Advanced Quantum Field Theory : Renormalization, Non-Abelian Gauge Theories and Anomalies

Lecture notes
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Abstract

These are lecture notes of an advanced quantum field theory course intended for graduate students in theoretical high energy physics who are already familiar with the basics of QFT. The first part quickly reviews what should be more or less known: functional integral methods and one-loop computations in QED and ϕ^4. The second part deals in some detail with the renormalization program and the renormalization group. The third part treats the quantization of non-abelian gauge theories and their renormalization with special emphasis on the BRST symmetry. The fourth part of the lectures, not contained in the present notes but based on `arXiv:0802.0634`, discusses gauge and gravitational anomalies, how to characterise them in various dimensions, as well as anomaly cancellations.

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PART I :

A QUICK REVIEW OF WHAT SHOULD BE KNOWN

1 Functional integral methods

1.1 Path integral in quantum mechanics

The usual description of quantum mechanics is in the Schrödinger picture where

$$[Q_a, P_b] = i\,\delta_{ab} \quad , \quad Q_a\,|q\rangle = q_a\,|q\rangle \quad , \quad P_a\,|p\rangle = p_a\,|p\rangle \quad , \quad \langle q\,|p\rangle = \prod_a \frac{e^{iq_a p_a}}{\sqrt{2\pi}} \ . \tag{1.1}$$

Go to the Heisenberg picture by $Q_a(t) = e^{iHt} Q_a e^{-iHt}$ and $P_a(t) = e^{iHt} P_a e^{-iHt}$. The eigenstates of these Heisenberg picture operators are

$$\begin{aligned} |q,t\rangle &= e^{iHt}\,|q\rangle \quad , \quad Q_a(t)\,|q,t\rangle = q_a\,|q,t\rangle \ , \\ |p,t\rangle &= e^{iHt}\,|p\rangle \quad , \quad P_a(t)\,|p,t\rangle = q_a\,|p,t\rangle \ . \end{aligned} \tag{1.2}$$

Note that these are *not* the Schrödinger states $|q\rangle$ or $|p\rangle$ evolved in time (which would be $e^{-iHt}\,|q\rangle$, resp. $e^{-iHt}\,|p\rangle$). It follows that $|q, t+\Delta t\rangle = e^{iH\Delta t}\,|q,t\rangle$ and $\langle q', t+\Delta t| = \langle q', t|\, e^{-iH\Delta t}$. Hence

$$\langle q', t+\Delta t\,|q,t\rangle = \langle q',t|\, e^{-iH\Delta t}\,|q,t\rangle = \langle q',t|\left(1 - iH\Delta t + \mathcal{O}(\Delta t^2)\right)|q,t\rangle \tag{1.3}$$

Now $H = H(P,Q) = e^{iHt} H(Q,P) e^{-iHt} = H(Q(t),P(t))$ and we assume that H is written with all P's to the right of all q's (by using $PQ = QP - i$ if necessary). Then one has

$$\langle q,t|\, H(Q(t),P(t))\,|p,t\rangle = H(q(t),p(t)\langle q,t\,|p,t\rangle \ , \tag{1.4}$$

so that

$$\begin{aligned} \langle q', t+\Delta t\,|q,t\rangle &= \int \left(\prod_a \mathrm{d}\,p_a\right) \langle q',t|\left(1 - iH(Q(t),P(t))\Delta t + \mathcal{O}(\Delta t^2)\right)|p,t\rangle\,\langle p,t\,|q,t\rangle \\ &= \int \left(\prod_a \mathrm{d}\,p_a\right) \langle q',t|\left(1 - iH(q'(t),p(t))\Delta t + \mathcal{O}(\Delta t^2)\right)|p,t\rangle\,\langle p,t\,|q,t\rangle \\ &= \int \left(\prod_a \mathrm{d}\,p_a\right) e^{-iH(q'(t),p(t))\Delta t + \mathcal{O}(\Delta t^2)} \prod_b \frac{e^{ip_a(q'_a - q_a)}}{2\pi} \ . \end{aligned} \tag{1.5}$$

Now one can take a finite interval $t'-t$ and let $\Delta t = \frac{t'-t}{N}$. We write $t_k = t + k\Delta t$ with $k = 0, \ldots N$ and $t_0 = t,\ t_N = t'$ as well as $q_N = q',\ q_0 = q$. Then

$$\begin{aligned} \langle q', t'\,|q,t\rangle &= \int \prod \mathrm{d}q_1^a \ldots \mathrm{d}q_{N-1}^a \langle q_N, t_N\,|q_{N-1}, t_{N-1}\rangle\,\langle q_{N-1}, t_{N-1}| \ldots |q_1,t_1\rangle\,\langle q_1,t_1\,|q_0,t_0\rangle \\ &= \int \prod \mathrm{d}q_1^a \ldots \mathrm{d}q_{N-1}^a \prod \frac{\mathrm{d}p_1^a}{2\pi} \ldots \frac{\mathrm{d}p_N^a}{2\pi} \exp\left\{ -i \sum_{k=1}^N H(q_k,p_k)\Delta t + i \sum_{k=1}^N p_k(q_k - q_{k-1}) \right\} \ . \end{aligned} \tag{1.6}$$

Then, for any "configuration" $\{q_0, q_1, \ldots q_N\}$ define an 'interpolating' $q(\tau)$, so that $q_{k+1} - q_k \simeq \dot{q}(\tau)\Delta\tau$. Also $\prod_{k,a} \mathrm{d}q_k^a \simeq \prod_a \mathcal{D}q_a$ and $\prod_{k,a} \frac{\mathrm{d}p_k^a}{2\pi} \simeq \prod_a \mathcal{D}p_a$, so that finally

$$\langle q', t' \, | q, t \rangle = \int_{q_a(t)=q_a,\ q_a(t')=q'_a} \prod_a \mathcal{D}q_a \prod_b \mathcal{D}p_b \exp\left\{ -i \int_t^{t'} \mathrm{d}\tau \, H(q(\tau), p(\tau)) + i \int_t^{t'} \mathrm{d}\tau \, p(\tau)\dot{q}(\tau) \right\} . \tag{1.7}$$

This can be easily generalized to yield not only transition amplitudes but also matrix elements of products of operators. Going through the same steps again for $\langle q', t' | \, \mathcal{O}_A(Q(t_A), P(t_A))\mathcal{O}_B(Q(t_B), P(t_B)) \ldots | q, t\rangle$ with $t_A \geq t_B \geq \ldots$, one easily sees that the path integral just gets $\mathcal{O}_A(q(t_A), p(t_A))\mathcal{O}_B(q(t_B), p(t_B)) \ldots$ inserted. Thus

$$\begin{aligned} &\langle q', t' | \, T \left\{ \mathcal{O}_A(Q(t_A), P(t_A))\mathcal{O}_B(Q(t_B), P(t_B)) \ldots \right\} | q, t\rangle \\ &\qquad = \int_{q_a(t)=q_a,\ q_a(t')=q'_a} \prod_a \mathcal{D}q_a \prod_b \mathcal{D}p_b \; \mathcal{O}_A(q(t_A), p(t_A)) \; \mathcal{O}_B(q(t_B), p(t_B)) \; \ldots \; \times \\ &\qquad\qquad \times \exp\left\{ -i \int_t^{t'} \mathrm{d}\tau \, H(q(\tau), p(\tau)) + i \int_t^{t'} \mathrm{d}\tau \, p(\tau)\dot{q}(\tau) \right\} . \end{aligned} \tag{1.8}$$

1.2 Functional integral in quantum field theory

An advantage of the canonical formalism is that unitarity is manifest, but Lorentz invariance is somewhat obscured (although guaranteed by general theorems). In the functional integral formalism with covariant Lagrangians to be discussed next, Lorentz invariance is manifest, but unitarity is not guaranteed, unless the formalism can be derived from the canonical one (and then extra terms might be present).

1.2.1 Derivation of the Hamiltonian functional integral

The path integral formula for matrix elements in quantum mechanics immediately generalizes – at least formally – to quantum field theory by the obvious generalizations of the labels a to include the position in space:

$$a \to (n, \vec{x}) \quad , \qquad \sum_a \to \sum_n \int \mathrm{d}^3x \quad , \qquad \text{etc.} \tag{1.9}$$

However, in field theory we do not want to compute transition amplitudes between eigenstates $|\psi(\vec{x})\rangle$ of the field operator $\Psi(\vec{x})$(the analogue of Q) but between in and out states having definite numbers of particles, or often simply between the in and out vacuum states. In order to obtain these one has to multiply the transition amplitudes obtained from generalizing (1.7) to field theory by the appropriate vacuum wave functions which for a real scalar e.g. are

$$\langle \phi(\vec{x}), \pm \, |\text{vac}, \pm \rangle = \mathcal{N} \exp\left\{ -\frac{1}{2} \int \mathrm{d}^3x \, \mathrm{d}^3y \int \frac{\mathrm{d}^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \sqrt{\vec{p}^{\,2} + m^2} \, \phi(\vec{x})\phi(\vec{y}) \right\} . \tag{1.10}$$

Note that, contrary to the exponentials appearing in the transition amplitudes or matrix elements, the exponential in (1.10) is *real*. Note also that it only contains 3-dimensional space integrals (if it were not for the $\sqrt{\vec{p}^2+m^2}$ the whole expression would collapse to a single $\int \mathrm{d}^3x$ integral), and in this sense it is infinitesimal as compared to the 4-dimensional space-time integrals in the exponents of the transition amplitudes or matrix elements. Hence we are let to expect that the effect of multiplying with (1.10) is only to add terms of the form $i\times$ (infinitesimally small) to the exponent. It can indeed be shown that they precisely provide the correct $i\epsilon$ terms that result in the correct Feynman propagator. Again, this was to be expected since this must be the role of the initial and final conditions imposed by $\langle\phi,\pm|\mathrm{vac},\pm\rangle$. Hence, one arrives at the functional integral representation for the time-ordered product of Heisenberg picture operators between the in and out vacuum states:

$$\begin{aligned}
&\langle \mathrm{vac},\ \mathrm{out}|\, T\Big\{\mathcal{O}_A\big(\Psi(t_A,\vec{x}_A),\Pi(t_A,\vec{x}_A)\big)\ \mathcal{O}_B\big(\Psi(t_B,\vec{x}_B),\Pi(t_B,\vec{x}_B)\big)\ldots\Big\}\,|\mathrm{vac},\ \mathrm{in}\rangle\\
&\quad = |\mathcal{N}|^2\int\prod_l \mathcal{D}\psi_l\prod_n\mathcal{D}\pi_n\ \ \mathcal{O}_A\big(\psi(t_A,\vec{x}_A),\pi(t_A,\vec{x}_A)\big)\ \mathcal{O}_B\big(\psi(t_B,\vec{x}_B),\pi(t_B,\vec{x}_B)\big)\ \ldots\ \times\\
&\qquad\times\exp\left\{i\int_{-\infty}^{\infty}\mathrm{d}\tau\ \left[\int\mathrm{d}^3x\sum_l\partial_0\psi_l(\tau,\vec{x})\pi_l(\tau,\vec{x})-H(\psi(\tau,\vec{x})\pi(\tau,\vec{x}))+i\epsilon\mathrm{-terms}\right]\right\},
\end{aligned} \tag{1.11}$$

where we have denoted the fields and their conjugate momenta as ψ_l and π_l while the corresponding Heisenberg picture operators are Ψ_l and Π_l. The functional measures can be thought of as being

$$\mathcal{D}\psi_l=\prod_{\tau,\vec{x}}\mathrm{d}\left(\psi_l(\tau,\vec{x})\right)\quad,\quad\mathcal{D}\pi_l=\prod_{\tau,\vec{x}}\mathrm{d}\left(\pi_l(\tau,\vec{x})\right)\ . \tag{1.12}$$

1.2.2 Derivation of the Lagrangian version of the functional integral

In many theories the Hamiltonian is a quadratic functional of the momenta π_l:

$$H(\psi(\tau,\vec{x})\pi(\tau,\vec{x}))=\frac{1}{2}\sum_{n,m}\int\mathrm{d}^3x\,\mathrm{d}^3y\ A_{n,\vec{x},m\vec{y}}(\psi)\pi_n(\tau,\vec{x})\pi_m(\tau,\vec{y})+\sum_n\int\mathrm{d}^3x\,B_{n\vec{x}}(\psi)\pi_n(\tau,\vec{y})+C(\psi)\ , \tag{1.13}$$

with a real, symmetric, positive and *non-singular* kernel $A_{n,\vec{x},m\vec{y}}(\psi)$. Then the functional integral over $\pi_n(\tau,\vec{x})$ in the vacuum to vacuum amplitude is gaussian and can be performed explicitly. More generally, if the $\mathcal{O}_A$ only depend on the fields ψ_l and not on the π_l, one can also perform the $\mathcal{D}\pi_n$ -integration in (1.11). Before giving the result it is useful to recall the following remark on gaussian integrations.

Let $f(x)$ be a quadratic form in $x^i,\ i=1,\ldots N$, i.e. $f(x)=\frac{1}{2}x^ia_{ij}x^j+b_ix^i+c$, with a real, symmetric, positive and *non-singular* matrix a. Then by straightforward computation ("completing the square")

$$\int\prod_i\mathrm{d}x^i\ e^{-f(x)}=(2\pi)^{N/2}\left(\det a\right)^{-1/2}e^{\frac{1}{2}b_i(a^{-1})^{ij}b_j-c}\ . \tag{1.14}$$

Now the exponent $\frac{1}{2}b_i(a^{-1})^{ij}b_j-c$ is just $-f(x_0)$ where x_0^i is the value which minimizes f. Indeed, $\partial f/\partial x^i=a_{ij}x^j+b_i$ and hence $x_0^i=-(a^{-1})^{ij}b_j$ and $f(x_0)=c-\frac{1}{2}b_i(a^{-1})^{ij}b_j$. This is just the statement that for a gaussian integration the saddle-point approximation is exact. Indeed, expanding $f(x)$ around its minimum we have $f(x)=f(x_0)+\frac{1}{2}(x-x_0)^ia_{ij}(x-x_0)^j$ from which follows immediately

$$\int\prod_i\mathrm{d}x^i\ e^{-f(x)}=(2\pi)^{N/2}(\det a)^{-1/2}e^{-f(x_0)}\ . \tag{1.15}$$

We now apply this remark to the quadratic form given by

$$\int_{-\infty}^{\infty} \mathrm{d}\tau \left[\int \mathrm{d}^3x \sum_l \partial_0\psi_l(\tau,\vec{x})\pi_l(\tau,\vec{x}) - H(\psi(\tau,\vec{x})\pi(\tau,\vec{x})) \right] , \tag{1.16}$$

with H given by (1.13). Note that for the second term there is a double integral $\mathrm{d}^3x\,\mathrm{d}^3y$ but only a single $\mathrm{d}\tau$ integral. We rewrite everything as full 4-dimensional integrals by adding a $\delta(\tau-\tau')$. Hence the corresponding kernel is $\mathcal{A}_{n\tau\vec{x},m\tau'\vec{y}}(\psi) = \delta(\tau-\tau')A_{n,\vec{x},m\vec{y}}(\psi)$. The saddle-point value of π_l extremizing (1.16) is the solution $\bar\pi_l$ of $\partial_0\psi_l = \frac{\delta H}{\delta\pi_l}$. But evaluating $\int \mathrm{d}^3x \sum_l \partial_0\psi_l\pi_l - H(\psi_l,\pi_l)$ at $\pi_l=\bar\pi_l$ is exactly doing the (inverse) Legendre transformation that gives back the Lagrange function:

$$\int \mathrm{d}^3x \sum_l \partial_0\psi_l\, \bar\pi_l(\psi,\partial_0\psi) - H(\psi_l,\bar\pi_l(\psi,\partial_0\psi)) = L(\psi_l,\partial_0\psi_l) \equiv \int \mathrm{d}^3x\, \mathcal{L}(\psi_l,\partial_\mu\psi_l) \ . \tag{1.17}$$

Putting everything together we find for Hamiltonians that are quadratic in the π_l:

$$\begin{aligned}
&\langle \text{vac, out}|\, T\Big\{\mathcal{O}_A\big(\Psi(t_A,\vec{x}_A)\big)\ \mathcal{O}_B\big(\Psi(t_B,\vec{x}_B)\big)\ldots\Big\}\,|\text{vac, in}\rangle \\
&\quad = |\mathcal{N}|^2 \int \prod_l \mathcal{D}\psi_l\ \big(\mathrm{Det}\,[2\pi i\mathcal{A}(\psi)]\,\big)^{-1/2}\ \mathcal{O}_A\big(\psi(t_A,\vec{x}_A)\big)\ \mathcal{O}_B\big(\psi(t_B,\vec{x}_B)\big)\ \ldots\ \times \\
&\qquad \times \exp\left\{ i\int \mathrm{d}^4x\, \mathcal{L}(\psi_l(x),\partial_\mu\psi_l(x)) + i\epsilon\text{−terms}\right\} .
\end{aligned} \tag{1.18}$$

A few remarks are in order:

- The overall constant $|\mathcal{N}|^2$ drops out when computing amplitudes that do not involve "vacuum bubbles", which is achieved by dividing by $\langle \text{vac, out}\,|\text{vac, in}\rangle$. This is the case in particular for the connected n-point amplitudes. Most of the times, this is implicitly understood, and we drop this factor, as well as other overall constants. Similarly, if $\mathcal{A}$ is field independent, $\mathrm{Det}\,[2\pi i\mathcal{A}(\psi)]$ is a constant and can be dropped. Moreover, even if it is field-dependent, it can be replaced by $\mathrm{Det}\,[2\pi i\mathcal{A}(\psi)] \times (\mathrm{Det}\,[2\pi i\mathcal{A}(0)])^{-1}$, which may be easier to handle.

- If $\mathcal{A}$ is field-dependent, e.g. $\mathcal{A}_{n\,x,m\,y}(\psi) = \alpha_{nm}(\psi(x))\,\delta^{(4)}(x-y)$ it gives a contribution to an "effective Lagrangian". To see this note that

  $$\mathrm{Det}\mathcal{A} = \exp\left[\mathrm{Tr}\log\mathcal{A}\right] \ . \tag{1.19}$$

  $\mathcal{A}$ is the quantum-mechanical operator whose matrix elements are

  $$\langle x,n|\,\mathcal{A}\,|y,m\rangle = \mathcal{A}_{n\,x,m\,y}(\psi) = \alpha_{nm}(\psi(x))\,\delta^{(4)}(x-y) = \alpha_{nm}(\psi(x))\ \langle x\,|y\rangle \ , \tag{1.20}$$

  with $\alpha(\psi(x))$ an ordinary matrix-valued function. It follows that

  $$\begin{aligned}
  \langle x,n|\log\mathcal{A}\,|y,m\rangle &= \big(\log\alpha(\psi(x))\big)_{nm}\ \langle x\,|y\rangle \\
  \Rightarrow \qquad \mathrm{Tr}\log\mathcal{A} &= \int \mathrm{d}^4x\ \langle x,n|\log\mathcal{A}\,|x,n\rangle = \int \mathrm{d}^4x\ \mathrm{tr}\,\big(\log\alpha(\psi(x))\big)\ \langle x\,|x\rangle \ ,
  \end{aligned} \tag{1.21}$$

  where tr is an ordinary matrix trace over the indices $n=m$, and $\langle x\,|x\rangle = \delta^{(4)}(0)$ is to be interpreted, as usual, as $\int \frac{\mathrm{d}^4p}{(2\pi)^4}$ (which is divergent, of course, and has to be regularized and renormalized). Thus

  $$\mathrm{Det}\mathcal{A} = \exp\left[\left(\int \frac{\mathrm{d}^4p}{(2\pi)^4}\right) \int \mathrm{d}^4x\ \mathrm{tr}\,\big(\log\alpha(\psi(x))\big)\right] , \tag{1.22}$$

  which can indeed be interpreted as an additional contribution to the Lagrangian.

- As just mentioned, one encounters diverging expressions and there is the need to regularize and renormalize as will be extensively discussed later-on. Actually, the need to renormalize occurs in any interacting theory, whether there are divergences or not. In particular, the fields that appear in the Lagrangian in the first place are so-called bare fields $\psi_{l,B}$. They are related to the renormalized fields $\psi_{l,R}$ by a multiplicative factor, $\psi_{l,B} = \sqrt{Z_l}\,\psi_{l,R}$. For the time being, it is understood that the fields ψ_l are bare fields, although we do not indicate it explicitly.

- In the presence of constraints, e.g. if some of the fields have vanishing canonical momentum, the corresponding π_l are absent in the Hamiltonian. Integrating over these π_l when deriving (1.18) formally still gives the r.h.s. of (1.18) but with the Lagrangian missing certain auxiliary fields. This can be cured by adding in the Hamiltonian formulation a constant factor which is an integral over the auxiliary fields. In the end one recovers (1.18) with the full Lagrangian.

