out that the linear perturbations $\{\delta\Psi_i\}_i$ can be expressed in terms of either $\delta\Psi_0$ or $\delta\Psi_4$ up to the change of M,a (in $\delta\Psi_2$).

Insight: the equations for $(r - ia\cos\theta)^4\delta\Psi_4$ and $\delta\Psi_0$ are separable. We call it ψ for s=-2 or +2. It obeys

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\phi^2}$$
$$- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\phi}$$
$$- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = T$$

Radial and polar Teukolsky equations

$$\psi(t,r,\theta,\phi) = \frac{1}{2\pi}\int d\omega e^{-i\omega t}\sum_{l=|s|}^{\infty}\sum_{m=-l}^{+l} e^{im\phi}R^s_{lm\omega}(r)S^s_{lm\omega}(\cos\theta).$$

The equation for $S^s_{lm\omega}(\cos\theta)$ is called the *spin weighted spheroidal harmonic* equation

$$\left[\frac{d}{dx}(1-x^2)\frac{d}{dx}\right]S^s_{lm\omega}(x) + \left[a^2\omega^2x^2 - 2a\omega sx + \mathcal{E}^s_{lm\omega} - \frac{m^2+2msx+s^2}{1-x^2}\right]S^s_{lm\omega}(x) = 0$$

where $x = \cos\theta$ and $\mathcal{E}^s_{lm\omega}$ is the separation constant. When $a = 0$, the dependence in ω drops out and the functions $S^s_{lm}(\cos\theta)$ reduce to spin-weighted spherical harmonics $Y^s_{lm}(\theta,\phi) = S^s_{lm}(\cos\theta)e^{im\phi}$ after inclusion of the Fourier ϕ factor. In this case, the angular separation constants $\mathcal{E}^s_{lm\omega} = \mathcal{E}^s_{lm}$ are known analytically to be $\mathcal{E}^s_{lm} = l(l+1)$.

The radial equation is the *radial Teukolsky equation*:

$$\Delta^{-s}\frac{\partial}{\partial r}(\Delta^{s+1}\frac{\partial R_{lm\omega}}{\partial r}) - V(r)R_{lm\omega}(r) = T_{lm\omega}(r)$$

with source $T_{lm\omega}(r)$ and potential

$$\begin{aligned}
V(r) &= -\frac{(K_{m\omega})^2 - 2si(r-M)K_{m\omega}}{\Delta} - 4si\omega r + \lambda_{\ell m\omega},\\
K_{m\omega} &\triangleq (r^2+a^2)\omega - ma,\\
\lambda_{\ell m\omega} &\triangleq \mathcal{E}_{lm\omega} - 2am\omega + a^2\omega^2 - s(s+1).
\end{aligned}$$

When $a = 0$, the m dependence drops out. This is a consequence of $SO(3)$ symmetry.

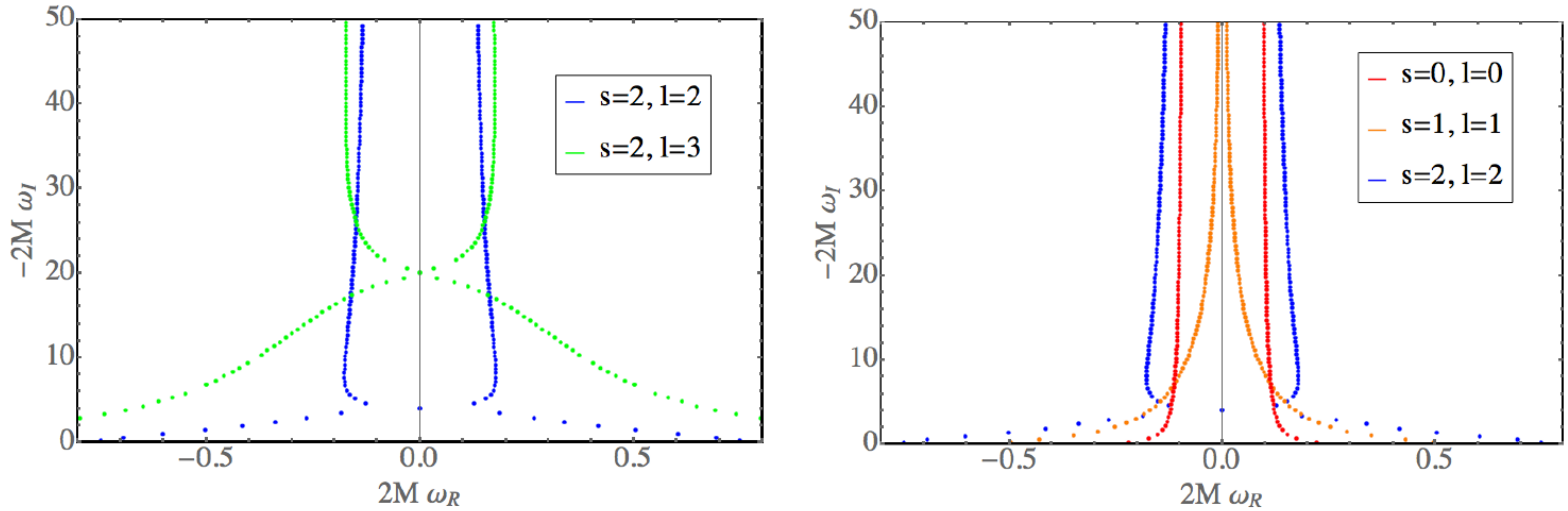
Radial and polar Teukolsky solutions

- Leaver method: continued fractions [1985]
- Any method that solves a boundary value ODE !
- Implementations in "Black Hole Perturbation Toolkit"

  http://bhptoolkit.org/

- Mathematica 12.1. now implements HeunG which solves the Spin weighted spheroidal harmonic equation.

Schwarzschild spectroscopy



(a) *Quasi-normal modes frequencies for gravitational perturbations* $(s = 2)$. (b) *Comparison of quasi-normal modes fundamental spectra* $l = |s|$ *for scalar, vector, and gravitational perturbations* $(s = 0, 1, 2)$.

Most weakly damped s=2 mode $M\omega = 0.3737 - 0.0890i.$

Adapted from

E. Berti, V. Cardoso, and A. O. Starinets, "Quasinormal modes of black holes and black branes", *Class. Quant. Grav.* **26** (2009) 163001, arXiv:0905.2975 [gr-qc].

"E. Berti's homepage, Ringdown." https://pages.jh.edu/~eberti2/ringdown/.

Kerr spectroscopy

- Most weakly damped s=2 mode $M\omega_{020} \approx 0.4437 - 0.0739(1 - a/M)^{0.3350}$

- Zeeman splitting (dependence upon m)

- Highly spinning behavior (Split between non-damped and zero-damped modes with half-integer real frequency). This is due to the near-horizon Kerr region with angular velocity $M\Omega = \frac{1}{2}$

Video on

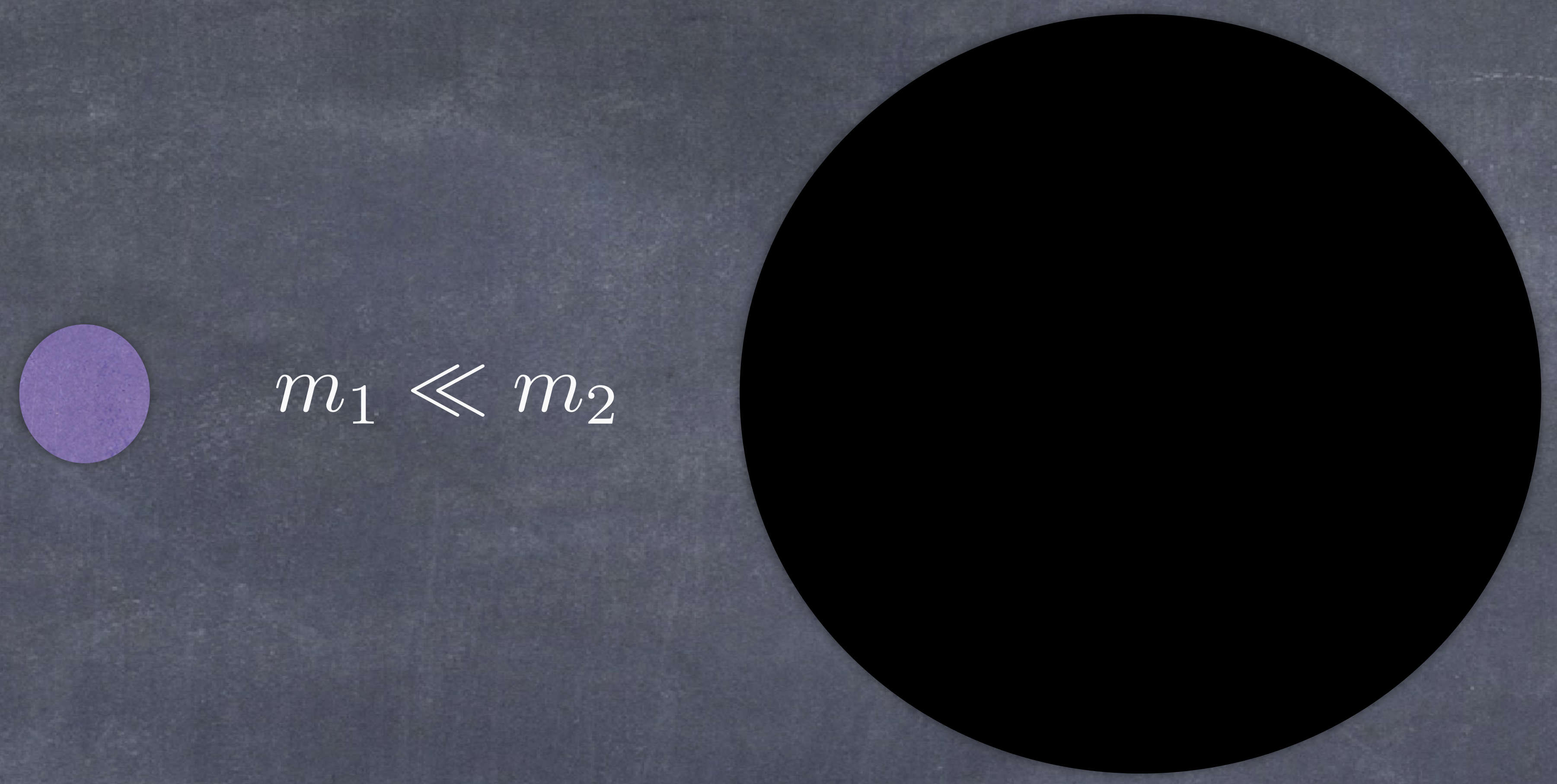
https://www.youtube.com/watch?v=lmXqtM4Ke9Q

[Cook, Zalutskiy, 1410.7698]

2.4. Mathisson-Papapetrou-Dixon theory

Or how to model finite objects without their gravitational backreaction?

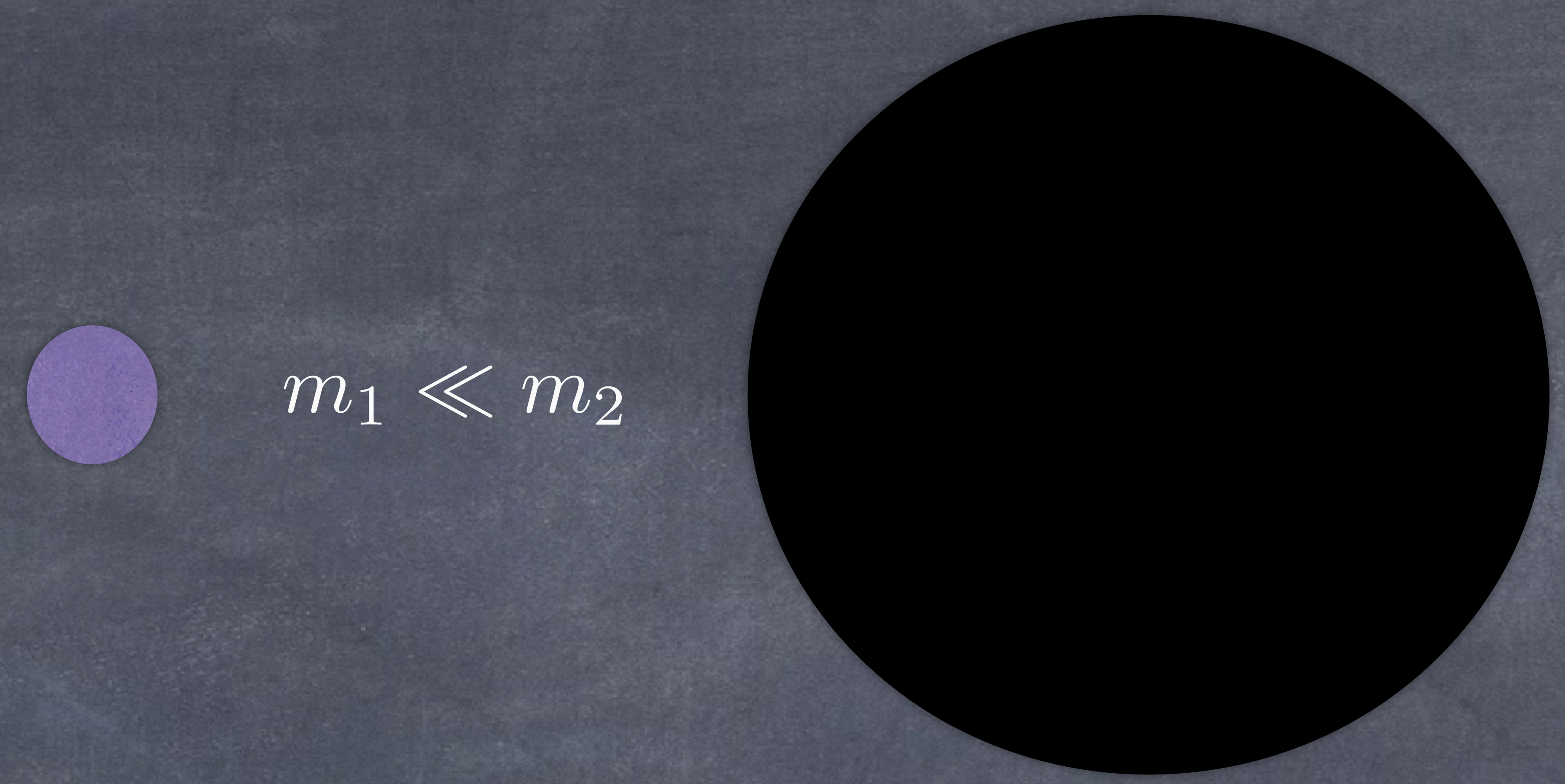
We consider the small ratio limit of the two-body problem



If body 1 is a point particle without any internal structure not gravitational backreaction, its motion is determined by geodesic motion in the metric generated by the body 2. [Einstein-Grommer, 1927]

This is not a postulate but a consequence of Einstein's equations. For a modern proof, see [Wald-Gralla, 2008]

This is very useful if the body 2 is a black hole because the metric is the Kerr metric depending only on M and J. It is also useful for a stationary metric like a neutron star, which is determined by two infinite sets of multipole moments.