- Functional integrals for anticommuting fields (fermions) can be defined similarly. The relevant formula for fermionic gaussian integrals is

$$\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\ \exp\Big(\overline{\psi}M\psi + \overline{\eta}\psi + \overline{\psi}\eta\Big) = \mathcal{N}\ \mathrm{Det}\, M\ \exp\Big(- \overline{\eta}M^{-1}\eta\Big)\ . \tag{1.23}$$

The power of the determinant is positive rather than negative because the integration variables are anticommuting. Furthermore, it is $+1 = 2\times\frac{1}{2}$ because the fields ψ and $\overline{\psi}$ are to be considered as independent fields (just as bosonic ϕ and $\phi^\dagger$ are considered independent). Another difference with the bosonic case is that the Hamiltonian is not quadratic in the momenta (they are anticommuting, too), e.g. for the Dirac field the free Hamiltonian density is $\mathcal{H} = -\pi\gamma^0(\gamma^j\partial_j + m)\psi$, where $\pi = -\overline{\psi}\gamma^0$. As a result, to pass from the Hamiltonian formalism to the Lagrangian one, one should not integrate the π but only rename $\pi = -\overline{\psi}\gamma^0$. The analogue of our bosonic formula (1.18) for Dirac fields is

$$\begin{aligned}
&\langle \mathrm{vac,\ out}|\, T\Big\{\mathcal{O}_A\big(\Psi(t_A,\vec{x}_A),\overline{\Psi}(t_A,\vec{x}_A)\big)\ \mathcal{O}_B\big(\Psi(t_B,\vec{x}_B),\overline{\Psi}(t_B,\vec{x}_B)\big)\ldots\Big\}\,|\mathrm{vac,\ in}\rangle\\
&\quad= |\mathcal{N}|^2\int\prod_l \mathcal{D}\psi_l\mathcal{D}\overline{\psi}_l\ \mathcal{O}_A\big(\psi(t_A,\vec{x}_A),\overline{\psi}(t_A,\vec{x}_A)\big)\ \mathcal{O}_B\big(\psi(t_B,\vec{x}_B),\overline{\psi}(t_B,\vec{x}_B)\big)\ \ldots\ \times\\
&\qquad\times\exp\left\{i\left[\int \mathrm{d}^4x\,\mathcal{L}(\overline{\psi},\psi,\partial_\mu\psi) + i\epsilon\mathrm{-terms}\right]\right\}\ .
\end{aligned}\tag{1.24}$$

1.2.3 Propagators

The free propagators – or simply propagators – are defined as

$$-i\Delta_{lk}(x,y) = \langle \mathrm{vac,\ out}|\, T(\Psi_l(x)\Psi_k(y))\,|\mathrm{vac,\ in}\rangle\,|_{\text{no interactions}}\ . \tag{1.25}$$

They are not to be confused with the "complete propagators" (denoted Δ')

$$-i\Delta'_{lk}(x,y) = \langle \mathrm{vac,\ out}|\, T(\Psi_l(x)\Psi_k(y))\,|\mathrm{vac,\ in}\rangle\ , \tag{1.26}$$

to be discussed later on. Recall that in a free theory we do not need to distinguish between the Heisenberg picture and interaction picture field operators $\Psi_l(x)$ and $\psi_l(x)$. Evidently, the free propagators are determined by the free part of the action, i.e. the part of the Lagrangian density that is quadratic in the fields. Hence, the computation of the propagators reduces to computing a Gaussian integral. As before, we consider the bosonic case where the free action is of the form[1]

$$\int \mathrm{d}^4x\, \mathcal{L}_0 = -\frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y \sum_{l,l'} \psi_l(x)\mathcal{D}_{l,l'}(x,y)\psi_{l'}(y)\ . \tag{1.27}$$

For a hermitean scalar field e.g. $\mathcal{D}(x,y) = (-\partial^\mu\partial_\mu + m^2 - i\epsilon)\delta^{(4)}(x-y)$. The general formula (1.18) (with $\mathcal{A} = 1$) gives

$$-i\Delta_{lk}(x,y) = \widetilde{\mathcal{N}} \int \prod_{l'} \mathcal{D}\psi_{l'}\ \psi_l(x)\,\psi_k(y)\exp\left\{i\int \mathrm{d}^4x\,\mathcal{L}_0\right\}. \tag{1.28}$$

In a free theory one has

$$1 = \langle \text{vac, out}\,|\text{vac, in}\rangle\,|_{\text{no interactions}} = \widetilde{\mathcal{N}} \int \prod_{l'} \mathcal{D}\psi_{l'}\ \exp\left\{i\int \mathrm{d}^4x\,\mathcal{L}_0\right\}\ , \tag{1.29}$$

which allows us to rewrite

$$-i\Delta_{lk}(x,y) = \frac{\int \prod_{l'} \mathcal{D}\psi_{l'}\ \psi_l(x)\,\psi_k(y)\exp\left\{i\int \mathrm{d}^4x\,\mathcal{L}_0\right\}}{\int \prod_{l'} \mathcal{D}\psi_{l'}\ \exp\left\{i\int \mathrm{d}^4x\,\mathcal{L}_0\right\}}\ . \tag{1.30}$$

Actually, in a free theory, it is not much more difficult to compute the n-point functions:

$$\begin{aligned}
&\langle \text{vac, out}|\, T(\Psi_{l_1}(x_1)\ldots\Psi_{l_n}(x_n))\,|\text{vac, in}\rangle\,|_{\text{no interactions}} \\
&\qquad = \frac{\int \prod_{l'} \mathcal{D}\psi_{l'}\ \psi_{l_1}(x_1)\ldots\psi_{l_n}(x_n)\exp\left\{i\int \mathrm{d}^4x\,\mathcal{L}_0\right\}}{\int \prod_{l'} \mathcal{D}\psi_{l'}\ \exp\left\{i\int \mathrm{d}^4x\,\mathcal{L}_0\right\}} \\
&\qquad = \big(Z_0[0]\big)^{-1}\,(-i)^n\,\frac{\delta}{\delta J_{l_1}(x_1)}\cdots\frac{\delta}{\delta J_{l_n}(x_n)} Z_0[J]\bigg|_{J=0}\ ,
\end{aligned} \tag{1.31}$$

where

$$Z_0[J] = \int \prod_{l'} \mathcal{D}\psi_{l'}\ \exp\left\{i\int \mathrm{d}^4x\,\big[\mathcal{L}_0(x) + J_l(x)\psi^l(x)\big]\right\}\ . \tag{1.32}$$

(One should not confuse the generating functional $Z_0[J]$ with the field renormalization factors Z_l.) With the quadratic $\mathcal{L}_0$ given by (1.27), the integral is Gaussian and one gets

$$\begin{aligned}
Z_0[J] &= \Big(\text{Det}\,\big[\frac{i\widehat{\mathcal{D}}}{2\pi}\big]\Big)^{-1/2}\ \exp\left(\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, J_l(x)\mathcal{D}^{-1}_{lk}(x,y)J_k(y)\right) \\
&= Z_0[0]\ \exp\left(\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\, J_l(x)\mathcal{D}^{-1}_{lk}(x,y)J_k(y)\right)\ .
\end{aligned} \tag{1.33}$$

We then get for the free propagator

$$-i\Delta_{lk}(x,y) = (-i)^2\,\big(Z_0[0]\big)^{-1}\,\frac{\delta}{\delta J_l(x)}\frac{\delta}{\delta J_k(y)}Z_0[J]\bigg|_{J=0} = -i\mathcal{D}^{-1}_{lk}(x,y)\ , \tag{1.34}$$

[1] As already mentioned, for the time being, our fields are bare fields. Indeed, the fact that the quadratic part of the action equals the free action is true for the bare fields with a bare mass parameter, while for the renormalized fields the quadratic part of the action contains the "free" part determining the free propagator, as well as a counterterm part which is at least of first order in the coupling constant. This will be discussed in detail in section 2.

or

$$\boxed{\Delta_{lk}(x,y) = (\mathcal{D}^{-1})_{lk}(x,y)\ .} \tag{1.35}$$

From translation invariance one has $\mathcal{D}_{l,k}(x,y) \equiv \mathcal{D}_{l,k}(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{ip(x-y)} \mathcal{D}_{l,k}(p)$ so that the inverse operator $(\mathcal{D}^{-1})_{lk}(x,y)$ is given by the Fourier transform of $(\mathcal{D}^{-1})_{lk}(p)$, which is the inverse matrix of $\mathcal{D}_{lk}(p)$:

$$\Delta_{lk}(x,y) \equiv \Delta_{lk}(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{ip(x-y)} (\mathcal{D}^{-1})_{lk}(p)\ . \tag{1.36}$$

For the scalar field with $\mathcal{D}(x,y) = (-\partial^\mu\partial_\mu + m^2 - i\epsilon)\,\delta^{(4)}(x-y)$ $= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{ip(x-y)} (p^2 + m^2 - i\epsilon)$ this leads to $\Delta(x-y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{ip(x-y)} \frac{1}{p^2+m^2-i\epsilon}$.

1.3 Green functions, S-matrix and Feynman rules

1.3.1 Vacuum bubbles and normalization of the Green functions

It is a most important result that the n-point Green functions $G_{(n)}^{l_1\ldots l_n}(x_1,\ldots,x_n)$ $= \langle \text{vac, out}|\, T\Big[\Psi_{l_1}(x_1)\ldots\Psi_{l_n}(x_n)\Big]\,|\text{vac, in}\rangle$ (where the Ψ_l are Heisenberg picture operators of the interacting theory) are given by the sum of all Feynman diagrams with n external lines terminating at $x_1,\ldots x_n$. We will now derive this result and at the same time obtain the Feynman rules from the functional integral formalism.

It will be useful to consider "normalized" n-point Green functions (or simply n-point functions) obtained by dividing by the 0-point function:

$$\boxed{\widehat{G}_{(n)}^{l_1\ldots l_n}(x_1,\ldots,x_n) = \frac{\langle \text{vac, out}|\, T\Big[\Psi_{l_1}(x_1)\ldots\Psi_{l_n}(x_n)\Big]\,|\text{vac, in}\rangle}{\langle \text{vac, out}\,|\text{vac, in}\rangle}\quad .} \tag{1.37}$$

Obviously, if the fields are bare fields, this is the so-called bare n-point function $\widehat{G}_{B\,(n)}$, while if the fields are renormalized fields, this is the so-called renormalized n-point function $\widehat{G}_{R\,(n)}$. Since $\Psi_{l,B} = \sqrt{Z_l}\,\Psi_{l,R}$ one simply has

$$\widehat{G}_{B\,(n)}^{l_1\ldots l_n}(x_1,\ldots,x_n) = \left[\prod_{r=1}^{n}\sqrt{Z_{l_r}}\right] \widehat{G}_{R\,(n)}^{l_1\ldots l_n}(x_1,\ldots,x_n)\ . \tag{1.38}$$

For the time being, we will concentrate on the bare n-point functions, although we will not indicate it explicitly.

We use the functional integral representation of the numerator and the denominator[2] in the Lagrangian formalism and obtain

$$\widehat{G}_{(n)}^{l_1\ldots l_n}(x_1,\ldots,x_n) = \frac{\int \mathcal{D}\psi\ \psi_{l_1}(x_1)\ldots\psi_{l_n}(x_n)\ e^{i\int \mathrm{d}^4 x\,\mathcal{L}(x)}}{\int \mathcal{D}\psi\ e^{i\int \mathrm{d}^4 x\,\mathcal{L}(x)}}\quad . \tag{1.39}$$

[2]In the following we simply write $\mathcal{D}\psi$ instead of $\prod_l \mathcal{D}\psi_l$

Note that the normalization constant $|\mathcal{N}|^2$ has been eliminated when dividing by

$$S_{\mathrm{vac,vac}} \equiv \langle \mathrm{vac,\ out}\,|\mathrm{vac,\ in}\rangle = |\mathcal{N}|^2 \int \mathcal{D}\psi\ e^{i\int \mathrm{d}^4x\,\mathcal{L}(x)}\ . \tag{1.40}$$

In the absence of time-varying external fields $S_{\mathrm{vac,vac}}$ is just a number. Contrary to a free field theory, however, in general this number is not just 1. Recall the definition of the in and out states: $|\mathrm{vac,in}\rangle$ is the state that resembles the vacuum $|0\rangle$ of particles without interactions if an observation is made at $t \to -\infty$. Recall also that the separation of H into H_0 and V must be such that H and H_0 have the same spectrum. In particular, $H\,|\mathrm{vac,in}\rangle = 0$ and $H_0\,|0\rangle = 0$. Hence $|\mathrm{vac,in}\rangle$ cannot contain any particles that would necessarily contribute a positive energy. We will suppose that the vacuum is unique[3] and stable, so that there are no transitions $\langle \alpha, \mathrm{out}\,|\mathrm{vac,in}\rangle$ for any $\alpha \neq \mathrm{vac}$. (For a unique vacuum, this follows from energy conservation.) Hence,

$$S_{\alpha,\mathrm{vac}} = S_{\mathrm{vac,vac}}\ \delta_{\alpha,\mathrm{vac}}\ . \tag{1.41}$$

Unitarity of the S-matrix implies

$$1 = \sum_\alpha |S_{\alpha,\mathrm{vac}}|^2 = |S_{\mathrm{vac,vac}}|^2 \quad \Rightarrow \quad S_{\mathrm{vac,vac}} \equiv \langle \mathrm{vac,\ out}\,|\mathrm{vac,\ in}\rangle = e^{i\gamma_{\mathrm{vac}}}\ . \tag{1.42}$$

It is instructive to compute $S_{\mathrm{vac,vac}}$ in perturbation theory and verify that it is a pure phase. Indeed, $S_{\mathrm{vac,vac}} = \langle 0|\,T\exp\left(-i\int \mathrm{d}^4x\,\mathcal{H}_{\mathrm{int}}(x)\right)|0\rangle$, which equals 1 plus all Feynman diagrams without external lines, cf Fig. 1. One can convince oneself that the sum of all such diagrams equals the exponential of the connected diagrams only:

$$S_{\mathrm{vac,vac}} = \exp\Big[\text{sum of all connected vacuum-vacuum diagrams}\Big] \tag{1.43}$$

In such a diagram, every propagator contributes a $-i$, and each vertex also gives a factor $-i$ (since

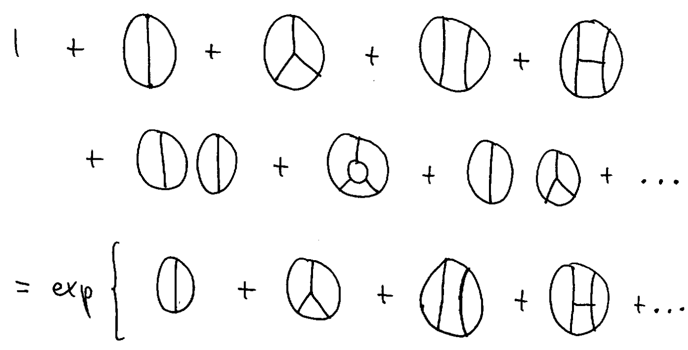


Figure 1: $S_{\mathrm{vac,vac}}$ is given by the sum of all vacuum bubbles which equals the exponential of the sum of all connected vacuum bubbles.

[3]In many theories with symmetries, the vacuum is degenerate. In this case the present discussion is slightly more complicated but can be adapted accordingly.

$\mathcal{H}_{\rm int}$ is real, but the vertex equals $-i$ times the numerical factor). Finally, each loop contributes an i due to the Wick rotation (to be discussed below). If we let I be the number of internal lines, V the number of vertices and L the number of loops, this yields a total factor

$$(-i)^I(-i)^V i^L = (-)^V i^{V-I+L} = (-)^V \ i \ , \tag{1.44}$$

where we used the diagrammatic identity

$$I - V = L - 1 \ , \tag{1.45}$$

valid for each connected component of a diagram. Thus, every connected vacuum-to-vacuum diagram is purely imaginary and $S_{\rm vac,vac}$ is indeed pure phase.

What is the effect of normalizing the Green functions as in (1.37), i.e. of dividing by $\langle {\rm vac,\ out}\,|{\rm vac,\ in}\rangle$? Suppose the numerator in (1.37) is given by the sum of all Feynman diagrams with n external lines (including propagators) terminating at $x_1, \ldots x_n$. This sum then corresponds to connected and disconnected diagrams. The disconnected diagrams, in particular, contain diagrams with vacuum-bubbles. There may be $0, 1, 2, \ldots$ vacuum bubbles. It is easy to convince oneself that the sum of all diagrams is the product of a) the sum of diagrams without vacuum-bubbles and of b) 1 plus the sum of all vacuum bubbles, i.e. of $S_{\rm vac,vac} = \langle {\rm vac,\ out}\,|{\rm vac,\ in}\rangle$. Thus $\widehat{G}_{(n)}$ as given by (1.37) should exactly be the sum of all diagrams (connected and disconnected) with n external lines (with their propagators) not containing any vacuum bubbles:

$$\boxed{\begin{array}{c}\widehat{G}_{(n)}^{l_1\ldots l_n}(x_1,\ldots x_n) \text{ is given by the sum of all Feynman diagrams with } n \text{ external lines} \\ \text{(with propagators) terminating at } x_1,\ldots x_n \text{ and not containing any vacuum bubbles.}\end{array}} \tag{1.46}$$

This is the result we will show starting from the identity (1.39). Actually, this result applies both to the bare and the renormalized Green functions, provided one uses the Feynman rules with bare propagators and interactions in the first case, and renormalized propagators and interactions (and counterterms) in the second case. This will become clearer in section 2.

One can also rewrite $\widehat{G}_{(n)}$ in a simpler-looking way. Indeed, still assuming a non-degenerate vacuum, $|{\rm vac,in}\rangle$ and $|{\rm vac,out}\rangle$ only differ by the phase factor $e^{i\gamma_{\rm vac}}$ as is easily seen from (1.41) and (1.42):

$$|{\rm vac,in}\rangle = \sum_\alpha |\alpha,{\rm out}\rangle\,\langle\alpha,{\rm out}\,|{\rm vac,in}\rangle = \sum_\alpha |\alpha,{\rm out}\rangle\, S_{\alpha,{\rm vac}} = e^{i\gamma_{\rm vac}}\,|{\rm vac,out}\rangle \ . \tag{1.47}$$

It follows that for any operator or product of operators M one has

$$\frac{\langle{\rm vac,out}|\,M\,|{\rm vac,in}\rangle}{\langle{\rm vac,out}\,|{\rm vac,in}\rangle} = \langle\Omega|\,M\,|\Omega\rangle = \left\langle\widetilde\Omega\right|M\left|\widetilde\Omega\right\rangle \quad , \quad |\Omega\rangle \equiv |{\rm vac,in}\rangle \ , \quad \left|\widetilde\Omega\right\rangle \equiv |{\rm vac,out}\rangle \ , \tag{1.48}$$

and hence

$$\widehat{G}_{(n)}^{l_1\ldots l_n}(x_1,\ldots,x_n) = \langle\Omega|\,T\Big[\Psi_{l_1}(x_1)\ldots\Psi_{l_n}(x_n)\Big]\,|\Omega\rangle \equiv \langle T\Big[\Psi_{l_1}(x_1)\ldots\Psi_{l_n}(x_n)\Big]\rangle_{\rm vac} \ . \tag{1.49}$$

1.3.2 Generating functional of Green functions and Feynman rules

Just as we defined $Z_0[J]$ for a free theory, eq. (1.32), the generating functional for the interacting theory is defined by

$$\boxed{Z[J] = \int \mathcal{D}\psi \ \exp\left\{ i \int \mathrm{d}^4x \left[\mathcal{L}(x) + J_l(x)\psi^l(x) \right] \right\} \ .} \tag{1.50}$$

Equation (1.39) can then be written as

$$\widehat{G}^{l_1 \ldots l_n}_{(n)}(x_1, \ldots, x_n) = \frac{1}{Z[0]} \ (-i)^n \ \frac{\delta}{\delta J_{l_1}(x_1)} \cdots \frac{\delta}{\delta J_{l_n}(x_n)} Z[J] \Bigg|_{J=0} \ . \tag{1.51}$$

We see that indeed $Z[J]$, or rather $Z[J]/Z[0]$, generates the n-point Green functions $\widehat{G}_{(n)}$ by successive functional derivatives. Conversely, the $\widehat{G}_{(n)}$ appear as the coefficients in the development of $Z[J]$ in powers of the J:

$$Z[J] = Z[0] \ \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n \, \widehat{G}^{l_1 \ldots l_n}_{(n)}((x_1, \ldots, x_n) \ iJ_{l_1}(x_1) \ldots iJ_{l_n}(x_n) \ . \tag{1.52}$$

To make the relation with the Feynman diagrams, recall that the sum of Feynman diagrams corresponds to a perturbative expansion in the coupling constant(s). So let us compute $Z[J]$ in perturbation theory. To do so, separate

$$\mathcal{L}\big(\psi(x), \partial_\mu\psi(x)\big) = \mathcal{L}_0\big(\psi(x), \partial_\mu\psi(x)\big) + \mathcal{L}_{\rm int}\big(\psi(x), \partial_\mu\psi(x)\big) \ , \tag{1.53}$$

with the free Lagrangian $\mathcal{L}_0$ given by the quadratic part, cf. (1.27), and develop $e^{i\int \mathcal{L}_{\rm int}}$ in a power series.[4] Hence

$$\begin{aligned} Z[J] &= \int \mathcal{D}\psi \sum_{N=0}^{\infty} \frac{i^N}{N!} \Big[\int \mathrm{d}^4x \, \mathcal{L}_{\rm int}\big(\psi(x), \partial_\mu\psi(x)\big) \Big]^N \ \exp\Big\{ i \int \mathrm{d}^4x \ \big[\mathcal{L}_0(x) + J_l(x)\psi^l(x)\big] \Big\} \\ &= \sum_{N=0}^{\infty} \frac{i^N}{N!} \Big[\int \mathrm{d}^4x \, \mathcal{L}_{\rm int}\big(-i \frac{\delta}{\delta J(x)}, -i\partial_\mu \frac{\delta}{\delta J(x)} \big) \Big]^N \ \int \mathcal{D}\psi \exp\Big\{ i \int \mathrm{d}^4x \ \big[\mathcal{L}_0(x) + J_l(x)\psi^l(x)\big] \Big\} \\ &= \sum_{N=0}^{\infty} \frac{i^N}{N!} \Big[\int \mathrm{d}^4x \, \mathcal{L}_{\rm int}\big(-i \frac{\delta}{\delta J(x)}, -i\partial_\mu \frac{\delta}{\delta J(x)} \big) \Big]^N \ Z_0[J] \ . \end{aligned} \tag{1.54}$$

$Z_0[J]$ is the generating functional of the free theory computed before, cf. eq. (1.33) with $\mathcal{D}^{-1}$ equal to $\Delta^{(0)}$:

$$\begin{aligned} Z_0[J] &= Z_0[0] \ \exp\left(\frac{i}{2} \int \mathrm{d}^4x \mathrm{d}^4y \ J_l(x) \Delta_{lk}(x,y) J_k(y) \right) \\ &= Z_0[0] \ \exp\left(\frac{1}{2} \int \mathrm{d}^4x \mathrm{d}^4y \ \big(iJ_l(x)\big)\big(-i\Delta_{lk}(x,y)\big)\big(iJ_k(y)\big) \right) \ . \end{aligned} \tag{1.55}$$

[4]As it stands, this applies to the computation of bare Green functions. To compute the renormalized Green functions, one simply takes the corresponding $\mathcal{L}_0$ while including all counterterms into $\mathcal{L}_{\rm int}$, even the quadratic ones. The bare and renormalized generating functionals then are the same provided one also defined $J_{B,l} = Z_l^{-1/2} J_{R,l}$ so that $J_{B,l}\psi_{B,l} = J_{R,l}\psi_{R,l}$.