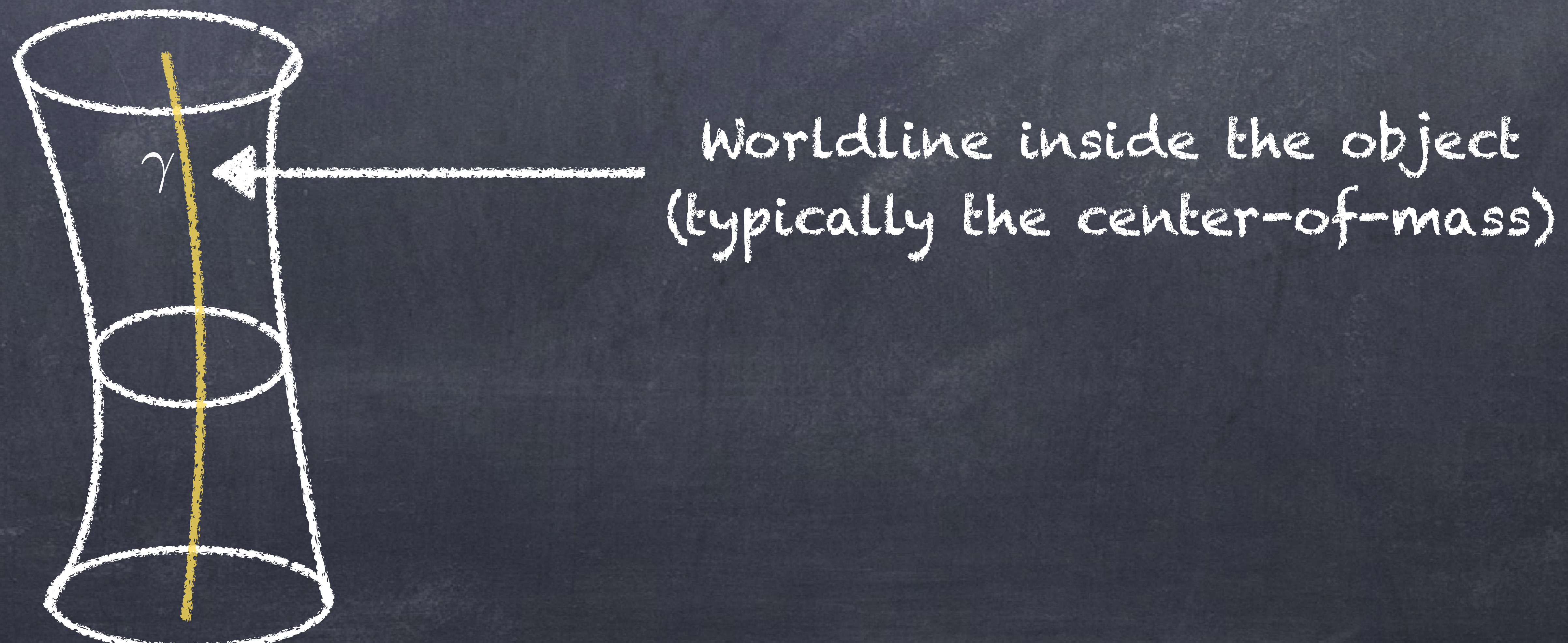


If body 1 is not a point particle but an extended object, a more general theory of motion is required

In addition, the body 1 gravitational backreacts and "gravitational self-force" corrections are also required.

In this lecture, we ignore the gravitational self-force. We assume that the background metric is the Kerr metric determined by the black hole (body 2).

We consider the space-time diagram



Stress-tensor for timelike geodesics

$$S = -m\int_{-\infty}^{\infty} d\tau$$

$$d\tau = \sqrt{-g_{\mu\nu}(x_*)dx_*^\mu dx_*^\nu} = \frac{d\tau}{d\lambda}d\lambda$$

$$S = -m\int_{-\infty}^{\infty} d\lambda \frac{d\tau}{d\lambda}\delta^{(4)}(x-x_*)$$

$$= -m\int_{-\infty}^{\infty} d\lambda \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\delta^{(4)}(x-x_*)$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}(x)}$$

$$= \frac{2}{\sqrt{-g}}(-m)\int_{-\infty}^{\infty} d\lambda \frac{1}{\frac{d\tau}{d\lambda}}\frac{1}{2}(-)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\delta^{(4)}(x-x_*)$$

$$u^\mu \equiv \frac{dx^\mu}{d\tau}$$

$$= m\int_{-\infty}^{\infty}\frac{\delta^{(4)}(x-x_*(\tau))}{\sqrt{-g}}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

$$T^{\mu\nu} = m\int_{-\infty}^{\infty}\frac{\delta^{(4)}(x-x_*(\tau))}{\sqrt{-g}}u^\mu(\tau)u^\nu(\tau)$$

Exercise

Show that the conservation of

$$T^{\mu\nu} = m\int_{-\infty}^{\infty} d\tau \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}} u^\mu(\tau)u^\nu(\tau)$$

$$\nabla_\nu T^{\mu\nu}(x) = 0$$

is equivalent to the geodesic equation.

Hint: use $$\nabla_\nu T^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}T^{\mu\nu}) + \Gamma^\mu_{\alpha\nu}T^{\alpha\nu}$$

Solution

$$\nabla_\nu T^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}T^{\mu\nu}) + \Gamma^\mu_{\alpha\nu}T^{\alpha\nu}$$

$$T^{\mu\nu} = m\int_{-\infty}^{\infty} d\tau \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}} u^\mu(\tau)u^\nu(\tau)$$

$$\nabla_\nu T^{\mu\nu} = m\int_{-\infty}^{\infty} \frac{d\tau}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\delta^{(4)}(x - x_*(\tau))u^\mu u^\nu + m\int_{-\infty}^{\infty} d\tau \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}}\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta$$

$$\frac{\partial}{\partial x^\mu}\delta^{(4)}(x - x_*(\tau)) = -\frac{\partial}{\partial x_*^\mu}\delta^{(4)}(x - x_*(\tau))$$

$$u^\mu(\tau)\frac{\partial}{\partial x^\mu}\delta^{(4)}(x - x_*(\tau)) = -\frac{d}{d\tau}\delta^{(4)}(x - x_*(\tau))$$

$$\nabla_\nu T^{\mu\nu} = m\int_{-\infty}^{\infty} \frac{d\tau}{\sqrt{-g}}\delta^{(4)}(x - x_*(\tau))\frac{d}{d\tau}u^\mu + m\int_{-\infty}^{\infty} d\tau \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}}\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta$$

$$\nabla_\nu T^{\mu\nu}(x) = 0 \quad \longleftrightarrow \quad \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0 \quad \longleftrightarrow \quad u^\nu \nabla_\nu u^\mu = 0$$

Extended objects contain additional multipoles:

- Spin
- Quadrupole
- 2^N pole

Their equation of motion follows from

$$\nabla_a T^{ab}_{body} = 0$$

$$P_t[\xi] = \int_{B_t} T^{ab}_{body}(x')\xi_a(x')dS_b$$

↑

??

Killing vector = Symmetry of spacetime

$$\mathcal{L}_\xi g_{\mu\nu} = 0$$

$$\xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\rho\mu} = 0$$

By covariance: $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$

In Minkowski, using Cartesian coordinates, $(ct, \vec{x})$

$$\partial_{(\mu}\xi_{\nu)} = 0$$

$$\xi^\mu = a^\mu + b^{[\mu\nu]} x^\nu$$

translations Lorentz

$$[\xi^\mu_{(a)}, \xi^\nu_{(b)}] = C^{(c)}_{(a)(b)} \xi^\mu_{(c)}$$

Poincaré algebra under the Lie bracket

In AdS_4/dS_4 (other two maximally symmetric spacetimes) -> 10 as well

$AdS_4 : \; SO(2,3)$ $dS_4 : \; SO(1,4)$

Schwarzschild: $\mathbb{R} \times SO(3)$

Kerr: $\mathbb{R} \times SO(2) \times \mathbb{Z}_2$

Exercise

Prove that a Killing vector obeys

$$\nabla_a \nabla_b \xi_c = R_{cbad}\xi^d$$

using the Misner-Thorne-Wheeler/Wald convention $[\nabla_a, \nabla_b]\xi^c = R^c{}_{dab}\xi^d$

Solution

Prove that a Killing vector obeys

$$\nabla_a \nabla_b \xi_c = R_{cbad}\xi^d$$

using the Misner-Thorne-Wheeler/Wald convention $[\nabla_a, \nabla_b]\xi^c = R^c{}_{dab}\xi^d$

$$\nabla_a(\nabla_b\xi_c + \nabla_c\xi_b) = 0 \longrightarrow \nabla_a\nabla_b\xi_c + \nabla_c\nabla_a\xi_b + [\nabla_a, \nabla_c]\xi_b = 0$$

$$\nabla_a\nabla_b\xi_c + \nabla_c\nabla_a\xi_b + R_{bdac}\xi^d = 0 \quad (1)$$

$a \mapsto c$, $b \mapsto a$, $c \mapsto b$

$$\nabla_c\nabla_a\xi_b + \nabla_b\nabla_c\xi_a + R_{adcb}\xi^d = 0 \quad (2)$$

$a \mapsto c$, $b \mapsto a$, $c \mapsto b$

$$\nabla_b\nabla_c\xi_a + \nabla_a\nabla_b\xi_c + R_{cdba}\xi^d = 0 \quad (3)$$

(1)-(2)+(3) =

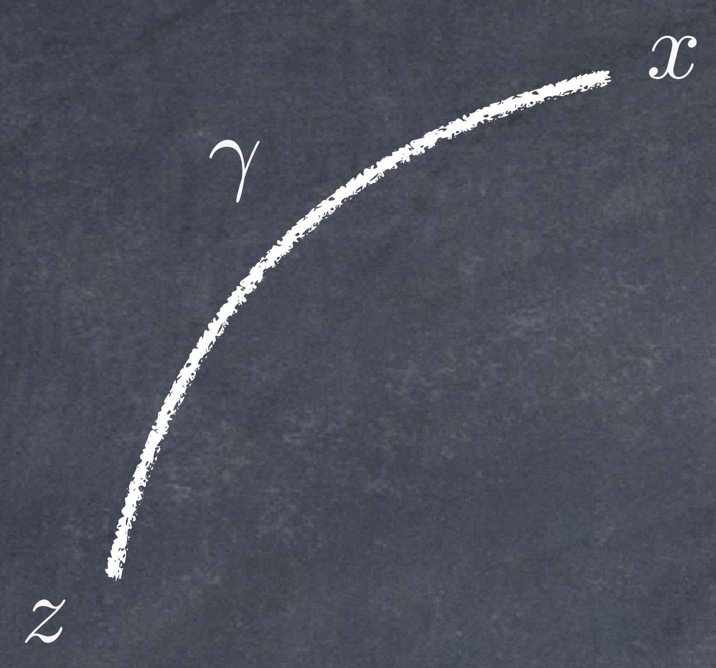
$$2\nabla_a\nabla_b\xi_c + (R_{bdac} - R_{adcb} + R_{cdba})\xi^d = 0$$

$$2\nabla_a\nabla_b\xi_c + (\underline{R_{acbd}} - R_{cbad} + \underline{R_{bacd}})\xi^d = 0$$

$$R_{cbad} + \underline{R_{bacd}} + \underline{R_{acbd}} = 0 \longrightarrow 2\nabla_a\nabla_b\xi_c + 2(-R_{cbad})\xi^d = 0$$

Tool 1: Killing transport along a curve

[Geroch, 1969]



Given $\begin{matrix} \xi_a(z) \\ L_{ab}(z) \end{matrix}$ we can build $\begin{matrix} \xi_a(x) \\ L_{ab}(x) \end{matrix}$ at a point x

by integrating along the curve the following ordinary differential equations,

$$v^a(x')\nabla_a \xi_b(x') = v^a(x') L_{ab}(z)$$
$$v^a(x')\nabla_a L_{bc}(x') = -R_{bcad}(x')\xi^d(z) v^a(x')$$

Tangent vector along the curve

$$v^a(x') = \frac{dx'^a}{d\lambda}$$

The equation is trivially obeyed at z. By construction, $L_{ab}(x') = L_{[ab]}(x')$ if $L_{(ab)}(x_0) = 0$ at any x_0

This defines $\begin{matrix} \xi_a(z) \\ L_{ab}(z) \end{matrix}$ along the curve. There is also a relationship among these quantities along the curve,

which are compatible with the Killing equation.

Tool 2: Bitensors

[Poisson, Pound, Vega, 1102.0529]
Note: They use $\sigma(z,x)$ instead of $\sigma(x,z)$.

Although there is no vector which preserves geodesic distances between all pairs of points in a general spacetime, there are vectors which preserve geodesic distances from a given worldline.

$\sigma(z,x')$ Synge function = 1/2 square of geodesic distance between z and x'

$$\sigma(z,x') = \frac{1}{2}(\lambda_{x'} - \lambda_z)\int_{\lambda_z}^{\lambda_{x'}} g_{\mu\nu}(y)\frac{dy^\mu}{d\lambda}\frac{dy^\nu}{d\lambda}d\lambda = \frac{1}{2}\epsilon(\lambda_{x'} - \lambda_z)^2$$

Constant: ϵ

Definition invariant under reparametrizations $\lambda \mapsto a\lambda + b$

It is a bitensor: scalar with respect to both z and x.

We can define vectors/tensors with respect to each point:

$$\sigma_a = \nabla_a = \partial_a\sigma(z,x'), \qquad \sigma_{ab} = \nabla_b\nabla_a\sigma(z,x')$$
$$\sigma_{a'} = \nabla_{a'}\sigma = \partial_{a'}\sigma(z,x'), \qquad \sigma_{aa'} = \nabla_{a'}\nabla_a\sigma(z,x')$$

In the coincident limit, $z \mapsto x'$

$$[\sigma] \equiv \lim_{z \mapsto x'} \sigma(z, x') = 0$$

Geodesic distance between a point and itself is 0

$$[\sigma_a] \equiv \lim_{z \mapsto x'} \sigma_a(z, x') = 0$$

because no odd tensor exists.

$$[\sigma_{a'}] = 0$$

$$[\sigma_{a'ab}] = 0$$

...

$$[\sigma_{ab}] = g_{ab}$$

$$[\sigma_{a'b'}] = g_{a'b'}$$

$$[\sigma_{a'b}] = -g_{a'b}$$

We need to prove it

Exercise

Use

$$\sigma(z,x') = \frac{1}{2}(\lambda_{x'} - \lambda_z)\int_{\lambda_z}^{\lambda_{x'}} g_{\mu\nu}(y)\frac{dy^\mu}{d\lambda}\frac{dy^\nu}{d\lambda}d\lambda = \frac{1}{2}\epsilon(\lambda_{x'} - \lambda_z)^2$$

to prove that

(i) $$\sigma(z+\delta z, x') - \sigma(z,x') = -(\lambda_{x'} - \lambda_z) g_{\mu\nu}\dot{z}^\mu \delta z^\nu$$
$$\sigma(z, x'+\delta x') - \sigma(z,x') = +(\lambda_{x'} - \lambda_z) g_{\mu'\nu'}\dot{x}'^{\mu'} \delta x'^{\nu'}$$