We see that $-i\delta/\delta J(x)$ acting on $Z_0[J]$ yields a propagator $-i\Delta(x,y)$ "attached" to a vertex at x (times $iJ(y)$ and integrated over d^4y). There are as many propagators attached to a vertex at x as there are fields in $\mathcal{L}_{\rm int}(x)$. All propagators are attached to some vertex or to an "external" $iJ(z_i)$. Obviously, a term of given order N in (1.54) corresponds to a diagram with N vertices. It is also not difficult to work out that the combinatorial factors accompanying a diagram are the usual ones. Hence, $Z[J]$ is the product of $Z_0[0]$ and the sum of all Feynman diagrams with an arbitrary number of external lines at the end of which are attached the factors $iJ(z_i)$ (integrated d^4z_i).

Let's look at an example. Take a hermitean scalar field with an interaction $\mathcal{L}_{\rm int} = -\frac{g}{24}\phi^4$, and compute $Z[J]$ up to first order in g, meaning we only keep the terms of order $N=0$ and $N=1$ in (1.54):

$$\begin{aligned} Z^{(g)}[J] &= Z_0[0]\left\{1 - i\frac{g}{24}\int \mathrm{d}^4x\,\big(-i\frac{\delta}{\delta J(x)}\big)^4\right\}\ \exp\big(\tfrac{1}{2}\textstyle\int \mathrm{d}^4x\mathrm{d}^4y\,(iJ_l(x))(-i\Delta_{lk}(x,y))(iJ_k(y))\big) \\ &= Z_0[0]\left\{1 - i\frac{g}{24}\int \mathrm{d}^4x\left[\big(\int \mathrm{d}^4z\,(-i\Delta(x,z)iJ(z)\big)^4 + 6(-i\Delta(x,x)\big(\int \mathrm{d}^4z\,(-i\Delta(x,z)iJ(z)\big)^2\right.\right. \\ &\qquad\left.\left. +3(-i\Delta(x,x))^2\right]\right\}\ \exp\big(\tfrac{1}{2}\textstyle\int \mathrm{d}^4x\mathrm{d}^4y\,(iJ_l(x))(-i\Delta_{lk}(x,y))(iJ_k(y))\big)\ . \end{aligned} \tag{1.56}$$

First, take $J=0$. At order g there is only one term and:

$$Z^{(g)}[0] = Z_0[0]\left\{1 - i\frac{g}{8}\int \mathrm{d}^4x\,(-i\Delta(x,x))^2\right\} \tag{1.57}$$

The term of order g corresponds to a single vertex with 4 lines, joined two by two (two loops). This is a vacuum-bubble diagram. The factor $-\frac{ig}{8}$ is in agreement with the usual combinatoric factor: the vertex gives a factor $-ig$ and the symmetry factor is $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$. More generally, $Z[0]$ is the sum of 1 and all vacuum-bubbles.

If one first takes the derivatives $\frac{\delta}{\delta(iJ(x_1))}\ldots\frac{\delta}{\delta(iJ(x_n))}$ of $Z[J]$ and only then sets $J=0$, one generates a sum of products of propagators $(-i\Delta)$ attached either to the external x_i or to internal $\widetilde{x}_i$ of vertices which are integrated. One sees that each vertex contributes i times the numerical factors in $\mathcal{L}_{\rm int}$, and the symmetry factors again are automatically generated. As explained above, this sum of all diagrams factorizes into a sum of diagrams without vacuum bubbles and the sum of 1 plus all vacuum bubbles. Thus, dividing by $Z[0]$ exactly eliminates these vacuum bubbles and we have shown (1.46) for the n-point Green functions $\widehat{G}_{(n)}$ as defined by the functional integral (1.51).

Let us come back to the example of the scalar theory with $\mathcal{L}_{\rm int} = -\frac{g}{24}\phi^4$. Here, we get for the

4-point function up to order g:

$$\frac{\delta}{\delta(iJ)(x_1)}\cdots\frac{\delta}{\delta(iJ)(x_4)}\left.\frac{Z^{(g)}[J]}{Z^{(g)}[0]}\right|_{J=0}$$

$$= -i\frac{g}{24}\int \mathrm{d}^4x\,\Big[24(-i\Delta(x,x_1))(-i\Delta(x,x_2))(-i\Delta(x,x_3))(-i\Delta(x,x_4))$$

$$+6(-i\Delta(x,x))\,2(-i\Delta(x,x_1))(-i\Delta(x,x_2))(-i\Delta(x_3,x_4)) + 5 \text{ permutations}\Big]. \quad (1.58)$$

The two terms correspond to the two diagrams shown in Figure 2.

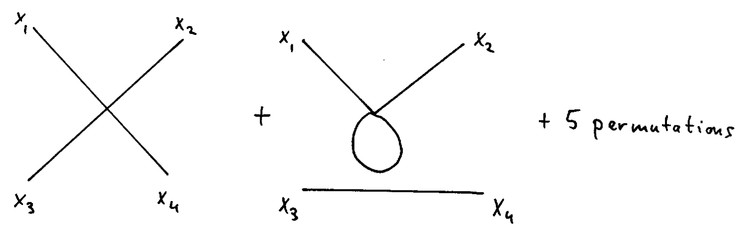


Figure 2: Diagrams corresponding to (1.58).

Loop-counting : It is sometimes convenient to introduce a loop-counting parameter λ by replacing the action $S \to \frac{1}{\lambda}S$ and $J \to \frac{1}{\lambda}J$. This multiplies all vertices by $\frac{1}{\lambda}$ and all propagators by λ. Each external line also gets a factor $\frac{1}{\lambda}$ from the $\frac{J}{\lambda}$. Thus external lines get a net factor λ^0, and the overall factor of a diagram is $\lambda^{I-V} = \lambda^{L-C}$, where I is the number of internal lines, V the number of vertices, L the number of loops and C the number of connected components of the diagram and we used (1.45). Thus for fixed C, λ is a loop-counting parameter. In particular, a connected diagram is accompanied by a factor λ^{L-1}. Note that the exponent in the functional integral is $\frac{i}{\hbar}(S+\int J\psi)$ if one does not use units where $\hbar = 1$. One sees that $\hbar$ is a loop-counting parameter, and the limit $\hbar \to 0$ isolates the diagrams with $L = 0$, i.e. tree diagrams. In this sense, tree amplitudes are referred to as classical, while loop corrections are quantum corrections. Note also that taking into account the tree and one-loop diagrams often is referred to as semi-calssical approximation.

1.3.3 Generating functional of connected Green functions

The n-point ($n > 0$) Green functions $\widehat{G}_{(n)}(x_1,\ldots,x_n)$ without vacuum-bubbles contain the important subclass of *connected* n-point Green functions $G^C_{(n)}(x_1,\ldots,x_n)$. They can be defined by an algebraic recursion relation: by definition $G^C_{(1)}(x) = \widehat{G}_{(1)}(x)$ and then $G^C_{(2)}(x_1,x_2) = \widehat{G}_{(2)}(x_1,x_2) - G^C_{(1)}(x_1)G^C_{(1)}(x_2)$, etc. One can show that this is equivalent to $G^C_{(n)}(x_1,\ldots,x_n)$ being the sum of the corresponding *connected* Feynman diagrams. The algebraic recursion relation is best summarized as a relation between generating functionals. Let $iW[J]$ be the generating functional of connected

Green functions, (cf. Fig. 3)

$$iW[J] = iW[0] + \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n \; G^{C,\, l_1 \ldots l_n}_{(n)}(x_1, \ldots, x_n)\; iJ_{l_1}(x_1) \ldots iJ_{l_n}(x_n) \; . \tag{1.59}$$

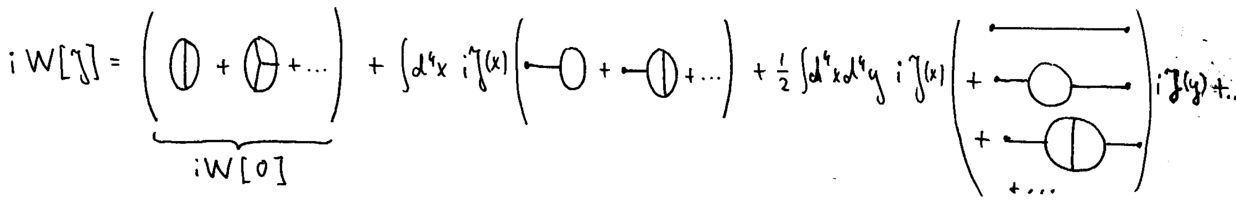


Figure 3: $W[J]$ is the generating functional of connected Green functions.

We separated the part $iW[0]$ which corresponds to connected 0 point Green function, i.e. to connected vacuum-bubbles. Note that for $n \geq 1$, the G^C_n cannot contain vacuum-bubbles. As one sees from Fig. 3 or the definition (1.59), the connected full propagator is given by

$$-i\Delta'_C(x,y) \equiv G^C_{(2)}(x,y) = -i \frac{\delta}{\delta J(x)} \frac{\delta}{\delta J(y)} W[J]\Big|_{J=0} \equiv -i\, W^{(2)}(x,y) \; . \tag{1.60}$$

Consider now $\exp\big(iW[0]\big) = 1 + iW[0] + \frac{1}{2}\big(W[0]\big)^2 + \ldots$. Here, $iW[0]$ contains all vacuum-bubbles with a single connected component, while $\frac{1}{2}\big(W[0]\big)^2$ contains all vacuum-bubble diagrams with two connected components (the factor $\frac{1}{2}$ is the appropriate symmetry factor for those diagrams having two identical components, while it is compensated by a factor 2 for the product of two different components), etc. Hence, $\exp\big(iW[0]\big)$ is the sum of 1 and all possible vacuum-bubble diagrams, connected or not, i.e. it equals $Z[0]$. In the same way one sees that $\exp\big(iW[J]\big)$ equals 1 plus the sum of all diagrams, connected or not, i.e. $Z[J]$:

$$\boxed{Z[J] = \exp\big(iW[J]\big) \; .} \tag{1.61}$$

Let's look at the example of connected 1- and 2-point functions. As already noted, the 1-point function without vacuum-bubbles is necessarily connected:

$$\widehat{G}_{(1)}(x) = G^C_{(1)}(x) \; . \tag{1.62}$$

Next, the relation (1.61) indeed leads to the correct relation between the 2-point functions (without vacuum-bubbles) $\widehat{G}_{(2)}$ and the connected 1- and 2-point functions $G^C_{(1)}$ and $G^C_{(2)}$:

$$\begin{aligned}
G^C_{(2)}(x,y) &= \frac{\delta}{\delta(iJ)(x)} \frac{\delta}{\delta(iJ)(y)} iW[J]\Big|_{J=0} = \frac{\delta}{\delta(iJ)(x)} \frac{\delta}{\delta(iJ)(y)} \log Z[J]\Big|_{J=0} \\
&= \tfrac{1}{Z[J]} \tfrac{\delta}{\delta(iJ)(x)} \tfrac{\delta}{\delta(iJ)(y)} Z[J]\Big|_{J=0} - \Big(\tfrac{1}{Z[J]} \tfrac{\delta}{\delta(iJ)(x)} Z[J]\Big)\Big|_{J=0} \Big(\tfrac{1}{Z[J]} \tfrac{\delta}{\delta(iJ)(y)} Z[J]\Big)\Big|_{J=0} \\
&= \widehat{G}_{(2)}(x,y) - \widehat{G}_{(1)}(x)\, \widehat{G}_{(1)}(y) \\
&= \widehat{G}_{(2)}(x,y) - G^C_{(1)}(x)\, G^C_{(1)}(y) \; .
\end{aligned} \tag{1.63}$$

Loop-counting : If one introduces the loop-counting parameter as before, one also has $W[J] = \sum_{L=0}^{\infty} \lambda^{L-1} W_L[J]$, where $W_L[J]$ is the L-loop contribution to $W[J]$. In the limit $\lambda \to 0$ one isolates the contributions of the tree-diagrams. On the other hand, in this limit, one can evaluate the functional integral in a saddle-point approximation (stationary phase) and then the integral is dominated by those ψ_J that solve $\frac{\delta S}{\delta \psi^l} + J_l = 0$. It follows that

$$W_0[J] = S[\psi_J] + \int \mathrm{d}^4 x J_l(x) \psi_J^l(x) \ , \tag{1.64}$$

i.e the tree contribution $W_0[J]$ is the (inverse) Legendre transform of the classical action.

1.3.4 Relation between Green functions and S-matrix

The basic quantity in particle physics is the S-matrix from which measurable transition rates like cross-sections and life-times can be extracted. The S-matrix elements are defined as

$$S_{\beta\alpha} = \langle \beta, \text{ out} \, | \alpha, \text{ in} \rangle \ , \tag{1.65}$$

and give the transition amplitudes between the in-states $|\alpha, \text{ in}\rangle$ and the out-states $|\beta, \text{out}\rangle$. Here, α and β are short-hand for a complete collection of momenta p_i, helicities σ_i and (anti)particle types n_i describing the state. Recall that the in-state $|\alpha, \text{ in}\rangle \equiv |p_1, \sigma_1, n_1; p_2, \sigma_2, n_2; \ldots \text{in}\rangle$ is a (time-independent Heisenberg-picture) state that looks, if an observation is made at $t \to -\infty$, as a collection of non-interacting particles with momenta p_i, helicities σ_i and of type n_i. A similar definition holds for the out-states with $t \to +\infty$.

To relate the S-matrix elements to the Green-functions, we first define the Fourier transform of the latter as

$$\widetilde{G}_{(n)}^{l_1 \ldots l_n}(p_1, \ldots p_n) = \int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_n \ e^{i \sum_{i=1}^n p_i x_i} \ G_{(n)}^{l_1 \ldots l_n}(x_1, \ldots x_n) \ , \tag{1.66}$$

with all momenta p_i^μ considered as *entering* the diagram. These momenta are *off-shell* and are those of the propagators associated with the external lines. S-matrix elements are computed between *on-shell* external states, i.e. precisely at those values of the momenta where the external propagators of the Green functions have poles. We will see in the next sections, that loop-corrections to the free propagators shift the pole from $p^2 = -m_B^2$ (m_B is the bare mass entering the Lagrangian) to $p^2 = -m^2$, where m must be interpreted as the physical mass. Thus the full propagators have poles at $p^2 = -m^2$. To get a finite result for on-shell external states, one obviously has to remove the full external propagators. This can be done by multiplying with the inverse *full* propagators $i(\Delta')^{-1}$. The result is called the amputated n-point Green function, cf. Fig. 4.

$$\widetilde{G}_{(n,\text{amp})}^{l_1 \ldots l_n}(p_1, \ldots p_n) = \left[\prod_{j=1}^n i(\Delta')^{-1}(p_j) \right] \ \widetilde{G}_{(n)}^{l_1 \ldots l_n}(p_1, \ldots p_n) \ . \tag{1.67}$$

Again, this definition holds with all Green functions and full propagators being the bare or renormalized ones.

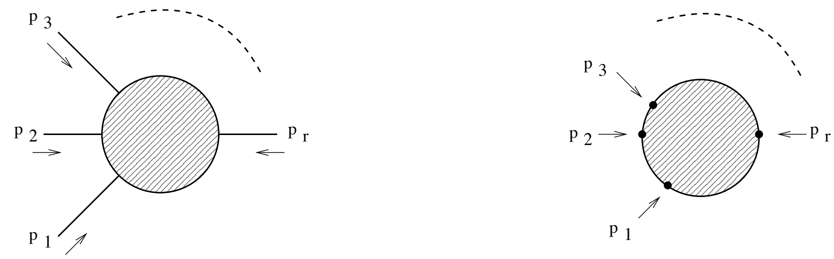


Figure 4: n-point Green function (left) and corresponding amputated n-point Green function (right)

It can be shown that the S-matrix elements are obtained from the *on-shell* amputated *renormalized* Green functions simply by multiplication with the appropriate "wave-functions" of the initial and final (anti)particles. More precicely, to obtain the S-matrix element with r (anti)particules in the initial state and $n-r$ in the final state : (i) take the corresponding amputated renormalized n-point Green function (with $\psi^\dagger$ for any initial particle or final antiparticle and ψ for any final particle or initial antiparticle), (ii) take the p_i on-shell for the initial (anti)particles, and similarly the $-p_j$ on-shell for the final (anti)particles, (iii) multiply with the appropriate wave-function factors $\frac{u(p_i,\sigma_i)}{(2\pi)^{3/2}}$ etc., that enter in the expansions of the corresponding free fields. Thus

$$\boxed{\begin{aligned} S_{p'_1,\sigma'_1,n'_1,\ldots;p_1,\sigma_1,n_1,\ldots} \quad &= \quad \left[\prod_{j=1}^{n-r} \frac{u^*_{l'_j}(p'_j,\sigma'_j)/v_{l'_j}(p'_j,\sigma'_j)}{(2\pi)^{3/2}}\right]\left[\prod_{i=1}^{r} \frac{u_{l_i}(p_i,\sigma_i)/v^*_{l_i}(p_i,\sigma_i)}{(2\pi)^{3/2}}\right] \\ &\times \widetilde{G}^{l_1\ldots l_r l'_1 \ldots l'_{n-r}}_{R\,(n,\text{amp})}(p_1,\ldots p_r,-p'_1,\ldots -p'_{n-r}) \ . \end{aligned}} \tag{1.68}$$

It follows from (1.38) that $\Delta'_B = Z\,\Delta'_R$ and, combining with the definition of the amputated Green function (1.67) one immediately sees that

$$\widetilde{G}^{l_1\ldots l_n}_{B\,(n,\text{amp})}(p_1,\ldots p_n) = \left[\prod_{j=1}^{n} Z_{l_j}^{-1/2}\right] \widetilde{G}^{l_1\ldots l_n}_{R\,(n,\text{amp})}(p_1,\ldots p_n) \ . \tag{1.69}$$

Thus we can rewrite the relation between the S-matrix elements and the amputated Green functions in terms of the bare amputated Green functions as

$$\boxed{\begin{aligned} S_{p'_1,\sigma'_1,n'_1,\ldots;p_1,\sigma_1,n_1,\ldots} \quad &= \quad \left[\prod_{j=1}^{n-r} \frac{u^*_{l'_j}(p'_j,\sigma'_j)/v_{l'_j}(p'_j,\sigma'_j)}{(2\pi)^{3/2}}\sqrt{Z_{l'_j}}\ \right]\left[\prod_{i=1}^{r} \frac{u_{l_i}(p_i,\sigma_i)/v^*_{l_i}(p_i,\sigma_i)}{(2\pi)^{3/2}}\sqrt{Z_{l_i}}\ \right] \\ &\times\ \widetilde{G}^{l_1\ldots l_r l'_1 \ldots l'_{n-r}}_{B\,(n,\text{amp})}(p_1,\ldots p_r,-p'_1,\ldots -p'_{n-r}) \ . \end{aligned}} \tag{1.70}$$

It is in this second form that the relation, first derived by Lehman, Symanzik and Zimmermann, is usually referred to as LSZ reduction formula. However, (1.68) has the advantage of expressing the finite S-matrix elements solely in terms of renormalized quantities that have a finite limit as the regularization is removed.