(ii) $$g^{ab}\sigma_a\sigma_b = 2\sigma$$

(iii) $$\sigma^a(\sigma_{ab} - g_{ab}) = 0$$

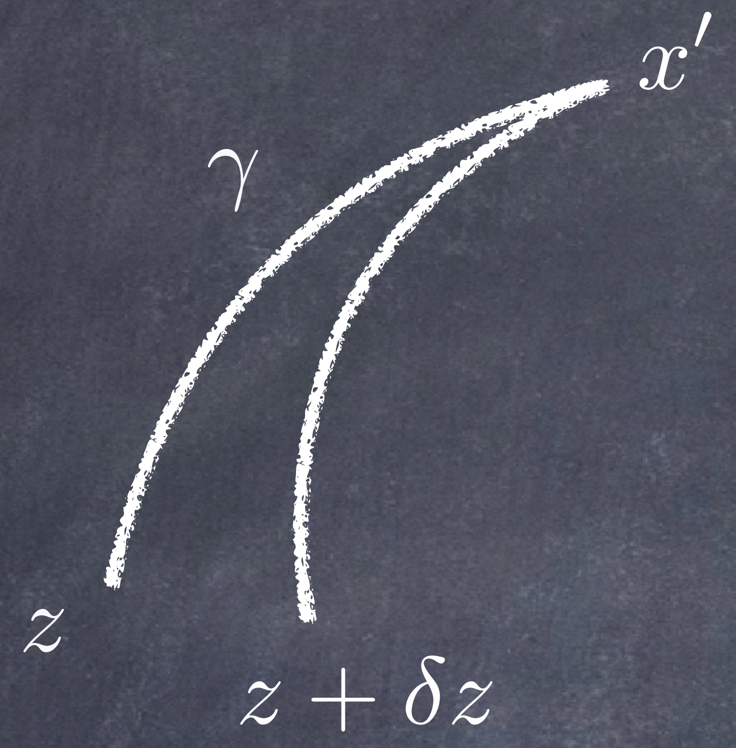
(iv) $$[\sigma_{ab}] = g_{ab}$$

(v) $$[\sigma_{a'b}] = -g_{a'b}$$

Solution

[Poisson, Pound, Vega, 1102.0529, pg 35-37]

(i)



$$\sigma(z,x') = \frac{1}{2}(\lambda_{x'}-\lambda_z)\int_{\lambda_z}^{\lambda_{x'}} g_{\mu\nu}(y)\frac{dy^\mu}{d\lambda}\frac{dy^\nu}{d\lambda}d\lambda = \frac{1}{2}\epsilon(\lambda_{x'}-\lambda_z)^2$$

Adjust parameter such that $\Delta\lambda \equiv \lambda_{x'}-\lambda_z = \lambda_{x'}-\lambda_{z+\delta z}$

$$\delta\sigma(z,x') = \Delta\lambda \int_{\lambda_z}^{\lambda_{x'}} \left(g_{\mu\nu}(y)\frac{dy^\mu}{d\lambda}\frac{d\delta y^\nu}{d\lambda} + \frac{1}{2}\partial_\lambda g_{\mu\nu}\frac{dy^\mu}{d\lambda}\frac{dy^\nu}{d\lambda}\delta y^\lambda \right) d\lambda$$

$$\delta\sigma(z,x') = \Delta\lambda\left[g_{\mu\nu}\frac{dy^\mu}{d\lambda}\delta y^\nu\right]_z^{x'} - \Delta\lambda\int_{\lambda_z}^{\lambda_{x'}}(g_{\mu\nu}\frac{d^2y^\mu}{d\lambda^2} + \Gamma^\sigma_{\mu\nu}g_{\sigma\lambda}\frac{dy^\mu}{d\lambda}\frac{dy^\lambda}{d\lambda})d\delta y^\nu d\lambda$$

$$\sigma(z+\delta z,x') - \sigma(z,x') = -(\lambda_{x'}-\lambda_z)g_{\mu\nu}\dot{z}^\mu\delta z^\nu$$

$$\sigma(z,x'+\delta x') - \sigma(z,x') = +(\lambda_{x'}-\lambda_z)g_{\mu'\nu'}\dot{x}'^{\mu'}\delta x'^{\nu'}$$

(ii)

$$\sigma_a(z,x') = -(\lambda_{x'}-\lambda_z)g_{ab}\dot{z}^b$$

$$g^{ab}\sigma_a\sigma_b = \Delta\lambda^2\dot{z}^a g_{ab}\dot{z}^b = \Delta\lambda^2\epsilon = 2\sigma$$

(iii) $g^{ab}\sigma_a\sigma_b = 2\sigma \rightarrow 2g^{ab}\sigma_{ac}\sigma_b = 2\sigma_c \rightarrow \sigma^a\sigma_{ac} = \sigma_c \rightarrow \sigma^a(\sigma_{ac} - g_{ac}) = 0$

(iv) $\sigma_a(z,x') = -(\lambda_{x'} - \lambda_z)g_{ab}\dot{z}^b \rightarrow \dot{z}^a(\sigma_{ac} - g_{ac}) = 0 \rightarrow [\sigma_{ab}] = g_{ab}$

(v) $g^{ab}\sigma_a\sigma_b = 2\sigma \rightarrow g^{ab}\sigma_{ac'}\sigma_b = \sigma_{c'} \rightarrow \sigma^a\sigma_{ac'} = \sigma_{c'}$

$\sigma_{a'}(z,x') = +(\lambda_{x'} - \lambda_z)g_{a'b'}\dot{x}'^{b'} \rightarrow -\dot{z}^a\sigma_{ac'} = +\dot{x}_{c'}$

$\rightarrow [\sigma_{a'b}] = -g_{a'b}$

Generalized Killing vector

[Harte, 2008]

We define a foliation $\{B_s\}_{s\in\mathbb{R}}$ around the curve γ

Given a pair $\begin{matrix}\xi_a(z)\\ L_{ab}(z)\end{matrix}$ along the curve γ we define ξ^a in the vicinity of γ as

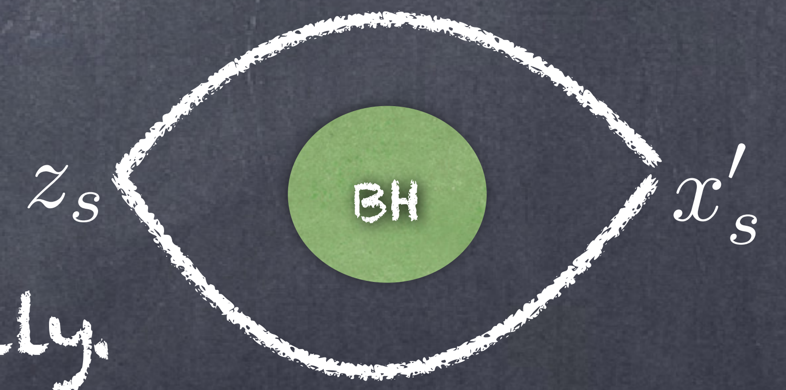
$$\mathcal{L}_\xi \sigma(z_s, x'_s) = 0 \qquad \forall z_s = \gamma \cap B_s \qquad \forall x'_s \in B_s$$