1.4 Quantum effective action

1.4.1 Legendre transform and definition of $\Gamma[\varphi]$

We already defined the generating functional $Z[J]$ of Green functions and the generating functional $W[J]$ of connected Green functions. They correspond to the sum of all Feynman diagrams and of connected diagrams only. Connected diagrams are more basic since all diagrams can be constructed from them. The algebraic relation was simply $Z[J] = e^{iW[J]}$. Here we will define yet another generating functional $\Gamma[\varphi]$ that generates an even smaller subclass of connected diagrams, namely the one-particle-irreducible diagrams, or 1PI for short. A 1PI diagram is a connected diagram that does not become disconnected by cutting a single line. (There is a slight subtlety with this definition for the 1PI 2-point diagram to be discussed below.) Since a tree diagram becomes disconnected by cutting a single line, tree diagrams are not 1PI. A one-loop diagram with the external propagators removed always is 1PI. Higher-loop diagrams may or may not be 1PI. For $n \geq 3$, a 1PI n-point diagram is also called an n-point *proper vertex*.

The functional $\Gamma[\varphi]$ is defined as the Legendre transform of $W[J]$. First, let

$$\phi^r_J(x) \equiv \frac{\delta}{\delta J_r(x)} W[J] = -i \frac{\delta}{\delta J_r(x)} \log Z[J] = \frac{1}{Z[J]} \Big(-i \frac{\delta}{\delta J_r(x)} Z[J] \Big) . \tag{1.71}$$

The expression on the r.h.s. is similar to the one-point Green function without vacuum bubbles (which is the connected one-point function) $\widehat{G}^r_{(1)}(x) \equiv G^C_{(1),\,r}(x)$ except that we have not set $J = 0$. Not setting $J = 0$ amounts to keeping the additional interaction terms $\phi^r J_r$ in the Lagrangian. Thus $\phi^r_J(x)$ is the connected one-point function in the presence of the additional interactions generated by the sources. This is also called the vaccum expectation value of the corresponding Heisenberg field Φ^r in the presence of the sources J:

$$\phi^r_J(x) = \langle \Phi^r(x) \rangle_{\text{vac},\,J} \equiv \langle \Omega | \, \Phi^r(x) \, | \Omega \rangle_J \equiv \frac{\langle \text{vac, out} | \, \Phi^r(x) \, | \text{vac, in} \rangle_J}{\langle \text{vac, out} \, | \text{vac, in} \rangle_J} . \tag{1.72}$$

One can invert the relation $\phi^r_J(x) = \frac{\delta}{\delta J_r(x)} W[J]$ to get $J_r(x)$ as a function of $\phi^r(x)$. More precisely, for every (c-number) function $\varphi^r(x)$, we let $j_{\varphi^r}(x)$ be the (c-number) function such that $\phi^r_J(x) = \varphi^r(r)$ if $J_r(x) = j_{\varphi^r}(x)$, i.e. $j_{\varphi^r}(x)$ is the current such that the vacuum expectation value of Φ^r equals $\varphi^r(x)$. We can now use φ as variable[5] and define the Legendre transform of W as

$$\Gamma[\varphi] = W[j_\varphi] - \int \mathrm{d}^4 x \, \varphi^r(x) j_{\varphi^r}(x) . \tag{1.73}$$

Γ is called the *quantum effective action*. Let us show why: one has

$$\frac{\delta}{\delta \varphi^s(y)} \Gamma[\varphi] = \int \mathrm{d}^4 x \, \frac{\delta j_{\varphi^r}(x)}{\delta \varphi^s(y)} \frac{\delta W[j_\varphi]}{\delta j_{\varphi^r}(x)} - j_{\varphi^s}(y) - \int \mathrm{d}^4 x \, \varphi^r(x) \frac{\delta j_{\varphi^r}(x)}{\delta \varphi^s(y)} \tag{1.74}$$

Now

$$\frac{\delta W[j_\varphi]}{\delta j_{\varphi^r}(x)} = \frac{\delta W[J]}{\delta J_r(x)} \bigg|_{J_r \,=\, j_{\varphi^r}} = \phi^r_J(x) \Big|_{J_r \,=\, j_{\varphi^r}} = \varphi^r(x) , \tag{1.75}$$

[5] Since $J_{B,s} = Z_s^{-1/2} J_{R,s}$ one obviously has $(\phi^s_J)_B = \sqrt{Z_s}\,(\phi^s_J)_R$ and thus also $\varphi^s_B = \sqrt{Z_s}\,\varphi^s_R$.

so that the first and third terms in (1.74) exactly cancel. Hence,

$$\frac{\delta}{\delta\varphi^s(y)}\Gamma[\varphi] = -j_{\varphi^s}(y) \ . \tag{1.76}$$

Suppose that for a given function φ one has $\frac{\delta\Gamma[\varphi]}{\delta\varphi^s(y)} = 0$, i.e. the corresponding j_{φ^r} vanishes. This means that the vacuum expectation values of the $\Phi^r(x)$, in the absence of any current, equal $\varphi^r(x)$. Conversely, the vacuum expectation values of Φ^r, for vanishing current, must be solutions of $\frac{\delta\Gamma[\varphi]}{\delta\varphi^s(y)} = 0$, i.e. be stationary points of $\Gamma[\varphi]$. This shows that Γ can indeed be interpreted as some quantum action.

Note that the preceding careful discussion usually is simply summarized as

$$\boxed{\frac{\delta W}{\delta J_r} = \varphi^r \quad , \quad \frac{\delta \Gamma}{\delta\varphi^r} = -J_r \quad , \quad \Gamma[\varphi] = W[J] - \int \mathrm{d}^4x\, \varphi^r(x) J_r(x)} \tag{1.77}$$

Note also that all these manipulations involving functional derivatives $\frac{\delta}{\delta J}$, $\frac{\delta}{\delta\varphi}$, etc remain valid for fermionic fields and sources, provided one correctly uses left or right derivatives, paying attention to the order of the fields. Thus one should define e.g $\frac{\delta_R W}{\delta J_r} = \varphi^r$ and $\frac{\delta_L \Gamma}{\delta\varphi^r} = -J_r$

1.4.2 $\Gamma[\varphi]$ as quantum effective action and generating functional of 1PI-diagrams

The interpretation of $\Gamma[\varphi]$ as quantum effective action is confirmed further if we recall that in the classical limit, i.e. at tree-level, $W[J]$ is just the inverse Legendre transform of the classical action, cf. (1.64). Since $\Gamma[\varphi]$ is the Legendre transform of $W[J]$, it follows that, in the classical limit, $\Gamma[\varphi]$ just is the classical action. Thus $\Gamma[\varphi]$ equals the classical action $S[\varphi]$ plus quantum-, i.e. loop-corrections. Actually, in a sense, $\Gamma[\varphi]$ captures all loop effects since one has the following property:

> One may compute $iW[J]$ as a sum of connected *tree* diagrams with vertices and propagators determined as if the action were $\Gamma[\varphi]$ rather than $S[\varphi]$.

To prove this, let us proceed as for the loop-counting above: we compute the generating functional of connected Green functions $W_\Gamma[J,\lambda]$ using as action $\Gamma[\phi]$ and having divided Γ and J by λ:

$$\exp\Big\{ iW_\Gamma[J,\lambda]\Big\} = \int \mathcal{D}\phi \exp\Big\{\frac{i}{\lambda}\Big(\Gamma[\phi] + \int \mathrm{d}^4x\, \phi^r(x) J_r(x)\Big)\Big\} \ . \tag{1.78}$$

If one does a perturbative (Feynman diagram) expansion of $W_\Gamma[J,\lambda]$, the propagators are given by the inverse of the quadratic piece in $\frac{\Gamma}{\lambda}$ and hence contribute a factor λ, while every vertex gets a factor $\frac{1}{\lambda}$ as does an external line. This yields an overall factor $\lambda^{I-V} = \lambda^{L-1}$ where L is the number of loops. Thus the loop-expansion of $W_\Gamma[J,\lambda]$ reads

$$W_\Gamma[J,\lambda] = \sum_{L=0}^{\infty} \lambda^{L-1} W_\Gamma^{(L)}[J,\lambda=1] \ . \tag{1.79}$$

One isolates the tree graphs ($L=0$) by taking the limit $\lambda \to 0$: $\lim_{\lambda\to 0}\Big(\lambda W_\Gamma[J,\lambda]\Big) = W_\Gamma^{(0)}[J,\lambda=1]$. But $iW_\Gamma^{(0)}[J,\lambda=1] = iW_\Gamma^{(0)}[J]$ is the sum of connected *tree* diagrams computed as if the action were $\Gamma[\phi]$. On the other hand, in the limit $\lambda \to 0$, one can use the stationary phase (saddle point) to evaluate (1.78) and get

$$\exp\left\{\frac{i}{\lambda}W_\Gamma^{(0)}[J]\right\} \sim \exp\left\{\frac{i}{\lambda}\Big(\Gamma[\phi_J] + \int \mathrm{d}^4x\,\phi_J^r(x)J_r(x)\Big)\right\} \quad \text{where} \quad \frac{\delta\Gamma}{\delta\phi}\bigg|_{\phi=\phi_J} = -J\ . \tag{1.80}$$

There is some constant of proportionality which has some finite limit as $\lambda \to 0$ and which contributes an order λ^0 piece to the exponent, but nothing at order $\frac{1}{\lambda}$. We see that $W_\Gamma^{(0)}$ is the (inverse) Legendre transform of Γ. On the other hand, the (inverse) Legendre transform of Γ is the ordinary $W[J]$. We conclude that

$$W[J] = W_\Gamma^{(0)}\ , \tag{1.81}$$

and the full generating functional of connected Green functions is indeed given as a sum of connected tree diagrams computed with propagators and vertices taken from the effective action Γ.

If we let

$$\Gamma[\varphi] = \sum_{n=0}^{\infty}\frac{1}{n!}\int \mathrm{d}^4x_1\ldots\mathrm{d}^4x_n\ \Gamma^{(n)}_{r_1\ldots r_n}(x_1,\ldots,x_n)\ \varphi^{r_1}(x_1)\ldots\varphi^{r_n}(x_n)\ , \tag{1.82}$$

the $\Gamma^{(n)}$ for $n \geq 3$ are the so-called proper vertices, and the complete (connected) propagators $G^C_{(2)}(x,y)$ are given (cf. (1.27) and (1.35)) by $-i\big(-\Gamma^{(2)}\big)^{-1}(x,y)$. This can also be seen more formally as

$$\begin{aligned} G^{C,r,s}_{(2)}(x,y) &= -i\frac{\delta}{\delta J_r(x)}\frac{\delta}{\delta J_s(y)}W[J] = -i\frac{\delta}{\delta J_r(x)}\phi_J^s(y) \\ \Gamma^{(2)}_{r,s}(x,y) &= \frac{\delta}{\delta\varphi^r(x)}\frac{\delta}{\delta\varphi^s(y)}\Gamma[\varphi] = -\frac{\delta}{\delta\varphi^r(x)}j_{\varphi^s}(y)\ . \end{aligned} \tag{1.83}$$

It follows that

$$\boxed{G^C_{(2)} \equiv -i\Delta' = i\big(\Gamma^{(2)}\big)^{-1}\ .} \tag{1.84}$$

Since an arbitrary connected diagram is obtained once and only once as a tree diagram using these complete propagators and proper vertices, the proper vertices must be one-particle irreducible (1PI) amputated n-point functions:

$\Gamma[\varphi]$ is the generating functional of one-particle irreducible (1PI) diagrams.

As an example, consider a hermitean scalar field. The full propagator is of the form $-i\Delta'(p) = -i\big(p^2+m^2-\Pi^*(p)\big)^{-1}$ so that $\Gamma^{(2)}(p) = -p^2-m^2+\Pi^*(p)$. Clearly, $-p^2-m^2$ is the contribution from the quadratic part of the classical action and Π^* contains the loop-contributions.

A few remarks:

- In later sections, we will be much concerned with possible divergences occurring in loop-diagrams and their cancellation by counterterms. Since a tree diagram is never divergent if the vertices and propagators are finite, it is clear that any diagram will be finite if the $\Gamma^{(n)}$ are. Hence the issue of renormalisation can be entirely discussed at the level of the $\Gamma^{(n)}$. More precisely, one can expand Γ in powers of the bare φ^s_B or of the renormalized φ^s_R related by the same relation as the fields ψ_B and ψ_R, namely

$$\varphi^s_B = \sqrt{Z_s}\,\varphi^s_R \ , \tag{1.85}$$

implying

$$\Gamma^{(n)}_{B\ r_1\ldots r_n}(x_1,\ldots,x_n) = \left[\prod_{j=1}^n Z_{r_j}^{-1/2}\right]\ \Gamma^{(n)}_{R\ r_1\ldots r_n}(x_1,\ldots,x_n) \ . \tag{1.86}$$

The $\Gamma^{(n)}_B$ and $\Gamma^{(n)}_R$ are called the bare and renormalized n-point vertex functions. The vertex functions that should be finite after removing the regularization are the $\Gamma^{(n)}_R$.

- Quite often one encounters a somewhat different notion of effective action: in a theory with two sorts of fields, say ϕ and ψ, one might only be interested in Green functions of one sort of fields, say the ϕ. This happens in particular if the other sort corresponds to very heavy particles that do not appear as asymptotic states in a scattering experiment, though they still do contribute to intermediate loops. Let $S[\phi,\psi] = S_1[\phi] + S_2[\psi] + S_{12}[\phi,\psi]$. We only introduce sources J for the ϕ and define

$$Z[J] = \int \mathcal{D}\phi\mathcal{D}\psi \exp\left\{i\Big(S[\phi,\psi]+\int \phi^r J_r\Big)\right\} = \int \mathcal{D}\phi \exp\left\{i\Big(S_1[\phi]+\widetilde{W}[\phi]+\int \phi^r J_r\Big)\right\} , \tag{1.87}$$

where

$$\exp\left\{i\widetilde{W}[\phi]\right\} = \int \mathcal{D}\psi \exp\left\{i\Big(S_2[\psi] + S_{12}[\phi,\psi]\Big)\right\} . \tag{1.88}$$

Then, for reasons that are obvious from (1.87), $S_1[\phi] + \widetilde{W}[\phi]$ is referred to as the effective action for the field ϕ obtained after integrating out the field ψ. Note that often $\widetilde{W}[\phi]$ still allows to obtain certain Green function of the ψ-field. Suppose e.g. that the coupling between the two sorts of fields is $S_{12}[\phi,\psi] \sim \phi\mathcal{F}(\psi)$. Then, by taking functional derivatives of $\widetilde{W}[\phi]$ with respect to ϕ one generates vacuum expectation values of time-ordered products of the $\mathcal{F}(\Psi)$. A standard example is spinor quantum electrodynamics with ψ playing the role of the fermions and ϕ of the gauge field. It is relatively easy to integrate out the fermions since they only appear quadratically in the action. This yields a determinant which can be exponentiated into $\widetilde{W}$ and is interpreted as a single fermion loop with arbitrarily many gauge fields attached.

- There is a different, sometimes more direct way to compute the quantum effective action $\Gamma[\varphi]$:

$$\exp\big(i\Gamma[\varphi]\big) = \int_{\text{1PI only}} \mathcal{D}\phi\ \exp\big(iS[\varphi+\phi]\big) \ , \tag{1.89}$$

where the subscript "1PI" instructs us to keep only 1PI diagrams in a perturbative evaluation of the functional integral. To see why this equation is correct, it is best to look at an example. Consider a scalar ϕ^4-theory with $S[\phi] = \int(-\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{g}{24}\phi^4)$. Then

$$S[\varphi+\phi] = S[\varphi]+\int(\partial^2\varphi-m^2\varphi-\frac{g}{6}\varphi^3)\phi-\frac{1}{2}\int((\partial\phi)^2+m^2\phi^2)-\int(\frac{g}{4}\varphi^2\phi^2+\frac{g}{6}\varphi\phi^3+\frac{g}{24}\phi^4)\ . \quad (1.90)$$

If one computes the functional integral (1.89) in perturbation theory one sees that (i) $S[\varphi]$ can be taken in front of the integral, (ii) the (free) ϕ-propagator is the same as before, (iii) one now has vertices with one, two, three and four ϕ-lines attached. However, the vertices with only one line attached cannot lead to 1PI-diagrams and we can drop the term linear in ϕ. Thus only the interactions quadratic, cubic and quartic in ϕ remain and they exactly generate all diagrams where at every vertex one has either two external φ and two internal ϕ-lines, or one external φ and three internal ϕ-lines or only four internal ϕ-lines. With the restriction to 1PI diagrams only, the perturbation theory will exactly yield the generating functional of all 1PI diagrams, connected or not, i.e. $\exp\big(i\Gamma[\varphi]\big)$. It should also be clear that the cubic and higher terms in ϕ only contribute to two- and higher-loop 1PI diagrams. Thus if we are only interested in the one-loop approximation to $\Gamma[\varphi]$ it is enough to keep only the part of the interactions that is quadratic in ϕ. On the other hand, this quadratic part cannot generate any contributions beyond one loop and the latter are necessarily 1PI. We have in general:

$$\begin{aligned} e^{i\Gamma_{1-\mathrm{loop}}[\varphi]} &= e^{iS[\varphi]}\int\mathcal{D}\phi\ \exp\left(\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y\ \phi[x]\,\frac{\delta^2 S[\varphi]}{\delta\varphi(x)\delta\varphi(y)}\,\phi(y)\right) \\ &= e^{iS[\varphi]}\ \left(\mathrm{Det}\frac{\delta^2 S[\varphi]}{\delta\varphi(x)\delta\varphi(y)}\right)^{\mp 1/2}\ , \end{aligned} \quad (1.91)$$

with the power of the determinant depending on whether ϕ is bosonic or fermionic.

1.4.3 Symmetries and Slavnov-Taylor identities

Symmetries of the classical action lead, via Noether's theorem, to conserved currents, at least classically, and in many cases also at the quantum level. Since the quantum effective action equals the classical action plus quantum corrections, one might expect that the former shares the symmetries of the latter. We will show when this is indeed the case.

Suppose that under the infinitesimal transformation

$$\phi^r(x) \to \phi'^r \equiv \phi^r(x) + \epsilon F^r(x,\phi) \quad (1.92)$$

the action *and* the functional integral measure are invariant:

$$S[\phi'] = S[\phi] \quad , \quad \mathcal{D}\phi' \equiv \prod_r \mathcal{D}\phi'^r = \mathcal{D}\phi \equiv \prod \mathcal{D}\phi^r\ . \quad (1.93)$$

One then has (suppressing the indices r)

$$\begin{aligned} Z[J] &= \int\mathcal{D}\phi\ e^{iS[\phi]+i\int\phi J} = \int\mathcal{D}\phi'\ e^{iS[\phi']+i\int\phi' J} = \int\mathcal{D}\phi\ e^{iS[\phi]+i\int(\phi+\epsilon F)J} \\ &= \int\mathcal{D}\phi\ e^{iS[\phi]+i\int\phi J}(1+i\epsilon\int FJ) = Z[J] + i\epsilon\int\mathcal{D}\phi\ \int FJ\ e^{iS[\phi]+i\int\phi J}\ , \end{aligned} \quad (1.94)$$

where we first renamed the integration variable from ϕ to ϕ' and then identified ϕ' with the transformed field (1.92). Hence,

$$0 = \int \mathcal{D}\phi \int \mathrm{d}^4x\, F^r(x,\phi) J_r(x)\, e^{iS[\phi]+i\int \phi J} = Z[J] \int \mathrm{d}^4x\, \langle F^r(x)\rangle_J\, J_r(x)\ , \tag{1.95}$$

for every J. Recall that $\frac{\delta\Gamma}{\delta\varphi^r} = -J_{\varphi,r}$ with $J_{\varphi,r}$ such that $\langle \Phi^r\rangle_J = \varphi^r$. Choosing J_r in (1.95) to equal this $J_{\varphi,r}$, we can rewrite (1.95) as

$$\boxed{\int \mathrm{d}^4x\, \langle F^r(x)\rangle_{J_\varphi}\, \frac{\delta\Gamma}{\delta\varphi^r(x)} = 0\ .} \tag{1.96}$$

This identity is called *Slavnov-Taylor identity.* It states that $\Gamma[\varphi]$ is invariant under $\varphi^r \to \varphi^r + \epsilon\langle F^r(x)\rangle_{J_\varphi}$. In general, for a non-linear transformation, $\langle F^r(x)\rangle_{J_\varphi}$ is different from $F^r(x,\varphi)$, and the symmetry of the quantum effective action is different ("quantum-corrected") from the symmetry of the classical action. For a *linear* classical symmetry, one can go further. Suppose now that $F^r(x,\phi) = f^r(x) + \int \mathrm{d}^4y\, t^r{}_s\, \phi^s(y)$. Then $\langle F^r(x)\rangle_{J_\varphi} = f^r + \int \mathrm{d}^4y\, t^r{}_s\, \langle\phi^s(y)\rangle_{J_\varphi} = f^r + \int \mathrm{d}^4y\, t^r{}_s\, \varphi^s(y) \equiv F^r(x,\varphi)$. In this case, (1.96) states that $\Gamma[\varphi]$ is invariant under $\varphi^r(x) \to {\varphi'}^r \equiv \varphi^r(x) + \epsilon F^r(x,\varphi)$:

> If the action and measure are invariant under a *linear* field transformation, then so is the quantum effective action $\Gamma[\varphi]$.

1.5 Functional integral formulation of QED

We will now apply the functional integral formalism to the particularly important example of quantum electrodynamics. We will consider Lagrangian densities of the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu + \mathcal{L}_{\rm matter}(\Psi^l, \partial_\mu\Psi^l)\ , \tag{1.97}$$

where J^μ is a conserved *matter* current (i.e. $\partial_\mu J^\mu = 0$ by the classical Euler-Lagrange equations). Lagrangians of the form (1.97) include in particular those of spinor electrodynamics, which describe the coupling of a charged spin $\frac{1}{2}$ Dirac field to the electromagnetic fields.[6]

1.5.1 Coulomb gauge

Due to the gauge symmetry $A_\mu \to A_\mu + \partial_\mu\lambda$ and $\Psi^l \to e^{iq_l\lambda}\Psi^l$, the "naive" canonical formalism does not apply. In particular, the canonical momenta are $\Pi^\mu = F^{\mu 0}$ and obviously then $\Pi^0 = 0$: A_0 has a vanishing canonical momentum. A vanishing momentum is a constraint on the canonical variables. One has to distinguish so-called first class and second class constraints. The first class constraints always correspond to a local (gauge) symmetry and can be eliminated by a gauge choice.

[6]They do not include scalar electrodynamics though, which has couplings $\sim \phi^\dagger\phi A_\mu A^\mu$ or, equivalently, in which case the current J^μ depends on A^μ.

Possibly remaining second class constraints can be dealt with either by Dirac quantization or by the functional integral formalism in the way we will see now.

We adopt the Coulomb gauge $\vec{\nabla}\cdot\vec{A}=0$. This fixes A^0 in terms of J^0 and thus eliminates A^0 and Π^0 as canonical pair, hence eliminates the first class constraints. It leaves as second class constraint the Coulomb gauge condition itself and a corresponding condition on the momenta: $\vec{\nabla}\cdot\vec{\Pi}_\perp=0$, where $\Pi_{\perp,j}=\Pi_j-\partial_j A^0=\dot{A}_j$. Upon working out the Hamiltonian one finds that it is given by

$$
\begin{aligned}
H(\vec{A},\vec{\Pi}_\perp,\Psi^l,P_l) &= \int \mathrm{d}^3x\left[\frac{1}{2}\vec{\Pi}_\perp^2+\frac{1}{2}(\vec{\nabla}\wedge\vec{A})^2-\vec{J}\cdot\vec{A}\right]+V_{\text{Coulomb}}+H_{\text{matter}}(\Psi^l,P_l)\ ,\\
V_{\text{Coulomb}} &= \int \mathrm{d}^3x\,\frac{1}{2}J^0A^0=\frac{1}{2}\int \mathrm{d}^3x\,\mathrm{d}^3y\,\frac{J^0(t,\vec{x})J^0(t,\vec{y})}{4\pi|\vec{x}-\vec{y}|}\ ,
\end{aligned}
\tag{1.98}
$$

where H_{matter} is the part of the Hamiltonian that does not depend on the gauge field or the $\Pi_\perp$.

Our starting point for the functional integral formulation is the Hamiltonian formalism. The two constraints $\vec{\nabla}\cdot\vec{A}=0$ and $\vec{\nabla}\cdot\vec{\Pi}_\perp=0$ will be enforced by inserting the factors $\prod_x\delta(\vec{\nabla}\cdot\vec{a})\equiv\prod_{\vec{x},t}\delta\big((\vec{\nabla}\cdot\vec{a})(\vec{x},t)\big)$ and $\prod_x\delta(\vec{\nabla}\cdot\vec{\pi})$ inside the integral. (We write a instead of A and $\vec{\pi}$ instead of $\vec{\Pi}_\perp$ for the integration variables.) To simplify the discussion suppose the matter Hamiltonian H_{matter} is quadratic in the P_l (with a constant matrix A_{lk}) and that the operators $\mathcal{O}_{A_j}$ do not depend on the P_l, so that they can be straightforwardly integrated out. Hence[7]

$$
\begin{aligned}
&\langle T\{\mathcal{O}_A\,\mathcal{O}_B\ldots\}\rangle_{\text{vac}}\\
&=\int\mathcal{D}\vec{a}\,\mathcal{D}\vec{\pi}\,\prod_l\mathcal{D}\psi_l\,\prod_x\delta(\vec{\nabla}\cdot\vec{a})\,\prod_x\delta(\vec{\nabla}\cdot\vec{\pi})\ \mathcal{O}_A\ \mathcal{O}_B\ \ldots\ \times\\
&\qquad\times\exp\left\{i\int\mathrm{d}^4x\left[\vec{\pi}\cdot\partial_0\vec{a}-\frac{1}{2}\vec{\pi}^2-\frac{1}{2}(\vec{\nabla}\wedge\vec{a})^2+\vec{j}\cdot\vec{a}+\mathcal{L}_{\text{matter}}\right]-i\int\mathrm{d}tV_{\text{Coulomb}}\right\}.
\end{aligned}
\tag{1.99}
$$

To appreciate the role of the $\delta(\ldots)$, recall the formula $\delta(f(x))=\sum_a\frac{1}{|f'(x_a)|}\delta(x-x_a)$ with the x_a being the solutions of $f(x)=0$. For N variables x^i, this reads $\delta^{(N)}(f^i(x))=\sum_a\frac{1}{|\det J(x_a)|}\delta^{(N)}(x^i-x^i_a)$ with $J^i_j=\partial f^i/\partial x^j$. Thus, $\prod_x\delta(\vec{\nabla}\cdot\vec{a})=\frac{1}{|\mathrm{Det}\partial_3|}\prod_x\delta\big(a_3+\partial_3^{-1}(\partial_1a_1+\partial_2a_2)\big)$, and we see that imposing the Coulomb gauge amounts to eliminating the functional integration over one out of the 3 fields $a_j(t,\vec{x})$, as expected. Similarly, the insertion of $\prod_x\delta(\vec{\nabla}\cdot\vec{\pi})$ eliminates the integration over the corresponding canonically conjugate momentum.

It is often useful to rewrite a functional δ as a functional integral over an auxiliary field, e.g.

$$
\prod_x\delta(\vec{\nabla}\cdot\vec{\pi})\ =\int\mathcal{D}f\,\exp\left\{i\int\mathrm{d}^4x\,f(x)\vec{\nabla}\cdot\vec{\pi}(x)\right\}\ .
\tag{1.100}
$$

We will also suppose that the operators $\mathcal{O}$ do not depend on the $\vec{\pi}$. Then the only part in (1.99)

[7] Above we denoted $\langle(\ldots)\rangle_{\text{vac}}=\langle\Omega|\,(\ldots)\,|\Omega\rangle=\frac{\langle\text{vac,out}|(\ldots)|\text{vac,in}\rangle}{\langle\text{vac,out}|\text{vac,in}\rangle}$. Since $\langle\text{vac},\text{out}\,|\text{vac},\text{in}\rangle=e^{i\gamma_{\text{vac}}}$ is just a (constant) phase, we will drop it together with other constants and simply write $\langle(\ldots)\rangle_{\text{vac}}$ instead of $\langle\text{vac},\text{out}|\,(\ldots)\,|\text{vac},\text{in}\rangle$, with the understanding that overall constants are either unimportant or should be fixed in the end by dividing by the same expression without the operators $\mathcal{O}_A\,\mathcal{O}_B\ldots$.

that does depend on the $\vec{\pi}$ is

$$\int\mathcal{D}\vec{\pi}\prod_x\delta(\vec{\nabla}\cdot\vec{\pi})\exp\left\{i\int\mathrm{d}^4x\left[\vec{\pi}\cdot\partial_0\vec{a}-\frac{1}{2}\vec{\pi}^2\right]\right\}=\int\mathcal{D}\vec{\pi}\,\mathcal{D}f\exp\left\{i\int\mathrm{d}^4x\left[-\vec{\nabla}f\cdot\vec{\pi}+\vec{\pi}\cdot\partial_0\vec{a}-\frac{1}{2}\vec{\pi}^2\right]\right\}$$
$$=\int\mathcal{D}f\exp\left\{i\int\mathrm{d}^4x\,\frac{1}{2}\left(\partial_0\vec{a}-\vec{\nabla}f\right)^2\right\}$$
$$=\exp\left\{i\int\mathrm{d}^4x\,\frac{1}{2}(\partial_0\vec{a})^2\right\}\int\mathcal{D}f\exp\left\{i\int\mathrm{d}^4x\left[-\frac{1}{2}f\vec{\nabla}^2f+f\vec{\nabla}\cdot\partial_0\vec{a}\right]\right\}\ ,\qquad(1.101)$$

up to an irrelevant overall multiplicative constant which we do not write explicitly. This expression (1.101) is to be inserted into the remaining integral. But then $\vec{a}$ is constrained by $\vec{\nabla}\cdot\vec{a}=0$ which implies $\vec{\nabla}\cdot\partial_0\vec{a}=0$ and the term $f\vec{\nabla}\cdot\partial_0\vec{a}$ in the exponent in the third line of (1.101) does not contribute. The integral over $\mathcal{D}f$ then only gives another irrelevant constant. We arrive at

$$\langle T\{\mathcal{O}_A\,\mathcal{O}_B\ldots\}\rangle_{\rm vac}=\int\mathcal{D}\vec{a}\,\prod_l\mathcal{D}\psi_l\prod_x\delta(\vec{\nabla}\cdot\vec{a})\ \mathcal{O}_A\ \mathcal{O}_B\ \ldots\ \times$$
$$\times\exp\left\{i\int\mathrm{d}^4x\left[\frac{1}{2}(\partial_0\vec{a})^2-\frac{1}{2}(\vec{\nabla}\wedge\vec{a})^2+\vec{j}\cdot\vec{a}+\mathcal{L}_{\rm matter}\right]-i\int\mathrm{d}tV_{\rm Coulomb}\right\}.\qquad(1.102)$$

1.5.2 Lorentz invariant functional integral formulation and α-gauges

Let us rewrite this functional integral in a manifestly Lorentz invariant form. First note that

$$\int\mathcal{D}a_0\exp\left\{i\int\mathrm{d}^4x\left[-a^0j^0+\frac{1}{2}(\vec{\nabla}a^0)^2\right]\right\}=\exp\left\{i\int\mathrm{d}^4x\,\frac{1}{2}j^0(\vec{\nabla}^2)^{-1}j^0\right\}$$
$$=\exp\left\{-i\int\mathrm{d}t\ \frac{1}{2}\int\mathrm{d}^3x\,\mathrm{d}^3y\,\frac{j^0(t,\vec{x})j^0(t,\vec{y})}{4\pi|\vec{x}-\vec{y}|}\right\}=\exp\left\{-i\int\mathrm{d}t\,V_{\rm Coulomb}\right\}\ .\qquad(1.103)$$

Furthermore,

$$\begin{aligned}-\frac{1}{4}f_{\mu\nu}f^{\mu\nu}&=-\frac{1}{2}\partial_\mu a_\nu\partial^\mu a^\nu+\frac{1}{2}\partial_\mu a_\nu\partial^\nu a^\mu\\&=\frac{1}{2}\partial_0a_i\partial_0a_i-\frac{1}{2}\partial_ia_j\partial_ia_j+\frac{1}{2}\partial_ia_0\partial_ia_0+\frac{1}{2}\partial_ia_j\partial_ja_i-\frac{1}{2}\partial_0a_i\partial_ia_0-\frac{1}{2}\partial_ia_0\partial_0a_i\\&=\frac{1}{2}(\partial_0\vec{a})^2+\frac{1}{2}(\vec{\nabla}a^0)^2-\frac{1}{2}(\vec{\nabla}\wedge\vec{a})^2-\vec{\nabla}(\partial_0\vec{a}a_0)+a_0\vec{\nabla}\cdot(\partial_0\vec{a})\ .\end{aligned}\qquad(1.104)$$

The last term of the last line vanishes due to the constraint, and the next to last term is a total derivative. Inserting the expression (1.103) of $\exp\left\{-i\int\mathrm{d}t\,V_{\rm Coulomb}\right\}$ into (1.102) one gets

$$\langle T\{\mathcal{O}_A\,\mathcal{O}_B\ldots\}\rangle_{\rm vac}=\int\prod_\mu\mathcal{D}a^\mu\ \prod_l\mathcal{D}\psi_l\prod_x\delta(\vec{\nabla}\cdot\vec{a})\ \mathcal{O}_A\ \mathcal{O}_B\ \ldots\ \times$$
$$\times\exp\left\{i\int\mathrm{d}^4x\left[\frac{1}{2}(\partial_0\vec{a})^2-\frac{1}{2}(\vec{\nabla}\wedge\vec{a})^2+\vec{j}\cdot\vec{a}-a^0j^0+\frac{1}{2}(\vec{\nabla}a^0)^2+\mathcal{L}_{\rm matter}\right]\right\}$$
$$=\int\prod_\mu\mathcal{D}a^\mu\ \prod_l\mathcal{D}\psi_l\prod_x\delta(\vec{\nabla}\cdot\vec{a})\ \mathcal{O}_A\ \mathcal{O}_B\ \ldots\ \exp\left\{i\big[S[a_\mu,\psi_l]\big]\right\}\ ,\qquad(1.105)$$

with

$$S[a_\mu, \psi_l] = \int \mathrm{d}^4x \, \mathcal{L}[a_\mu, \psi_l] = \int \mathrm{d}^4x \, \left(-\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + j_\mu a^\mu + \mathcal{L}_{\text{matter}} \right) . \tag{1.106}$$

Now everything is manifestly Lorentz invariant except the insertion $\prod_x \delta(\vec{\nabla} \cdot \vec{a})$ which fixes the gauge. Let us now suppose that not only the action $S[a_\mu, \psi_l]$ is gauge invariant but also the operators $\mathcal{O}_A \, \mathcal{O}_B \, \ldots$, e.g.[8] $\mathcal{O}_1(x) = F_{\mu\nu}(x) F^{\mu\nu}(x)$ or $\mathcal{O}_2 = \exp\left(\oint \mathrm{d}x^\mu A_\mu(x) \right)$. Moreover, we will assume that the product of the mesures $\prod_\mu \mathcal{D}a_\mu$ et $\prod_l \mathcal{D}\psi_l$ is gauge invariant. One can show rather easily that $\prod_\mu \mathcal{D}a_\mu$ is gauge invariant, but the invariance of $\mathcal{D}\psi_l$ is not always warranted. As we will see later-on, in the presence of chiral fermions, this measure generally is *not* invariant and one has an anomaly. Different chiral fermions contribute additively to the anomaly and, in a consistent theory, the sum of all anomalous contributions must vanish so that $\prod_l \mathcal{D}\psi_l$ indeed is gauge invariant. With these assumptions, the only gauge non-invariant term in (1.105) is the gauge-fixing term $\prod_x \delta(\vec{\nabla} \cdot \vec{a})$. Recall that the gauge transformations act as

$$a_\mu \to a_{\mu\Lambda} = a_\mu + \partial_\mu \Lambda \quad , \quad \psi_l \to \psi_{l,\Lambda} = e^{i q_l \Lambda} \psi_l \ , \tag{1.107}$$

with $\Lambda = \Lambda(x)$ completely arbitrary. It could even depend on the a_μ themselves.[9]

One can rewrite the functional integral (1.105) by first changing the names of the integration variables from a_μ and ψ_l to $a_{\mu\Lambda}$ and $\psi_{l,\Lambda}$, then identifying the latter with the gauge transformed fields (1.107). The gauge invariance of the action and the operators $\mathcal{O}$ gives

$$\langle T\{\mathcal{O}_A \, \mathcal{O}_B \ldots\}\rangle_{\text{vac}} = \int \prod_\mu \mathcal{D}a^\mu_\Lambda \, \prod_l \mathcal{D}\psi_{l,\Lambda} \prod_x \delta(\vec{\nabla} \cdot \vec{a}_\Lambda) \ \mathcal{O}_A \ \mathcal{O}_B \ \ldots \ \exp\{iS[a_\mu, \psi_l]\} . \tag{1.108}$$

Since the Λ-dependence came about by a simple change of integration variables, we know that the expression on the r.h.s. actually does not depend on Λ, whatever this function may be. Let us choose

$$\Lambda(t, \vec{x}) = \widetilde{\Lambda}(t, \vec{x}) - \int \mathrm{d}^3y \, \frac{\partial_0 a^0(t, \vec{y})}{4\pi |\vec{y} - \vec{x}|} \ , \tag{1.109}$$

with an a_μ independent $\tilde{\Lambda}$.

Let us check what happens to the measure $\prod_\mu \mathcal{D}a_\mu$ under this field-dependent gauge transformation. One has

$$\begin{aligned} a_{\mu\Lambda}(t, \vec{x}) &= a_\mu(t, \vec{x}) + \partial_\mu \widetilde{\Lambda}(t, \vec{x}) - \frac{\partial}{\partial x^\mu} \int \mathrm{d}^3y \, \frac{\partial_0 a^0(t, \vec{y})}{4\pi |\vec{y} - \vec{x}|} \\ &= a_\mu(t, \vec{x}) + \partial_\mu \widetilde{\Lambda}(t, \vec{x}) + \frac{\partial}{\partial x^\mu} \int \mathrm{d}^3y \, \mathrm{d}t' \left(\frac{\partial}{\partial t'} \delta(t - t') \right) \frac{a^0(t', \vec{y})}{4\pi |\vec{y} - \vec{x}|} \ , \end{aligned} \tag{1.110}$$

[8] The definition of composite operators like $F_{\mu\nu}(x) F^{\mu\nu}(x)$ requires some normal order type prescription preserving the gauge invariance. In practice, one most often computes $\langle T\{\mathcal{O}_A \, \mathcal{O}_B \ldots\}\rangle_{\text{vac}}$ with $\mathcal{O}_{A_i}$ that are *not* gauge invariant, as e.g. the propagator $\langle T\{A_\mu(x) A_\nu(y)\}\rangle_{\text{vac}}$. Nevertheless, such gauge non-invariant quantities should only appear at an intermediate stage, and the final result should be gauge invariant.