This vector exists. Proof: $\mathcal{L}_\xi \sigma(z_s, x'_s) = 0 \longrightarrow \mathcal{L}_\xi \sigma^a(z_s, x'_s) = 0$

$$\xi^{a'}(x'_s)\nabla_{a'}\sigma^a + \xi^b(z_s)\nabla_b\sigma^a - \nabla_b\xi^a(z_s)\sigma^b = 0$$

$$\xi^{a'}(x'_s)\nabla_{a'}\sigma^a = -\xi^b(z_s)\nabla_b\sigma^a + \nabla_b\xi^a(z_s)\sigma^b$$

Define the inverse $H^{b'}{}_a(x'_s, z_s)$ such that $H^{b'}{}_a(x'_s, z_s)\nabla_{a'}\sigma^a(z_s, x'_s) = -\delta^{b'}_{a'}$



This inverse exists and is unique locally around γ . Due to caustics, it is not unique globally.

$$\xi^{b'}(x'_s) = H^{b'}{}_a(x'_s, z_s)\left(\xi^b(z_s)\nabla_b\sigma^a - \nabla_b\xi^a(z_s)\sigma^b\right)$$

$$\xi^{a'}(x') = \Xi^{a'a}(x', z)\xi_a(z) + \Xi^{a',ab}(x', z)\nabla_a\xi_b(z)$$

$$\Xi^{a'a}(x', z) = H^{a'}_b(x', z)\sigma^{ba}(z, x')$$

$$\Xi^{a',ab}(x', z) = -H^{a'b}(x', z)\sigma^a(z, x')$$

We can write

$$\xi^{a'}(x') = \Xi^{a'a}(x',z)\xi_a(z) + \Xi^{a',ab}(x',z)\nabla_a\xi_b(z)$$

$$H^{b'}{}_a(x'_s,z_s)\nabla_{a'}\sigma^a(z_s,x'_s) = -\delta^{b'}_{a'}$$

$$\Xi^{a'a}(x',z) = H^{a'}{}_b(x',z)\sigma^{ba}(z,x')$$

$$\Xi^{a',ab}(x',z) = -H^{a'b}(x',z)\sigma^a(z,x')$$

equivalently by

$$\xi^{a'}(x') = \Xi^{a'a}(x',z)\xi_a(z) + \Xi^{a',ab}(x',z)L_{ab}(z)$$

Indeed, in the coincident limit, $z \mapsto x'$

$$[\sigma^a{}_{a'}] = -\delta^a_{a'} \quad [H^{b'}{}_a] = \delta^{b'}_a \quad [\sigma^{ba}] = g^{ba} \quad [\sigma^a] = 0 \quad \longrightarrow \quad [\Xi^{a'a}] = g^{a'a} \quad [\Xi^{a',ab}] = 0 \quad \longrightarrow \quad \xi^{a'}(x') = \xi^{a'}(x')$$

$$\nabla_{b'}\xi^{a'}(x') = \nabla_{b'}\Xi^{a'a}(x',z)\xi_a(z) + \nabla_{b'}\Xi^{a',ab}(x',z)L_{ab}(z)$$

$$[\nabla_{b'}\Xi^{a'a}] = 0 \quad [\sigma^a] = 0 \quad [\nabla_{b'}\Xi^{a',ab}] = -[H^{a'b}][\sigma^a{}_{b'}] = \delta^{a'b}\delta^a_{b'} \quad \longrightarrow \quad \nabla_{a'}\xi_{b'}(x') = L_{a'b'}(x')$$

Finally, using the property of Killing transport $L_{ab}(z) = L_{[ab]}(z)$ after choosing $\nabla_{(a}\xi_{b)}(z_0) = 0$ for any $z_0 \in \gamma$

$$\xi^{a'}(x') = \Xi^{a'a}(x',z)\xi_a(z) + \Xi^{a',[ab]}(x',z)L_{ab}(z)$$

[Harte, private communication, 2020]

Exercise

Check that the Killing vectors of Minkowski spacetime obey

$$\xi^{a'}(x') = \Xi^{a'a}(x', z)\xi_a(z) + \Xi^{a',[ab]}(x', z)\nabla_a \xi_b(z)$$

with

$$\Xi^{a',a}(x', z) = \delta^{a',a}$$
$$\Xi^{a',ab}(x', z) = (x' - z)^{[a}\delta^{b]a'}$$

Solution

Check that the Killing vectors of Minkowski spacetime obey

$$\xi^{a'}(x') = \Xi^{a'a}(x', z)\xi_a(z) + \Xi^{a',[ab]}(x', z)\nabla_a \xi_b(z)$$

with

$$\Xi^{a',a}(x', z) = \delta^{a',a}$$
$$\Xi^{a',ab}(x', z) = (x' - z)^{[a}\delta^{b]a'}$$

Explicitly,

$$\xi^a(z) = A^a + B^{[ab]}z^b$$

$$\xi^{a'}(x') = A^a + B^{[a'b']}x'^{b'} = A^a + B^{[a'b']}z^{b'} + (x'^{b'} - z^{b'})B^{a'b'}$$

Therefore,

$$\xi^{a'}(x') = \xi^{a'}(z) + (x' - z)^a \nabla_a \xi^{a'}(z)$$

Generalized Killing charges

$$P_t[\xi] = \int_{B_t} T^{a'b'}_{body}(x')\xi_{a'}(x')dS_{b'}$$

Using the decomposition

$$\xi^{a'}(x') = \Xi^{a'a}(x',z)\xi_a(z) + \Xi^{a',[ab]}(x',z)\nabla_a\xi_b(z)$$

we obtain

$$P_t[\xi] = p^a(z_t,t)\xi_a(z_t) + \frac{1}{2}S^{ab}(z_t,t)\nabla_{[a}\xi_{b]}(z_t)$$

where

$$p^a(z_t,t) = \int_{B_t} T^{a'b'}_{body}(x')\Xi_{a'}{}^{a}(x',z_t)dS_{b'}$$

is the momentum vector with respect to the reference worldline at foliation time t