[9] A familiar example of Λ depending on a_μ is the transformation that allows oneself to go to a given gauge, e.g. $\Lambda(t, \vec{x}) = \frac{1}{4\pi} \int \mathrm{d}^3y \frac{\vec{\nabla} \cdot \vec{a}(t, \vec{y})}{|\vec{x} - \vec{y}|}$ to go to Coulomb gauge.

so that

$$\begin{aligned}\frac{\delta a_{0\Lambda}(t,\vec{x})}{\delta a_0(t',\vec{y})} &= \delta^{(4)}(x-y)+\frac{1}{4\pi|\vec{y}-\vec{x}|}\partial_t\partial_{t'}\delta(t-t')\\ \frac{\delta a_{i\Lambda}(t,\vec{x})}{\delta a_0(t',\vec{y})} &= \frac{\partial}{\partial x^i}\frac{1}{4\pi|\vec{y}-\vec{x}|}\partial_{t'}\delta(t-t') \qquad , \qquad \frac{\delta a_{\mu\Lambda}(t,\vec{x})}{\delta a_i(t',\vec{y})}=\delta^i_\mu\,\delta^{(4)}(x-y)\ ,\end{aligned} \tag{1.111}$$

resulting in a non-trivial Jacobian.

$$\prod_\mu \mathcal{D}a_{\mu\Lambda}=\prod_\mu \mathcal{D}a_\mu\ \times\ \mathrm{Det}\left(\delta^{(4)}(x-y)-\frac{1}{4\pi|\vec{y}-\vec{x}|}\delta''(t-t')\right)\ . \tag{1.112}$$

Although non-trivial, this Jacobian only contributes an irrelevant field- and $\widetilde{\Lambda}$-independent constant to the functional integral (which we drop as usual). Similarly, in the absence of anomalies, $\prod_l \mathcal{D}\psi_{l,\Lambda}=\prod_l \mathcal{D}\psi_l$. Thus the only effect of this gauge transformation with Λ is

$$\delta(\vec{\nabla}\cdot\vec{a}_\Lambda)=\delta(\vec{\nabla}\cdot\vec{a}+\vec{\nabla}^2\widetilde{\Lambda}+\partial_0 a^0)=\delta(\partial_\mu a^\mu+\vec{\nabla}^2\widetilde{\Lambda})\ , \tag{1.113}$$

which allows to write (1.108) as

$$\langle T\big\{\mathcal{O}_A\,\mathcal{O}_B\ldots\big\}\rangle_{\rm vac}=\int\prod_\mu \mathcal{D}a^\mu\ \prod_l \mathcal{D}\psi_l\prod_x \delta(\partial_\mu a^\mu+\vec{\nabla}^2\widetilde{\Lambda})\ \mathcal{O}_A\ \mathcal{O}_B\ \ldots\ \exp\left\{iS[a_\mu,\psi_l]+i\epsilon{-}\text{terms}\right\}. \tag{1.114}$$

By construction, both sides of this equation are independent of $\widetilde{\Lambda}$. We can multiply both sides by $\exp\left[-i\frac{\alpha}{2}\int \mathrm{d}^4x\,(\vec{\nabla}^2\widetilde{\Lambda})^2\right]$ (with $\alpha>0$) and integrate $\mathcal{D}\widetilde{\Lambda}=\left(\mathrm{Det}\vec{\nabla}^2\right)^{-1}\mathcal{D}(\vec{\nabla}^2\widetilde{\Lambda})$. On the l.h.s. this results in yet another irrelevant constant factor. Interchanging the order of integrations on the r.h.s., we finally arrive at

$$\boxed{\langle T\big\{\mathcal{O}_A\,\mathcal{O}_B\ldots\big\}\rangle_{\rm vac}=\int\prod_\mu \mathcal{D}a^\mu\ \prod_l \mathcal{D}\psi_l\ \mathcal{O}_A\ \mathcal{O}_B\ \ldots\ \exp\left\{iS_{\rm eff}[a_\mu,\psi_l]\right\},} \tag{1.115}$$

with

$$\boxed{S_{\rm eff}[a_\mu,\psi_l]=S[a_\mu,\psi_l]-\frac{\alpha}{2}\int \mathrm{d}^4x\,(\partial_\mu a^\mu)^2\ ,} \tag{1.116}$$

where the parameter α is often called the gauge parameter. Starting from the *manifestly unitary* canonical formalism in Coulomb gauge, we have obtained a *manifestly Lorentz invariant* functional integral representation of the vacuum expectation values of time-ordered products of gauge invariant Heisenberg operators. As already noted, we will use this equation (1.115) to compute $\langle T\big\{\mathcal{O}_A\,\mathcal{O}_B\ldots\big\}\rangle_{\rm vac}$ even if the $\mathcal{O}$ are not gauge invariant. In this case, one has to remember that the result is unphysical and depends on the gauge-parameter α. Nevertheless, any final physical result (like S-matrix elements) must be gauge invariant and independent of α.

Let us now determine the propagator of the gauge field $\langle T\big[A_\mu(x)A_\nu(y)\big]\rangle_{\rm vac}^{\rm libre} = -i\Delta_{\mu\nu}(x,y)$. According to (1.36), the propagator is given by the inverse of the quadratic part of the action $S_{\rm eff}$:

$$\begin{aligned} S_{\rm eff}\big|_{\rm quadratic} &= \int {\rm d}^4x \left[-\frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{\alpha}{2}(\partial_\mu a^\mu)^2\right] + (i\epsilon - {\rm terms}) \\ &= \frac{1}{2}\int {\rm d}^4x\, a^\mu \left[\eta_{\mu\nu}\partial_\rho\partial^\rho - (1-\alpha)\partial_\mu\partial_\nu\right] a^\nu + (i\epsilon - {\rm terms}) \\ &\equiv -\frac{1}{2}\int {\rm d}^4x\, {\rm d}^4y\ a^\mu(x)\mathcal{D}_{\mu\nu}(x,y)a^\nu(y)\ , \end{aligned} \tag{1.117}$$

with

$$\begin{aligned} \mathcal{D}_{\mu\nu}(x,y) &= \left[-\eta_{\mu\nu}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial x_\rho} + (1-\alpha)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - i\epsilon\,\eta_{\mu\nu}\right]\delta^{(4)}(x-y) \\ &= \int \frac{{\rm d}^4q}{(2\pi)^4}\left[\eta_{\mu\nu}q^2 - (1-\alpha)q_\mu q_\nu - i\epsilon\,\eta_{\mu\nu}\right] e^{iq(x-y)}\ . \end{aligned} \tag{1.118}$$

The propagator is $-i\Delta_{\mu\nu}(x,y)$ where $\Delta = \mathcal{D}^{-1}$, i.e.

$$\Delta_{\mu\nu}(x,y) \equiv \Delta_{\mu\nu}(x-y) = \int \frac{{\rm d}^4q}{(2\pi)^4}\Delta_{\mu\nu}(q)e^{iq(x-y)}\ , \tag{1.119}$$

with

$$\boxed{\Delta_{\mu\nu}(q) = \frac{\eta_{\mu\nu}}{q^2 - i\epsilon} + \frac{1-\alpha}{\alpha}\frac{q_\mu q_\nu}{(q^2-i\epsilon)^2}\ .} \tag{1.120}$$

As expected for a gauge-dependent quantity, the propagator depends explicitly on α. Note that the limit $\alpha \to 0$ is singular since it would remove the gauge-fixing. The choice $\alpha = 1$ is called Feynman gauge and yields $\Delta_{\mu\nu}(q) = \frac{\eta_{\mu\nu}}{q^2-i\epsilon}$ which is particularly simple, while $\alpha \to \infty$ gives $\Delta_{\mu\nu}(q) = \frac{\eta_{\mu\nu}}{q^2-i\epsilon} - \frac{q_\mu q_\nu}{(q^2-i\epsilon)^2}$ and is called the Landau or Lorenz gauge (since $\alpha \to \infty$ strictly enforces the Lorenz gauge condition $\partial_\mu a^\mu = 0$).

1.5.3 Feynman rules of spinor QED

Let us now specify the matter part of the action to be that of an electron Dirac field (of charge $q = -e$ with $e > 0$) interacting with the electromagnetic field:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \overline{\psi}(\partial\!\!\!/ + ieA\!\!\!/ + m)\psi\ , \tag{1.121}$$

and

$$\mathcal{L}_{\rm eff} = \mathcal{L} - \frac{\alpha}{2}(\partial_\mu a^\mu)^2\ . \tag{1.122}$$

<u>The Feynman rules for S-matrix elements</u> then are:

- photon propagator : $\dfrac{-i}{(2\pi)^4}\left(\dfrac{\eta_{\mu\nu}}{q^2-i\epsilon} + \dfrac{1-\alpha}{\alpha}\dfrac{q_\mu q_\nu}{(q^2-i\epsilon)^2}\right)$,

- electron/positron propagator : $\frac{-i}{(2\pi)^4}\frac{1}{i\not{k}+m-i\epsilon} \equiv \frac{-i}{(2\pi)^4}\frac{(-i\not{k}+m)}{k^2+m^2-i\epsilon}$
- vertex : $(2\pi)^4 e\gamma^\mu\delta^{(4)}(k-k'+q)$,
- initial photon : $\frac{e_\mu}{(2\pi)^{3/2}\sqrt{2p^0}}$, final photon : $\frac{e^*_\mu}{(2\pi)^{3/2}\sqrt{2p^0}}$,
- initial electron : $\frac{u}{(2\pi)^{3/2}}$, final electron : $\frac{\bar{u}}{(2\pi)^{3/2}}$,
- initial positron : $\frac{\bar{v}}{(2\pi)^{3/2}}$, final positron : $\frac{v}{(2\pi)^{3/2}}$,
- integrate over all *internal* four-momenta.

The Feynman rules for Green-functions are the same, except that one associates propagators to the external lines instead of the initial/final particle wave-function factors u, v or ϵ.

Most of the integrations over internal momenta are fixed by the $\delta^{(4)}$'s from the vertices. Of course, one overall $\delta^{(4)}$ only enforces conservation of the external four-momenta and thus cannot serve to fix any internal momentum. Thus the number of unconstrained internal momenta is $I-V+1$ if the number of vertices is V and the number of internal lines I. We have already seen that there is the general topological relation (1.45) between I, V and the number of independent loops L in a diagram, $I-V=L-1$. It follows that in any Feynman diagram there are exactly L unconstrained four-momenta to be integrated, one for every loop.

Note that in spinor QED all vertices are tri-valent (3 lines attached). This gives another relation between V, I and the number E of external lines: $3V=2I+E$. Thus in spinor QED

$$3V=2I+E \quad , \quad I-V=L-1 \quad \Rightarrow \quad V=2L+E-2\ , \tag{1.123}$$

and for a given S-matrix element or given Green function (fixed number of external lines) one gets an additional factor of e^2 for every additional loop: one sees very clearly that the perturbative expansion is an expansion in the number of loops and the expansion parameter is the fine structure constant α (not to be confused with the gauge parameter) [10]

$$\alpha=\frac{e^2}{4\pi}\simeq\frac{1}{137}\ . \tag{1.125}$$

[10] One can argue that the expansion parameter for a given S-matrix element is $\frac{\alpha}{4\pi}$ rather than α: every vertex contributes a factor $(2\pi)^4 e$ and every internal line a $(2\pi)^{-4}$. Every integration over a loop momentum d^4k can be expected to give a factor π^2 (the angular integration is estimated to give the volume $2\pi^2$ and $k^3\mathrm{d}k=\frac{1}{2}k^2\mathrm{d}k^2$ gives another $\frac{1}{2}$). Altogether, one has a factor

$$(2\pi)^{4V}e^V(2\pi)^{-4I}\pi^{2L}=(2\pi)^4e^{E-2}\left(\frac{e^2}{16\pi^2}\right)^L=(2\pi)^4e^{E-2}\left(\frac{\alpha}{4\pi}\right)^L\ , \tag{1.124}$$

so that every loop can be expected to yield a factor $\frac{e^2}{16\pi^2}=\frac{\alpha}{4\pi}\simeq 6\times10^{-4}\ll1$.

2 A few results independent of perturbation theory

2.1 Structure and poles of Green functions

There are a few statements that can be made about the structure of the various Green functions independently of any explicit (perturbative) computation, just based on arguments of symmetry, in particular Poincaré invariance. Consider the Fourier transform of a general n-point (Green) function[11]

$$\widehat{G}_{(n)}(q_1, \ldots q_n) \equiv \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n e^{i\sum_r q_n x_n} \langle T\big(\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\big)\rangle_{\rm vac} \ . \tag{2.1}$$

Recall that $\langle T(\ldots)\rangle_{\rm vac} = \langle\Omega|\, T(\ldots)\,|\Omega\rangle$ where $\langle\Omega|$ and $|\Omega\rangle$ are both the in-vacuum. (In perturbation theory, this would be given by the sum of all the corresponding Feynman diagrams with n external lines but excluding all diagrams with vacuum bubbles.) From translational invariance, this Green function must be a product of $\delta^{(4)}(\sum_r q_r)$ times some $\widetilde{G}_{(n)}(q_1, \ldots q_n)$. The latter may contain pieces which are again proportional to some $\delta^{(4)}$ (corresponding to a disconnected part of the Green function) and pieces without such further $\delta^{(4)}$-singularities, but with various poles and branch cuts in various combinations of the momenta. We will concentrate on the poles and their residues. As an example, consider a free scalar theory where $\widetilde{G}_{(2)}$ is just the propagator with a pole at $q_1^2 = q_2^2 = -m^2$ and residue $-i$.

Here we will establish the general structure of the 2-point Green functions close to their poles and then just state the corresponding result for the n-point functions. To begin with, we write explicitly

$$\begin{aligned}\widehat{G}_{(2)}(q_1, q_2) = \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, e^{iq_1x_1+iq_2x_2}\Big[\, &\theta(x_1^0 - x_2^0)\, \langle\Omega|\, \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\,|\Omega\rangle \\ &+ \theta(x_2^0 - x_1^0)\, \langle\Omega|\, \mathcal{O}_2(x_2)\mathcal{O}_1(x_1)\,|\Omega\rangle\Big] \ ,\end{aligned} \tag{2.2}$$

We now insert a complete set of states in the in-basis of the Hilbert space. This basis contains, besides the in-vacuum, the one-particle states $\left|\Psi^{\rm in}_{\vec p,\sigma,n}\right\rangle$, as well as all the multi-particle states. These one-particle states correspond to the physical particles with masses m_n that one can measure as $m_n^2 = -p^2 \equiv -p_\mu p^\mu$ and where $P_\mu \left|\Psi^{\rm in}_{\vec p,\sigma,n}\right\rangle = p_\mu \left|\Psi^{\rm in}_{\vec p,\sigma,n}\right\rangle$. Thus

$$\mathbf{1} = |\Omega\rangle\,\langle\Omega| + \sum_{n,\sigma} \int \mathrm{d}^3p\ \left|\Psi^{\rm in}_{\vec p,\sigma,n}\right\rangle \left\langle\Psi^{\rm in}_{\vec p,\sigma,n}\right| + \ldots \ , \tag{2.3}$$

where $+\ldots$ indicates all the contributions from multi-particle states. These are defined as states depending on the total momentum $\vec p_{\rm tot}$, as well as at least one more continuous variable. Thus

$$\begin{aligned}\langle\Omega|\, \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\,|\Omega\rangle &= \langle\Omega|\, \mathcal{O}_1(x_1)\,|\Omega\rangle\, \langle\Omega|\, \mathcal{O}_2(x_2)\,|\Omega\rangle \\ &+ \sum_{n,\sigma}\int \mathrm{d}^3p\ \langle\Omega|\, \mathcal{O}_1(x_1)\left|\Psi^{\rm in}_{\vec p,\sigma,n}\right\rangle \left\langle\Psi^{\rm in}_{\vec p,\sigma,n}\right| \mathcal{O}_2(x_2)\,|\Omega\rangle + \ldots \ .\end{aligned} \tag{2.4}$$

By translational invariance one has

$$\langle\Omega|\, \mathcal{O}_1(x_1)\left|\Psi^{\rm in}_{\vec p,\sigma,n}\right\rangle = \langle\Omega|\, e^{-iP_\mu x_1^\mu}\mathcal{O}_1(0)e^{iP_\mu x_1^\mu}\left|\Psi^{\rm in}_{\vec p,\sigma,n}\right\rangle = e^{ip_\mu x_1^\mu}\,\langle\Omega|\, \mathcal{O}_1(0)\left|\Psi^{\rm in}_{\vec p,\sigma,n}\right\rangle \ , \tag{2.5}$$

[11] Here we use the same notation $\widehat{G}_{(n)}$ for $\widehat{G}_{(n)}(x_1, \ldots x_n)$ and its Fourier transform $\widehat{G}_{(n)}(q_1, \ldots q_n)$. Also, we consider general Heisenberg operators $\mathcal{O}_j$ rather than just the "elementary" fields Ψ_{l_j}, since most of the argument does not depend on the form of the operators.

as well as $\langle\Omega|\,\mathcal{O}_1(x_1)\,|\Omega\rangle = \langle\Omega|\,\mathcal{O}_1(0)\,|\Omega\rangle$. Most often, the $\mathcal{O}_j$ transform non-trivially under the Lorentz group or some internal symmetry group (that leaves the vacuum invariant) in which case $\langle\Omega|\,\mathcal{O}_j(0)\,|\Omega\rangle = 0$. In general, one has

$$\begin{aligned}\langle\Omega|\,\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\,|\Omega\rangle &= \langle\Omega|\,\mathcal{O}_1(0)\,|\Omega\rangle\,\langle\Omega|\,\mathcal{O}_2(0)\,|\Omega\rangle \\ &+ \sum_{n,\sigma}\int \mathrm{d}^3p\, e^{ip(x_1-x_2)}\underbrace{\langle\Omega|\,\mathcal{O}_1(0)\,\big|\Psi^{\rm in}_{\vec p,\sigma,n}\big\rangle}_{\equiv\, M_{\mathcal{O}_1}(\vec p,\sigma,n)}\underbrace{\big\langle\Psi^{\rm in}_{\vec p,\sigma,n}\big|\,\mathcal{O}_2(0)\,|\Omega\rangle}_{\equiv\, M^*_{\mathcal{O}_2^\dagger}(\vec p,\sigma,n)}+\ldots\ . \end{aligned} \tag{2.6}$$

Let us insist that the p^0 are "on-shell", i.e. $p^0=\sqrt{\vec p^{\,2}+m_n^2}\equiv\omega_n(\vec p)$. When inserted into (2.2), the first line of (2.6), if non-vanishing, yields a contribution $\sim\int \mathrm{d}^4x_1\mathrm{d}^4x_2 e^{-iq_1x_1-iq_2x_2}\sim\delta^{(4)}(q_1)\delta^{(4)}(q_2)$ corresponding to a disconnected piece. Concentrate now on the contributions of the one-particle states. Writing

$$\theta(x_1^0-x_2^0) = -\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi i}\frac{e^{-i\omega(x_1^0-x_2^0)}}{\omega+i\epsilon}\ , \tag{2.7}$$

they are

$$\begin{aligned}\widehat G_{(2)}(q_1,q_2)\Big|_{\rm one\ particle} &= \sum_{n,\sigma}\frac{i}{2\pi}\int\frac{\mathrm{d}\omega}{\omega+i\epsilon}\,\mathrm{d}^3p\int \mathrm{d}^4x_1\,\mathrm{d}^4x_2\ e^{iq_1x_1+iq_2x_2}\ \times\\ &\times\Big[e^{ip(x_1-x_2)}e^{-i\omega(x_1^0-x_2^0)}M_{\mathcal{O}_1}(\vec p,\sigma,n)M^*_{\mathcal{O}_2^\dagger}(\vec p,\sigma,n)+\ e^{-ip(x_1-x_2)}e^{+i\omega(x_1^0-x_2^0)}M_{\mathcal{O}_2}(\vec p,\sigma,n)M^*_{\mathcal{O}_1^\dagger}(\vec p,\sigma,n)\Big]\\ &= i(2\pi)^7\delta^{(4)}(q_1+q_2)\sum_{n,\sigma}\int\frac{\mathrm{d}\omega}{\omega+i\epsilon}\,\mathrm{d}^3p\ \times\\ &\Big[\delta^{(3)}(\vec p-\vec q_2)\delta(\omega-q_2^0+p^0)M_{\mathcal{O}_1}(\vec q_2,\sigma,\nu)M^*_{\mathcal{O}_2^\dagger}(\vec q_2,\sigma,n)+\delta^{(3)}(\vec p-\vec q_1)\delta(\omega-q_1^0+p^0)M_{\mathcal{O}_2}(\vec q_1,\sigma,n)M^*_{\mathcal{O}_1^\dagger}(\vec q_1,\sigma,n)\Big]\\ &= i(2\pi)^7\delta^{(4)}(q_1+q_2)\sum_{n,\sigma}\left[\frac{M_{\mathcal{O}_1}(\vec q_2,\sigma,n)M^*_{\mathcal{O}_2^\dagger}(\vec q_2,\sigma,n)}{q_2^0-\omega_n(\vec q_2)+i\epsilon}+\frac{M_{\mathcal{O}_2}(\vec q_1,\sigma,\nu)M^*_{\mathcal{O}_1^\dagger}(\vec q_1,\sigma,n)}{q_1^0-\omega_n(\vec q_1)+i\epsilon}\right]\end{aligned} \tag{2.8}$$

This expression clearly exhibits the poles due to the one-particle intermediate states. The poles are at $q_1^0=-q_2^0=\pm\omega_n(\vec q_1)=\pm\sqrt{m_n^2+\vec q_1^{\,2}}$, i.e. on the mass shell of the intermediate physical particle. One can show that the multi-particle intermediate states do not lead to poles but to branch cuts.