$$S^{ab}(z_t,t) = \int_{B_t} T^{a'b'}_{body}(x')\Xi_{a'}{}^{ab}(x',z_t)dS_{b'}$$

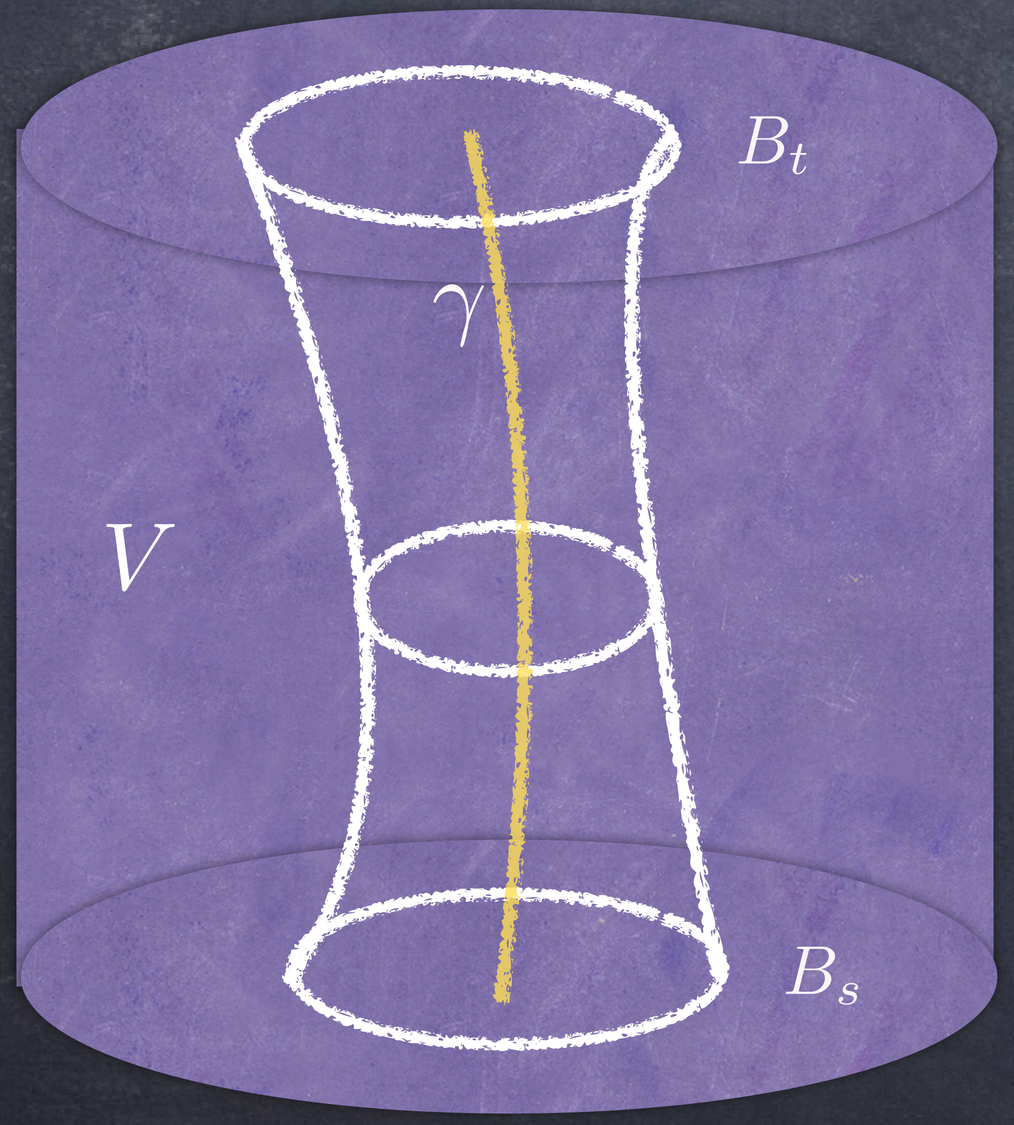
is the Lorentz charge tensor with respect to the reference worldline at foliation time t

Flux-balance law

The current $J^a \equiv T^{ab}_{body}(x)\xi_b(x)$ obeys $\nabla_a J^a = T^{ab}_{body}(x)\nabla_a\xi_b(x) = \frac{1}{2}T^{ab}_{body}\mathcal{L}_\xi g_{ab}$

Stokes' theorem implies

$$\int_V \sqrt{-g}\nabla_a J^a = \int_V \partial_a(\sqrt{-g}J^a) = \int_{\partial V} \sqrt{-g}J^a dS_a$$



$$\int_s^t dt' \int_{B_{t'}} d^3x\sqrt{-g}T^{ab}_{body}\mathcal{L}_\xi g_{ab} = P_t[\xi] - P_s[\xi]$$

Infinitesimally,

$$\frac{d}{dt}P_t[\xi] = \int_{B_t} d^3x\sqrt{-g}T^{ab}_{body}\mathcal{L}_\xi g_{ab} \equiv F_t[\xi]$$

These are the Mathisson-Papapetrou-Dixon equations

Mathisson-Papapetrou-Dixon equations

$$\frac{d}{dt}P_t[\xi] = \int_{B_t} d^3x\sqrt{-g}T^{ab}\mathcal{L}_\xi g_{ab} \equiv F_t[\xi] \qquad P_t[\xi] = p^a(z_t,t)\xi_a(z_t) + \frac{1}{2}S^{ab}(z_t,t)\nabla_{[a}\xi_{b]}(z_t)$$

Using the decomposition:

$$\frac{Dp^a}{Dt}\xi_a(z_t) + p^{[a}\dot{z}_t^{b]}\nabla_b\xi_a(z_t) + \frac{1}{2}\frac{DS^{ab}}{Dt}\nabla_a\xi_b(z_t) + \frac{1}{2}S^{ab}\dot{z}_t^c\nabla_c\nabla_a\xi_b(z_t) = F^a\xi_a(z_t) + \frac{1}{2}N_{ab}\nabla_{[a}\xi_{b]}(z_t)$$

use

$$\nabla_c\nabla_a\xi_b = -R_{bacd}\xi^d$$

True independently for $\xi_a(z_t),\ \nabla_{[a}\xi_{b]}(z_t)$

We obtain

$$\frac{Dp^a}{Dt} = \frac{1}{2}S^{bc}R_{bcd}{}^a\dot{z}_t^d + F^a$$
$$\frac{DS^{ab}}{Dt} = 2p^{[a}\dot{z}_t^{b]} + N^{ab}$$

Final variables :

$$p^a,\ S^{ab},\ \dot{z}_t^a \equiv v^a,\ J^{abcd},\ldots$$

The MPD equations arise because a generalized Killing vector which is purely translational at a point becomes a combination of translations and Lorentz transformations at a later time under Killing transport. This leads to a mixing of momenta and Lorentz charges.

If ξ^a is a background exact Killing vector, $\frac{d}{dt}P_t[\xi] = \int_{B_t} d^3x\sqrt{-g}T^{ab}\mathcal{L}_\xi g_{ab} \equiv F_t[\xi]$

Then $P_t[\xi] = p^a(z_t,t)\xi_a(z_t) + \frac{1}{2}S^{ab}(z_t,t)\nabla_{[a}\xi_{b]}(z_t)$ is exactly conserved. In Kerr, there are 2 conserved quantities, as for geodesics. Analogue of Carter's constant??

In order to close the system of equations

$$\frac{Dp^a}{Dt} = \frac{1}{2}S^{bc}R_{bcd}{}^a\dot{z}_t^d + F^a$$
$$\frac{DS^{ab}}{Dt} = 2p^{[a}\dot{z}_t^{b]} + N^{ab}$$

we need equations of state to define the stresses and torques, as well as the "spin supplementary conditions" that fix the center-of-mass:

Tulczyjew: $S^{ab}p_b = 0$ (Other option: Mathisson: $S^{ab}v_b = 0$)

Proposition: If t is proper time, we can solve for $\dot{z}_t^a$ in terms of p^a, S^{ab}, F^a, N^{ab} [Steinhoff, Puetzfeld, PRD 86, 044033 (2012)]

We can then define the intrinsic angular momentum: $S_a = \frac{1}{2\mu}\epsilon_{abcd}p^bS^{cd}$, $\mu^2 \equiv -p^ap_a$

As well as the spin length: $S^2 = \frac{1}{2}S_{ab}S^{ab} = S^aS_a$

Exercise

Show that the Mathisson-Papapetrou equations reduced with the Tulczyjew condition where t is proper time

$$\frac{Dp^a}{Dt} = \frac{1}{2} S^{bc} R_{bcd}{}^a v^d$$
$$\frac{DS^{ab}}{Dt} = 2p^{[a} v^{b]}$$

$$S^{ab} p_b = 0$$

lead to

$$v^a = N \left[\frac{p^a}{\mu} + \frac{1}{2\mu^2 \Delta} S^{ab} \frac{p^c}{\mu} R_{bcde} S^{de} \right]$$

where

$$\Delta = 1 + \frac{1}{4\mu^2} R_{abcd} S^{ab} S^{cd}$$

$$N^{-2} = 1 - \frac{1}{4\Delta^2 \mu^4} S_{ab} p_c S_{de} R^{bcde} S^{af} p^g S^{hi} R_{fghi}$$

Hint: prove $R_{abcd} S^{ae} S^{bf} = \frac{1}{2} R_{abcd} S^{ab} S^{ef}$

First derived by [Ehlers, Rudolph, GRG 8, 197 (1977)]