We will be mostly interested in the case where the Heisenberg operators $\mathcal{O}_j$ correspond to the elementary fields Ψ_l appearing in the Lagrangian, specifically $\mathcal{O}_1=\Psi_l$ and $\mathcal{O}_2=\Psi_k^\dagger$ so that the above result reads

$$\widehat G_{(2)}(q_1,q_2)\Big|_{\rm poles}= i(2\pi)^7\delta^{(4)}(q_1+q_2)\sum_{n,\sigma}\left[\frac{M_{\Psi_l}(\vec q_2,\sigma,n)M^*_{\Psi_k}(\vec q_2,\sigma,n)}{q_2^0-\omega_n(\vec q_2)+i\epsilon}+\frac{M_{\Psi_k^\dagger}(\vec q_1,\sigma,n)M^*_{\Psi_l^\dagger}(\vec q_1,\sigma,n)}{q_1^0-\omega_n(\vec q_1)+i\epsilon}\right]\ . \tag{2.9}$$

Let us compare with the result that would have been obtained in a free theory of a field of species n^* and mass m_* where $\Psi_l(x)=\psi_l(x)=\sum_\sigma\int\frac{\mathrm{d}^3p}{(2\pi)^{3/2}}\big(u_l(\vec p,\sigma,n^*)a(\vec p,\sigma,n^*)e^{ipx}+v_l(\vec p,\sigma,n^*)a_c^\dagger(\vec p,\sigma,n^*)e^{-ipx}\big)$. In this case, the only intermediate states that contribute are the one-particle states of species n^* created by $a^\dagger$ and $a_c^\dagger$. Furthermore, $M_{\psi_l}(\vec q_2,\sigma,n^*)=\frac{1}{(2\pi)^{3/2}}\,u_l(\vec q_2,\sigma,n^*)$ and $M_{\psi_k^\dagger}(\vec q_1,\sigma,n^*)=\frac{1}{(2\pi)^{3/2}}\,v_k^*(\vec q_1,\sigma,n^*)$, so that

$$\begin{aligned}\widehat G^{\rm free}_{(2)}(q_1,q_2) &= i(2\pi)^4\delta^{(4)}(q_1+q_2)\sum_\sigma\left[\frac{u_l(\vec q_2,\sigma,n^*)u_k^*(\vec q_2,\sigma,n^*)}{q_2^0-\omega_{n^*}(\vec q_2)+i\epsilon}+\frac{v_l(\vec q_1,\sigma,n^*)v_k^*(\vec q_1,\sigma,n^*)}{q_1^0-\omega_{n^*}(\vec q_1)+i\epsilon}\right]\\ &\equiv -i(2\pi)^4\delta^{(4)}(q_1+q_2)\ \Delta^{m_*}_{lk}(q_2)\end{aligned} \tag{2.10}$$

where $-i\Delta^{m_*}_{lk}(q)$ is the usual free propagator with mass m_*. The similarity between (2.9) and (2.10) is no coincidence. Indeed, by Lorentz invariance, the matrix element $M_{\Psi_l}(\vec q_1,\sigma,n)$ is constraint to equal the corresponding $u_l(\vec q_1,\sigma,n)$, up to a normalization, and similarly for the $M_{\Psi_k^\dagger}(\vec q_1,\sigma,n)$ and $v_l^*(\vec q_1,\sigma,n)$.

(Recall that for every irreducible representation of the Lorentz group one can determine the corresponding coefficients u_l and v_l solely from the transformation properties – up to a normalization). Hence:

$$M_{\Psi_l}(\vec{q}_2,\sigma,n)=N_\Psi^n\,\frac{u_l(\vec{q}_2,\sigma,n)}{(2\pi)^{3/2}}\quad,\quad M_{\Psi_l^\dagger}(\vec{q}_1,\sigma,n)=N_{\Psi^\dagger}^n\,\frac{v_l^*(\vec{q}_1,\sigma,n)}{(2\pi)^{3/2}}\ ,\tag{2.11}$$

where the normalization constants N_Ψ^n and $N_{\Psi^\dagger}^n$ may differ at most by a phase. In (2.9), the contributions to the residue of a given pole at some[12] $q_2^2=-m_*^2$ come from those one-particle states n that have a mass m_n equal to m_*.

Combining the results (2.9), (2.11) and (2.10), we finally get for the behaviour of the 2-point function:

$$\begin{aligned}\widehat{G}^{lk}_{(2)}(q_1,q_2)\Big|_{\text{pole at }q_1^2=-m_*^2} &\sim (2\pi)^4\delta^{(4)}(q_1+q_2)\Big[\sum_{n\,|\,m_n=m_*}|N_\Psi^n|^2\Big](-i)\,\Delta_{lk}^{m_*}(q_1)\\ &= |\mathcal{N}_\Psi^{m_*}|^2\ \widehat{G}^{lk}_{(2)\text{free},m_*}(q_1,q_2)\ .\end{aligned}\tag{2.12}$$

The lesson to remember is the following: in general, the 2-point function of the interacting theory is very complicated, with branch cuts and poles. Equation (2.12) states that, as $q_2^2\to-m^2$, where m is the mass of a physical one-article state such that $\langle\Omega|\,\Psi_l(0)\,\big|\Psi^{\text{in}}_{\vec{p},\sigma,n}\big\rangle\neq0$, the 2-point function behaves as the 2-point function of a free field of mass m, up to a normalization constant.

These results can be generalized to an arbitrary n-point function depending on momenta $q_1,\ldots q_n$: Such an n-point function has a pole whenever, for any subset I of $\{1,\ldots n\}$, the combination $q_I=\sum_{j\in I}q_j$ is such that $q_I^2=-m^2$ with m being equal to the mass of any one-particle state $\big|\Psi^{\text{in}}_{\vec{p},\sigma,n}\big\rangle$ that has non-vanishing matrix elements with $\prod_{j\in I}\mathcal{O}_j^\dagger\,|\Omega\rangle$ and with $\prod_{j\notin I}\mathcal{O}_j\,|\Omega\rangle$. More precisely, if we suppose $I=\{1,\ldots r\}$, $q\equiv q_I=q_1+\ldots+q_r=-q_{r+1}-\ldots-q_n$ then, as $q^0\to\sqrt{\vec{q}^2+m^2}$

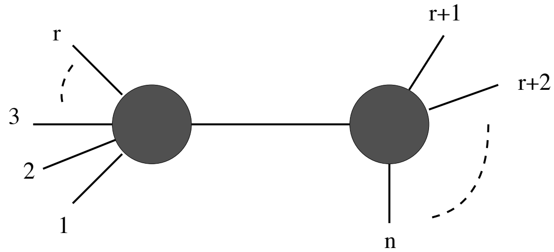
$$G\sim\frac{-2i\sqrt{\vec{q}^{\,2}+m^2}}{q^2+m^2-i\epsilon}(2\pi)^7\delta^{(4)}(q_1+\ldots+q_n)\sum_\sigma M_{0|q\sigma}(q_2,\ldots q_r)M_{q,\sigma|0}(q_{r+2},\ldots q_n)\ ,\tag{2.13}$$

with

$$\begin{aligned}(2\pi)^4\delta^{(4)}(\sum_{s=1}^r q_s-p)\,M_{0|p\sigma}(q_2,\ldots q_r)&=\int\mathrm{d}^4x_1\ldots\mathrm{d}^4x_r\,e^{i\sum_{s=1}^r q_sx_s}\times\langle\Omega|\,T\big(\mathcal{O}_1(x_1)\ldots\mathcal{O}_r(x_r)\big)\,|\Psi_{p,\sigma}\rangle\\ (2\pi)^4\delta^{(4)}(\sum_{s=r+1}^n q_s+p)\,M_{p\sigma|0}(q_{r+2},\ldots q_n)&=\int\mathrm{d}^4x_{r+1}\ldots\mathrm{d}^4x_n\,e^{i\sum_{r+1}^n q_sx_s}\times\\ &\qquad\times\langle\Psi_{p,\sigma}|\,T\big(\mathcal{O}_{r+1}(x_{r+1})\ldots\mathcal{O}_n(x_n)\big)\,|\Omega\rangle\ .\end{aligned}\tag{2.14}$$

Again, the proof uses only translation invariance, the causal structure implied by the time-ordering and the fact that multiparticle intermediate states produce branch cuts rather than poles. Note that the above pole structure is exactly what one expects from a Feynman diagram with a single internal line for a particle of mass m connecting a part of the diagram, with the first r operators $\mathcal{O}_i$ attached, to another part, with the last $n-r$ operators $\mathcal{O}_i$ attached, as shown in the figure. However, the above property is much more general in that the particle of mass m need not be one corresponding to an elementary field in the Lagrangian but could correspond to a complicated bound state.

[12]This is an abuse of language: when we say a pole at $q^2=-m^2$, since $\frac{-1}{q^2+m^2}=\frac{1}{2\omega_m(\vec{q})}\left(\frac{1}{q^0+\omega_m(\vec{q})}-\frac{1}{q^0-\omega_m(\vec{q})}\right)$, we really mean a pole at $q^0=\omega_m(\vec{q})$ and a pole at $q^0=-\omega_m(\vec{q})$.



2.2 Complete propagators, the need for field and mass renormalization

In the above formula (2.12) the 2-point function on the left-hand-side is the Fourier transform of $\langle T\big(\Psi_l(x_1)\Psi_k^\dagger(x_2)\big)\rangle_{\rm vac}$ where the $\Psi_l(x)$ are the Heisenberg operators that evolve with the full Hamiltonian. This is also referred to as the full or complete propagator, while on the right-hand-side appears the free propagator as entering the Feynman rules. More precicely, the Heisenberg operators Ψ_l correspond to the fields as they appear in the (interacting) Lagrangian and are accordingly normalized. Such fields will be called bare fields and we write $\Psi_{l,B}(x)$. Actually, in most theories (at least in perturbation theory) the only one-particle states $\Psi^{\rm in}_{\vec p,\sigma,n}$ that are such that $\big\langle \Psi^{\rm in}_{\vec p,\sigma,n}\big|\, \Psi_{l,B}(0)\,|\Omega\rangle \neq 0$ all have the same mass, and then there is only a pole at $q_1^2=-m^2$, with m being in general different from the mass parameter appearing in the Lagrangian of the bare field $\Psi_{l,B}$ and which we call the bare mass m_B. Hence, we can rewrite (2.12) as

$$\int {\rm d}^4x_1\, {\rm d}^4x_2\, e^{iq_1x_1+iq_2x_2}\langle T\big(\Psi_{l,B}(x_1)\Psi^\dagger_{k,B}(x_2)\big)\rangle_{\rm vac}\Big|_{q_1^2\to -m^2} \sim -i\Delta^m_{lk}(q_1)\,|N_\psi|^2\,(2\pi)^4\delta^{(4)}(q_1+q_2)\ . \quad (2.15)$$

We can get rid of the factor $|N_\psi|^2$ on the right-hand-side by deviding by it and defining

$$\Psi_{l,R}=\frac{1}{N_\psi}\Psi_{l,B}\ . \quad (2.16)$$

Then, *close to its pole* at $q_1^2=-m^2$, the two-point function of $\Psi_{l,R}$ behaves as the two-point function of a free field with mass m. A field with this behaviour is called a *renormalized field* and m the *renormalized mass*. In the sequel, to simplify the notation, we will not write the subscript R for the renormalized fields and masses. We will study in some detail how this goes for the different types of fields.

2.2.1 Example of a scalar field ϕ

Call the interacting real scalar ϕ_B with Lagrangian

$$\mathcal{L}=-\frac{1}{2}(\partial_\mu\phi_B)^2-\frac{1}{2}m_B^2\phi_B^2-V_B(\phi_B)\ . \quad (2.17)$$

There is no reason to expect that ϕ_B has a correctly normalized two-point function or that this function has a pole at $-m_B^2$. Let

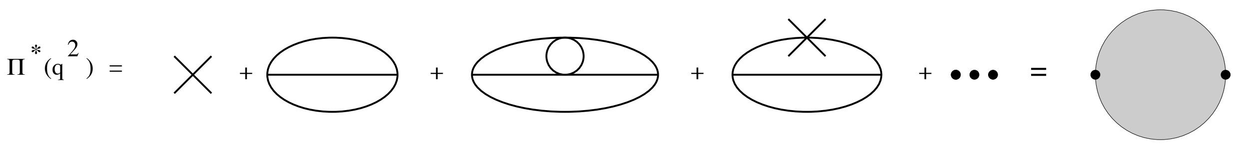
$$\boxed{\begin{aligned}\phi_B &= \sqrt{Z}\,\phi \quad \Leftrightarrow \quad \phi=Z^{-1/2}\,\phi_B\\ m^2 &= m_B^2+\delta m^2\ ,\end{aligned}} \quad (2.18)$$

and require that ϕ is correctly normalized, i.e. that $\langle\Omega|\,\phi(0)\,\big|\Psi^{\rm in}_{\vec q}\big\rangle = \frac{1}{(2\pi)^{3/2}2\sqrt{\vec q^{\,2}+m^2}}$ and that its two-point function have a pole at $-m^2$. This will fix Z and δm^2 as functions of m_B^2 and the couplings. Then ϕ is called the renormalized field and m the renormalized mass. A straightforward rewriting of the Lagrangian yields

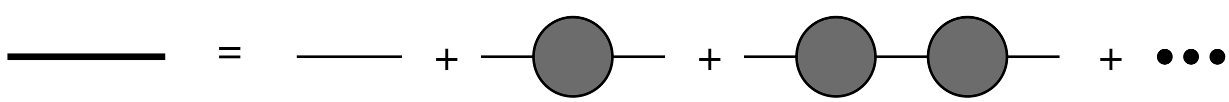
$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}Z(\partial_\mu\phi)^2 - \frac{1}{2}(m^2-\delta m^2)Z\phi^2 - V_B(\sqrt{Z}\phi)\\ &= \underbrace{\left(-\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2\right)}_{\mathcal{L}_0} + \underbrace{\left(-\frac{1}{2}(Z-1)\big((\partial_\mu\phi)^2+m^2\phi^2\big) + \frac{1}{2}Z\delta m^2\phi^2 - V_B(\sqrt{Z}\phi)\right)}_{\mathcal{L}_1} .\end{aligned} \tag{2.19}$$

The strategy is to treat $\mathcal{L}_0$ as the free part of the Lagrangian and $\mathcal{L}_1$ as the interaction. All the terms in $\mathcal{L}_1$ involving factors of $(Z-1)$ or δm^2 will be called "counterterms".

The full propagator of the renormalized field is called $\Delta'(q)$. It is conveniently expressed in terms of the one-particle irreducible propagator. In general, a one-particle irreducible (1PI) diagram is a connected diagram that will not become disconnected by just cutting a single line. More precisely, let $i(2\pi)^4\Pi^*(q^2)$ be the complete one-particle irreducible propagator of ϕ with two external free propagators $-\frac{i}{(2\pi)^4}\frac{1}{q^2+m^2-i\epsilon}$ removed. Graphically, with a ϕ^4 interaction Π^* is given by



The first term equals $-(Z-1)(q^2+m^2)+Z\delta m^2$ and is entirely due to the counterterms, while the other terms involve loops (possibly including counterterms inside the loops) and contribute to $\Pi^*_{\rm loops}(q^2)$. The full propagator then is related to the one-particle irreducible propagator[13] by (see Figure below)



$$\begin{aligned}&-\frac{i}{(2\pi)^4}\Delta'(q)\\ &\quad= -\frac{i}{(2\pi)^4}\frac{1}{q^2+m^2-i\epsilon} + \left(-\frac{i}{(2\pi)^4}\frac{1}{q^2+m^2-i\epsilon}\right) i(2\pi)^4\Pi^*(q^2)\left(-\frac{i}{(2\pi)^4}\frac{1}{q^2+m^2-i\epsilon}\right)+\ldots\\ &\quad= -\frac{i}{(2\pi)^4}\frac{1}{q^2+m^2-i\epsilon}\left(1-\frac{\Pi^*(q^2)}{q^2+m^2-i\epsilon}\right)^{-1} = -\frac{i}{(2\pi)^4}\frac{1}{q^2+m^2-\Pi^*(q^2)-i\epsilon} .\end{aligned} \tag{2.20}$$

In summary:

$$\boxed{\Delta'(q) = \left(q^2+m^2-\Pi^*(q^2)-i\epsilon\right)^{-1}\ ,} \tag{2.21}$$

[13] To simplify the discussion, we exclude the possibility of tri-linear self-interactions $\sim\phi^3$ which would lead to "tadpole" diagrams.

with

$$\Pi^*(q^2) = -(Z-1)(q^2+m^2) + Z\delta m^2 + \Pi^*_{\rm loop}(q^2) \ . \tag{2.22}$$

The so far arbitrary constants Z and δm^2 must be fixed by the normalization requirements: we required that Δ' has a pole at $q^2 = -m^2$. This implies

$$\boxed{\Pi^*(-m^2) = 0 \ .} \tag{2.23}$$

Correct normalization of the two-point function translates into the residue of the pole of Δ' be one:

$$\begin{aligned} \Delta'(q)\Big|_{q^2=-m^2+\delta q^2} &= \Big(\delta q^2 - \Pi^*(-m^2+\delta q^2) - i\epsilon\Big)^{-1} = \left(\delta q^2 - \delta q^2 \frac{{\rm d}\Pi^*}{{\rm d}q^2}(-m^2) + \mathcal{O}((\delta q^2)^2) - i\epsilon\right)^{-1} \\ &\sim \left(1 - \frac{{\rm d}\Pi^*}{{\rm d}q^2}(-m^2)\right)^{-1} \frac{1}{\delta q^2 - i\epsilon} \ , \end{aligned} \tag{2.24}$$

which leads to

$$\boxed{\frac{\rm d}{{\rm d}q^2}\Pi^*(q^2)\Big|_{q^2=-m^2} = 0 \ .} \tag{2.25}$$

Inserting (2.22) into (2.23) and (2.25) yields

$$Z\delta m^2 = -\Pi^*_{\rm loop}(-m^2) \tag{2.26}$$

$$Z = 1 + \frac{\rm d}{{\rm d}q^2}\Pi^*_{\rm loop}(q^2)\Big|_{q^2=-m^2} \ , \tag{2.27}$$

which determines Z and δm^2 in terms of $\Pi^*_{\rm loop}$.

Clearly, in any *generic* interacting theory, $\Pi^*_{\rm loop}$ will be non-vanishing (and in particular non-vanishing at $q^2 = -m^2$ with a non-vanishing first derivative). Hence, in such a generic interacting theory there will always be renormalization of the wave-function ($Z \neq 1$) and of the mass ($\delta m^2 \neq 0$). The necessity of renormalization has nothing to do with diverging, infinite loop-integrals, but is a generic feature of interacting theories in order to have correctly normalized two-point functions with poles at physical values of q^2.

It is interesting to substitute the values (2.26) and (2.27) of Z and $Z\delta m^2$ into eq. (2.22) for $\Pi^*(q^2)$ to get

$$\boxed{\Pi^*(q^2) = \Pi^*_{\rm loop}(q^2) - \Pi^*_{\rm loop}(-m^2) - \frac{\rm d}{{\rm d}q^2}\Pi^*_{\rm loop}(q^2)\Big|_{q^2=-m^2}(q^2+m^2) \ .} \tag{2.28}$$

We see that the (renormalized) one-particle irreducible complete propagator Π^* is given by its loop contribution $\Pi^*_{\rm loop}$ with its two first terms in a Taylor series expansion around $q^2 = -m^2$ subtracted!

To be completely clear, let us insist that $-i\Delta'$ is the full propagator of the renormalized field ϕ, related to the full propagator $-i\Delta'_B$ of the bare field ϕ_B by

$$-i\Delta'(x-y) = \langle T\phi(x)\phi(y)\rangle = \frac{1}{Z}\langle T\phi_B(x)\phi_B(y)\rangle = \frac{1}{Z}(-i)\Delta'_B(x-y) \ . \tag{2.29}$$

From our previous relations one finds

$$\begin{aligned}\Delta'(q) &= \Big(q^2+m^2-\Pi^*(q^2)-i\epsilon\Big)^{-1} = \Big(q^2+m^2+(Z-1)(q^2+m^2)-Z\delta m^2-\Pi^*_{\rm loop}(q^2)-i\epsilon\Big)^{-1} \\ &= \frac{1}{Z}\Big(q^2+m^2-\delta m^2-\frac{1}{Z}\Pi^*_{\rm loop}(q^2)-i\epsilon\Big)^{-1} = \frac{1}{Z}\Big(q^2+m_B^2-\Pi^*_{\rm B,\,loop}(q^2)-i\epsilon\Big)^{-1} , \end{aligned} \tag{2.30}$$

where we defined

$$\Pi^*_{\rm B,\,loop} = \frac{1}{Z}\Pi^*_{\rm loop} \ . \tag{2.31}$$

Comparing (2.29) and (2.30) we see that

$$\boxed{\begin{aligned}\Delta'_B(q) &= \Big(q^2+m_B^2-\Pi^*_{\rm B,\,loop}(q^2)-i\epsilon\Big)^{-1} \\ \Delta'(q) &= \Big(q^2+m^2-\Pi^*(q^2)-i\epsilon\Big)^{-1} \\ \Delta'_B(q) &= Z\ \Delta'(q)\ .\end{aligned}} \tag{2.32}$$

In particular, in the bare propagator Δ'_B one has $\Pi^*_{\rm B,\,loop}$, missing the contributions from the counterterms. Nevertheless the relation between the full bare propagator and the full renormalized propagator is very simple: they only differ by the factor Z. Let us insist that the renormalization conditions (2.23) and (2.25) are such that the renormalized propagator satisfies

$$\Delta'(q) = \frac{1}{q^2+m^2-i\epsilon}\Big[1+\mathcal{O}(q^2+m^2)\Big] = \Delta(q)\Big[1+\mathcal{O}(q^2+m^2)\Big] \ , \tag{2.33}$$

i.e. up to corrections that vanish on shell, the full renormalized propagator equals the free propagator.