Solution

$$\frac{Dp^a}{Dt} = \frac{1}{2} S^{bc} R_{bcd}{}^a v^d$$
$$\frac{DS^{ab}}{Dt} = 2p^{[a} v^{b]}$$

$$\frac{D}{Dt}(S^{ab} p_b) = 0 \longrightarrow v^a = -\frac{p_b v^b}{\mu^2} p^a - \frac{1}{2\mu^2} S^{ab} S^{cd} R_{cdfb} v^f$$

$$v^a \equiv \frac{N}{\mu} p^a + B^a{}_f v^f, \qquad B^a{}_f \equiv -\frac{1}{2\mu^2} S^{ab} S^{cd} R_{cdfb}$$

$$\longrightarrow v^a = \frac{N}{\mu}\left(p^a + B^a{}_f p^f\right) + B^a{}_f B^f{}_g v^g$$

We will solve for N later.

$$S^{ab} p_b = 0 \longrightarrow S^{ab} = \epsilon^{abcd} p_c S_d, \qquad p^a S_a = 0 \longrightarrow \epsilon_{abce} S^{ab} S^{cd} \sim (p_c S_e - p_e S_c) S^{cd} = 0$$

$$\longrightarrow S^{[ab} S^{c]d} = 0 \xrightarrow{R_{(abc)d} = 0} R_{abcd} S^{ae} S^{bf} = \frac{1}{2} R_{abcd} S^{ab} S^{ef} \longrightarrow B^a{}_f B^f{}_g v^g = \frac{B^b{}_b}{2} B^a{}_f v^f$$

$$\longrightarrow v^a = \frac{N}{\mu}\left(p^a + B^a{}_f p^f \sum_{n=0}^{\infty} \frac{(B^b{}_b)^n}{2^n}\right) = \frac{N}{\mu}\left(p^a + \frac{B^a{}_f p^f}{1 - B^b{}_b/2}\right)$$

$$v^a v_a = -1 \longrightarrow \text{Deduce N}$$

Note: instead of proper time, one could fix $v^a p_a = -\mu$ which results in another N [Dixon, 1970]

see e.g. [Susuki, Maeda, PRD 55(8), 4848 (1997)][Santos, Batista, PRD 101, 104049 (2020)]

Exercise

Using the Mathisson-Papapetrou equations reduced with the Tulczyjew condition

$$\frac{Dp^a}{Dt} = \frac{1}{2} S^{bc} R_{bcd}{}^{a} v^d$$
$$\frac{DS^{ab}}{Dt} = 2p^{[a} v^{b]}$$

$$S^{ab} p_b = 0$$

$$v^a = N \left[\frac{p^a}{\mu} + \frac{1}{2\mu^2 \Delta} S^{ab} \frac{p^c}{\mu} R_{bcde} S^{de} \right]$$

prove that $\mu^2 \equiv -p_a p^a$ as well as $S^2 \equiv \frac{1}{2} S_{ab} S^{ab}$ are conserved.

Solution

Using the Mathisson-Papapetrou equations reduced with the Tulczyjew condition

$$\frac{Dp^a}{Dt} = \frac{1}{2} S^{bc} R_{bcd}{}^a v^d$$
$$\frac{DS^{ab}}{Dt} = 2p^{[a} v^{b]}$$

$$S^{ab} p_b = 0$$

$$v^a = N \left[\frac{p^a}{\mu} + \frac{1}{2\mu^2 \Delta} S^{ab} \frac{p^c}{\mu} R_{bcde} S^{de} \right]$$

prove that $\mu^2 \equiv -p_a p^a$ as well as $S^2 \equiv \frac{1}{2} S_{ab} S^{ab}$ are conserved.

We have $S_{ab} \frac{DS^{ab}}{dt} = 0$ because $R_{abcd} p^c p^d = 0$

$p_a \frac{Dp^a}{dt} = 0$ because $X_a S^{ab} X_b = 0$ $\quad X_a \equiv R_{abcd} p^b S^{cd}$

Stress-energy tensor

"Skeletization": reduce the stress-tensor to its lowest multipole moments on the worldline: [Mathisson, 1937][Schwartz, Théorie des distributions, 1950] [Tulczyjew, 1959] [Dixon, 1973]

$$T^{ab}(x) = \int_{-\infty}^{\infty} d\tau \left[p^{(a}v^{b)} \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}} - \nabla_c \left(S^{c(a}v^{b)} \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}} \right) \right]$$

$$v^a \equiv \frac{dx_*^a(\tau)}{d\tau}$$

Exercise

Prove that $\nabla_b T^{ab} = 0$ is equivalent to the Mathisson-Papapetrou equations

$$\frac{Dp^a}{Dt} = \frac{1}{2} S^{bc} R_{bcd}{}^a v^d$$

$$\frac{DS^{ab}}{Dt} = 2p^{[a}v^{b]}$$

Trick: use an arbitrary test vector ϕ_a

$$0 = \int d^4x \sqrt{-g} \nabla_a (T^{ab}\phi_b) = \int d^4x \sqrt{-g} \nabla_a T^{ab} \phi_b + \int d^4x \sqrt{-g} T^{ab} \nabla_{(a}\phi_{b)}$$

Trick: use an arbitrary test vector ϕ_a

$$0 = \int d^4x\sqrt{-g}\nabla_a(T^{ab}\phi_b) = \int d^4x\sqrt{-g}\nabla_a T^{ab}\phi_b + \int d^4x\sqrt{-g}T^{ab}\nabla_{(a}\phi_{b)}$$

Therefore, conservation of the stress-tensor is equivalent to

$$0 = \int d^4x\sqrt{-g}T^{ab}\nabla_{(a}\phi_{b)} = \int d\tau\left(p^{(a}v^{b)}\nabla_a\phi_b + S^{c(a}v^{b)}\nabla_c\nabla_a\phi_b\right)$$

Use $\quad p^{(a}v^{b)} = p^{[a}v^{b]} + v^a p^b, \quad S^{ca}\nabla_c\nabla_a\phi_b = \frac{1}{2}S^{ca}R_{cab}{}^{d}\phi_d \qquad v^a\nabla_a = \frac{D}{D\tau}$ and integrate by parts to get

$$\int d^4x\sqrt{-g}T^{ab}\nabla_{(a}\phi_{b)} = -\frac{1}{2}\int d\tau\left[2(\frac{Dp^a}{D\tau} - \frac{1}{2}S^{bc}v^d R_{bcd}{}^{a})\phi_a + (\frac{DS^{ab}}{D\tau} - 2p^{[a}v^{b]})\nabla_{[a}\phi_{b]}\right]$$

The two test $\phi_a, \quad \nabla_{[a}\phi_{b]}$ are independent on the worldine:

$$\tilde{\phi}_a = \phi_a + \nabla_a\chi, \qquad \nabla_{[a}\phi_{b]} = \nabla_{[a}\tilde{\phi}_{b]}$$

Fermi normal coordinates

$$\chi = \alpha(t) + \beta_i(t)x^i$$

This is therefore equivalent to

$$\frac{Dp^a}{Dt} = \frac{1}{2}S^{bc}R_{bcd}{}^{a}v^d$$

$$\frac{DS^{ab}}{Dt} = 2p^{[a}v^{b]}$$

[Special thanks to Harte, private communication]

Remarks

- Quadrupole and higher multipoles are not constrained by stress-tensor conservation, but by the internal dynamics [Dixon,1980]
- Dynamics of a spinning particle in Schwarzschild or Kerr is chaotic [Susuki,Maeda,1997][Hartl,2003]