2.2.2 Example of a Dirac field

The Lagrangian is

$$\mathcal{L} = -\bar\psi_B(\partial\!\!\!/ + m_B)\psi_B - V_B(\psi_B) \ , \tag{2.34}$$

with

$$\boxed{\psi = Z_2^{-1/2}\,\psi_B \quad \Leftrightarrow \quad \psi_B = \sqrt{Z_2}\,\psi \quad , \quad m = m_B+\delta m \ .} \tag{2.35}$$

As above, we rewrite $\mathcal{L}$ as

$$\begin{aligned}\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \quad , \qquad \mathcal{L}_0 &= -\bar\psi(\partial\!\!\!/ + m)\psi \\ \mathcal{L}_1 &= -(Z_2-1)\bar\psi(\partial\!\!\!/ + m)\psi + Z_2\delta m\bar\psi\psi - V_B(\sqrt{Z_2}\psi) \ .\end{aligned} \tag{2.36}$$

Denote by $i(2\pi)^4\Sigma^*(k\!\!\!/)$ the one-particle irreducible complete fermion propagator.[14] Let the complete propagator be $\frac{-i}{(2\pi)^4}S'(k)$. Then

$$S'(k) = \frac{1}{ik\!\!\!/+m-i\epsilon} + \frac{1}{ik\!\!\!/+m-i\epsilon}\Sigma^*(k\!\!\!/)\frac{1}{ik\!\!\!/+m-i\epsilon} + \ldots = \frac{1}{ik\!\!\!/+m-\Sigma^*(k\!\!\!/)-i\epsilon} \ , \tag{2.37}$$

[14] When we write $\Sigma^*(k\!\!\!/)$ we mean the following: we will see that Σ^* is of the form $\Sigma^* = f(k^2)\,ik\!\!\!/ + g(k^2)\,\mathbf{1} = f(k\!\!\!/^2)\,ik\!\!\!/ + g(k\!\!\!/^2)\,\mathbf{1}$, which is indeed a function of $k\!\!\!/$.

and

$$\Sigma^*(\not k) = -(Z_2 - 1)(i\not k + m) + Z_2\delta m + \Sigma^*_{\rm loop}(\not k) \ . \tag{2.38}$$

The correct normalization for the complete propagator (pole at $k^2 = -m^2$ and the residue condition) yields

$$\boxed{\Sigma^*(im) = 0 \quad , \quad \frac{\partial}{\partial \not k}\Sigma^*(\not k)\Big|_{\not k = im} = 0 \ .} \tag{2.39}$$

This fixes Z_2 and δm as

$$Z_2\ \delta m \ = \ -\Sigma^*_{\rm loop}(im) \ \ , \tag{2.40}$$

$$Z_2 \ = \ 1 - i\frac{\partial}{\partial \not k}\Sigma^*_{\rm loop}(\not k)\Big|_{\not k = im} \ . \tag{2.41}$$

Again, if we insert these values for Z_2 and $Z_2\,\delta m$ into (2.38) we get

$$\boxed{\Sigma^*(\not k) = \Sigma^*_{\rm loop}(\not k) - \Sigma^*_{\rm loop}(im) - \frac{1}{i}\frac{\partial}{\partial \not k}\Sigma^*_{\rm loop}(\not k)\Big|_{\not k = im}(i\not k + m) \ ,} \tag{2.42}$$

showing again that the renormalized $\Sigma^*(\not k)$ is given by its loop-contributon with its two first terms in a Taylor expansion around $\not k = im$ subtracted.

2.3 Charge renormalization and Ward identities

The Lagrangian for charged particles is invariant under phase rotations of the associated fields. This leads, as usual to a conserved current J^μ and associated conserved charge $Q = \int {\rm d}^3x\, J^0$ which, upon quantization, become operators acting on the states. Since Q commutes with the Hamiltonian we can take all (one-particle) states $|\Psi_{p,\sigma,n}\rangle$ to be eigenstates of Q with

$$Q\,|\Psi_{p,\sigma,n}\rangle = q_n\,|\Psi_{p,\sigma,n}\rangle \ \ , \tag{2.43}$$

as well as $Q\,|\Omega\rangle = 0$. The eigenvalue q_n is called the charge of the particle. On the other hand, in the Lagrangian $\mathcal{L}$ appear parameters $\tilde q_l$ via the covariant derivatives $D_\mu\psi_l = (\partial_\mu - i\tilde q_l A_\mu)\psi_l$ of ψ_l. How are they related?

From the definition of $J^0 = \frac{\partial\mathcal{L}[\psi_l, D_\mu\psi_l]}{\partial A_0} = \frac{\partial\mathcal{L}}{\partial\partial_0\psi_l}(-i\tilde q_l)\psi_l = -i\tilde q_l P_l\psi_l$ and the canonical commutation relations we get $[Q,\Psi_l] = -\tilde q_l\Psi_l$. Hence

$$0 = \langle\Omega|\,Q\,\Psi_l\,|\Psi_{p,\sigma,n}\rangle = \langle\Omega|\,(\Psi_l\,Q + [Q,\Psi_l])\,|\Psi_{p,\sigma,n}\rangle = (q_n - \tilde q_l)\,\langle\Omega|\,\Psi_l\,|\Psi_{p,\sigma,n}\rangle \ \ . \tag{2.44}$$

Thus, whenever $\langle\Omega|\,\Psi_l\,|\Psi_{p,\sigma,n}\rangle \neq 0$ we must have $\tilde q_l = q_n :$ the charge q_n as measured by the eigenvalue of Q equals the parameter $\tilde q_n$ appearing in the covariant derivatives of ψ_n in the Lagrangian. Suppose now we rescale $A_\mu \to \gamma A_\mu = A'_\mu$. Then $J^\mu \to (J')^\mu = \gamma^{-1}J^\mu$ and hence $q_n \to q'_n = \gamma^{-1}q_n$. According to the previous argument, then also $\tilde q_n \to \tilde q'_n = \gamma^{-1}\tilde q_n$. As a result, $\tilde q'_n A'_\mu = \tilde q_n A_m$ and the covariant derivative remains unchanged. It follows that if A_μ is renormalized by some multiplicative factor, all charges are renormalized by the inverse of this factor:

$$\boxed{A^\mu = Z_3^{-1/2}\, A_B^\mu \quad \Leftrightarrow \quad q_l = \sqrt{Z_3}\; q_{B,l} \quad \forall\, l\ .} \tag{2.45}$$

The charge renormalization is the same for all fields! Of course, this is related to the gauge invariance which forces all charged particles to couple in the same way – via the covariant derivative – to the electromagnetic field A_μ. In particular (2.45) shows that if $q_{B,l} = q_{B,l'}$ then also $q_l = q_{l'}$, even if ψ_l and $\psi_{l'}$ have very different non-electromagnetic couplings like e.g. a proton and a positron, as shown in Fig. 5 for a quark and a positron/electron. One sometimes writes $q_B = \sqrt{Z_q} q$ so that (2.45) implies

$$Z_q Z_3 = 1\ . \tag{2.46}$$

Figure 5: Contributions to the electromagnetic vertex function for a quark (left) and an electron/positron (right). The solid lines are fermion propagators, the wavy lines photon propagators and the dashed lines gluon propagators.

One defines the *vertex function* $\Gamma^\mu_{mn}(p,p')$ by

$$\int \mathrm{d}^4x\, \mathrm{d}^4y\, \mathrm{d}^4z\, e^{-ipx-iky+ilz}\, \langle\Omega|\, T\big(J^\mu(x)\Psi_n(y)\bar\Psi_m(z)\big)\, |\Omega\rangle$$
$$\equiv -iq\, S'_{nn'}(k)\Gamma^\mu_{n'm'}(k,l)S'_{m'm}(l)\ (2\pi)^4\, \delta^{(4)}(p+k-l)\ . \tag{2.47}$$

It follows from this definition that Γ^μ is the sum of all vertex graphs (with the two complete Dirac propagators removed, and also no photon propagator): it is the one-particle irreducible 3-point function. To lowest order (free fields) the l.h.s. of (2.47) is $\frac{1}{i\not k+m-i\epsilon}(-iq\gamma^\mu)\frac{1}{i\not l+m-i\epsilon}\delta^{(4)}(p+k-l)$, so that

$$\Gamma^\mu\Big|_{\text{tree}} = \gamma^\mu\ . \tag{2.48}$$

Above we have derived identities due to the universal coupling of the electromagnetic field through the covariant derivative. One of the tools was the commutation relation of the charge operator with the Heisenberg picture quantum fields. Similarly, we now derive a relation between the vertex function and the full fermion propagators, known as *Ward identity*. Using $\partial_\mu J^\mu = 0$ and $[J^0(t,\vec x), \Psi_n(t,\vec y)] =$

$-q\delta^{(3)}(\vec{x}-\vec{y})\,\Psi_n(t,\vec{y})$ we get

$$\begin{aligned}\frac{\partial}{\partial x^\mu}T\Big(J^\mu(x)\Psi_n(y)\bar\Psi_m(z)\Big) &= T\Big(\partial_\mu J^\mu(x)\Psi_n(y)\bar\Psi_m(z)\Big)+\\ &\quad+\delta(x^0-y^0)T\Big([J^0(x),\Psi_n(y)]\bar\Psi_m(z)\Big)+\delta(x^0-z^0)T\Big(\Psi_n(y)[J^0(x),\bar\Psi_m(z)]\Big)\\ &= 0-q\,\delta^{(4)}(x-y)T\big(\Psi_n(y)\bar\Psi_m(z)\big)+q\,\delta^{(4)}(x-z)T\big(\Psi_n(y)\bar\Psi_m(z)\big)\ . \qquad (2.49)\end{aligned}$$

This, together with the definition (2.47), and recalling that $\langle\Omega|\,T\big(\Psi_n(y)\bar\Psi_m(z)\big)\,|\Omega\rangle = -iS'_{nm}(y-z) = -i\int\frac{\mathrm{d}^4q}{(2\pi)^4}e^{iq(y-z)}S'_{nm}(q)$, yields

$$(l-k)_\mu S'(k)\Gamma^\mu(k,l)S'(l) = iS'(l)-iS'(k)\ , \qquad (2.50)$$

or

$$\boxed{(l-k)_\mu\Gamma^\mu(k,l) = i{S'}^{-1}(k)-i{S'}^{-1}(l)\ ,} \qquad (2.51)$$

which is known as the generalized Ward identity. The original form of the Ward identity is obtained by letting $l\to k$ so that

$$\Gamma^\mu(k,k) = -i\frac{\partial}{\partial k_\mu}{S'}^{-1}(k) = \gamma^\mu + i\frac{\partial}{\partial k_\mu}\Sigma^*(\not k)\ . \qquad (2.52)$$

Due to (2.39), the last term vanishes on-shell, i.e. for $\not k = im$, and hence when evaluated between on-shell spinors one simply has

$$\bar u(k)\Gamma^\mu(k,k)u(k) = \bar u(k)\gamma^\mu u(k)\ , \qquad (2.53)$$

so that radiative corrections to the vertex function for the interaction of an on-shell fermion with a zero-momentum photon vanish. But this is exactly the way the electric charge of particles is defined, and we find again that $q\gamma^\mu A_\mu$ is not renormalized. Similarly, (2.51) leads to

$$(l-k)_\mu\,\bar u(k)\Gamma^\mu(k,l)u(l) = 0\ . \qquad (2.54)$$

2.4 Photon propagator and gauge invariance

Gauge invariance implied that the current J^μ is conserved, $\partial_\mu J^\mu = 0$. Then, much as above, one has

$$\begin{aligned}\frac{\partial}{\partial x^\mu}T\Big(J^\mu(x)J^\nu(y)J^\rho(z)\ldots\Big) &= \delta(x^0-y^0)T\Big([J^0(x),J^\nu(y)]J^\rho(z)\Big)\\ &\quad+\delta(x^0-z^0)T\Big(J^\mu(x)[J^0(y),J^\rho(z)]\Big)+\ldots\ . \qquad (2.55)\end{aligned}$$

Now J^ν is a neutral operator and hence $[J^0(x),J^\nu(y)] = 0$. This could be violated by so-called Schwinger terms from defining J^ν properly in terms of the elementary fields at coinciding points. For Dirac fermions and using dimensional regularization, however, no such Schwinger terms arise and one indeed has

$$[J^0(x),J^\nu(y)] = 0 \qquad (2.56)$$

as an operator identity. As a consequence, the r.h.s. of (2.55) vanishes. Let

$$M^{\mu\nu\ldots}_{\beta\alpha}(q_1,q_2,\ldots)=\int \mathrm{d}^4x_1\,\ldots e^{-iq_1x_1-\ldots}\left\langle \Psi^-_\beta\right| T\Big(J^\mu(x_1)J^\nu(x_2)\ldots\Big)\left|\Psi^+_\alpha\right\rangle\ . \tag{2.57}$$

Then (2.55) implies

$$(q_i)_\mu M^{\mu\nu\ldots}_{\beta\alpha}(q_1,q_2,\ldots)=0\ . \tag{2.58}$$

Now any S-matrix element in electrodynamics is of the form

$$\begin{aligned} S_{\beta\alpha}\ &\sim\ \int \mathrm{d}^4q_1\,\mathrm{d}^4q_2\ldots\Delta_{\mu_1\nu_1}(q_1)\Delta_{\mu_2\nu_2}(q_2)\ldots\epsilon^*_{\rho_1}(k_1)\epsilon^*_{\rho_2}(k_2)\ldots\epsilon_{\sigma_1}(l_1)\epsilon_{\sigma_2}(l_2)\ldots\times\\ &\qquad\times\widehat{M}^{\mu_1\nu_1\mu_2\nu_2\ldots\rho_1\rho_2\ldots\sigma_1\sigma_2\ldots}_{\beta\alpha}(q_1,q_2,\ldots,k_1,k_2\ldots,l_1,l_2\ldots)\ , \end{aligned} \tag{2.59}$$

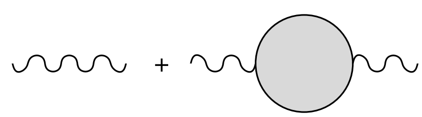
with $\widehat{M}$ being the matrix element of all the matter currents as defined above but with all electromagnetic interactions turned off. Clearly, (2.58) still holds for $\widehat{M}$ and we see that the S-matrix is unchanged if we replace any of the photon propagators $\Delta_{\mu\nu}(q)$ or any of the polarization vectors according to

$$\begin{aligned} \Delta_{\mu\nu}(q)\ &\to\ \Delta_{\mu\nu}(q)+a_\mu q_\nu+q_\mu b_\nu\ ,\\ \epsilon_\rho(k)\ &\to\ \epsilon_\rho(k)+c\,k_\rho\ , \end{aligned} \tag{2.60}$$

with arbitrary four-vectors a_μ, b_μ or scalar c.

The *complete photon propagator* necessarily is given by

$$\Delta'_{\mu\nu}(q)=\Delta_{\mu\nu}(q)+\Delta_{\mu\rho}(q)M^{\rho\sigma}(q)\Delta_{\sigma\nu}(q)\quad : \tag{2.61}$$



where $M^{\rho\sigma}\ \sim\ \sum_n \widehat{M}^{\rho\sigma\mu_1\ldots\mu_{2n}}_{\text{vac,vac}}\Delta_{\mu_1\mu_2}\ldots\Delta_{\mu_{2n-1}\mu_{2n}}$. Indeed, the first term in (2.61) just is the free propagator, and all higher-order corrections are summarized in the second term. Of course, $M^{\rho\sigma}$ is *not* one-particle irreducible. Obviously, it follows from (2.58) that $q_\rho M^{\rho\sigma}=M^{\rho\sigma}q_\sigma=0$. In a general gauge we had $\Delta_{\mu\nu}(q)=\frac{1}{q^2-i\epsilon}\left(\eta_{\mu\nu}-\xi(q^2)\frac{q_\mu q_\nu}{q^2}\right)$ and hence,

$$\Delta'_{\mu\nu}(q)=\Delta_{\mu\nu}(q)+\frac{1}{(q^2-i\epsilon)^2}M_{\mu\nu}\ . \tag{2.62}$$

Then $q^\mu\Delta'_{\mu\nu}(q)=q^\mu\Delta_{\mu\nu}(q)=\frac{1-\xi(q^2)}{q^2-i\epsilon}q_\nu$.

In terms of the complete one-particle irreducible photon propagator $\Pi^*_{\mu\nu}=\Pi^*_{\nu\mu}$ one has (with obvious index contractions)

$$\boxed{\Delta'=\Delta+\Delta\Pi^*\Delta+\ldots=(1-\Delta\Pi^*)^{-1}\Delta\ ,} \tag{2.63}$$

but also

$$\Delta'=\Delta+\Delta\Pi^*\Delta+\Delta\Pi^*\Delta\Pi^*\Delta+\ldots=\Delta+\Delta\Pi^*(\Delta+\Delta\Pi^*\Delta+\ldots)=\Delta+\Delta\Pi^*\Delta'\ . \tag{2.64}$$

Contracting the last equation with q_μ on the left yields[15] $0 = \frac{1-\xi(q^2)}{q^2-i\epsilon} q^\nu \Pi^*_{\nu\rho} \Delta'^{\rho\sigma}$, hence $q^\nu \Pi^*_{\nu\rho} = 0$. It follows that

$$\boxed{\Pi^*_{\mu\nu}(q) = (q^2 \eta_{\mu\nu} - q_\mu q_\nu)\pi(q^2) \ .} \tag{2.65}$$

Then $(\Delta\Pi^*)^{\rho\sigma} = \frac{1}{q^2-i\epsilon} \Pi^{*\rho\sigma}$ so that $(1-\Delta\Pi^*)^{\mu\nu} = \frac{1}{q^2-i\epsilon}\left(q^2\eta^{\mu\nu}(1-\pi) + q^\mu q^\nu \pi\right)$ whose inverse is $(1-\Delta\Pi^*)^{-1}_{\mu\nu} = \frac{1}{1-\pi(q^2)}\left(\eta_{\mu\nu} - \pi(q^2)\frac{q_\mu q_\nu}{q^2}\right)$. Using this in (2.63) gives

$$\Delta'_{\mu\nu} = \left[(1-\Delta\Pi^*)^{-1}\Delta\right]_{\mu\nu} = \frac{\eta_{\mu\nu} - [\xi(q^2)(1-\pi(q^2)) + \pi(q^2)]\frac{q_\mu q_\nu}{q^2}}{(1-\pi(q^2))(q^2-i\epsilon)} \equiv \frac{\eta_{\mu\nu} - \widetilde{\xi}(q^2)\frac{q_\mu q_\nu}{q^2}}{(1-\pi(q^2))(q^2-i\epsilon)} \tag{2.66}$$

As before, we require that the complete photon propagator should have a pole at the physical mass and close to this pole be normalized as the free propagator. Of course, we expect the physical mass to be still zero, although this needs to be verified. Here we do not prove this statement but give evidence for it. Indeed, since $\Pi^*_{\mu\nu}$ is one-particle irreducible it is *not* expected to have any poles at $q^2=0$. It certainly has branch cut singularities and it could, at least in principle, also have poles at $q^2 = -M^2$ due to intermediate *bound* states of mass M. However, we do not expect the latter to have zero mass. If $\Pi^*_{\mu\nu}$ has no pole at $q^2=0$, then $\pi(q^2)$ does not have such a pole either, and then $\Delta'_{\mu\nu}$ keeps its pole at $q^2=0$. Note that *if* $\pi(q^2)$ had a pole at $q^2=0$, say $\pi(q^2) \sim \frac{a}{q^2} - b$ then $\Delta' \sim \frac{1}{q^2(1-a/q^2+b)} = \frac{1}{q^2-a+bq^2} = \frac{1}{1+b}\frac{1}{q^2-\frac{a}{1+b}}$ would have its pole shifted to $q^2 = \frac{a}{1+b}$. On the other hand, if $\pi(q^2)$ has a pole at $q^2=-M^2$ with $M^2 \neq 0$, say $\pi(q^2) \sim \frac{a}{q^2+M^2} - b$ then Δ' keeps its pole at $q^2=0$, but there is an additional pole that appears at $q^2 = -M^2 + \frac{a}{1+b}$. Clearly, this is undesirable, too. Henceforth we assume that indeed $\pi(q^2)$ does not have any pole. Then, for the correctly normalized A_μ the residue of Δ' at the pole should be $\eta_{\mu\nu} - \xi(q^2)\frac{q_\mu q_\nu}{q^2}$ which requires

$$\boxed{\pi(0) = 0 \qquad \text{and} \qquad \pi(q^2) \text{ should not have a pole .}} \tag{2.67}$$

As for the scalar or Dirac fields we can rewrite the (bare) Lagrangian, originally written in terms of the bare fields A^μ_B, in terms of the renormalized fields A^μ and the constant Z_3, and then separate a free and an interaction part. In order to do so, we also need to start with a bare parameter α_B for the "gauge-fixing" term:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F^B_{\mu\nu} F^{\mu\nu}_B - \frac{\alpha_B}{2}(\partial_\mu A^\mu_B)^2 + \ldots = -\frac{1}{4} Z_3\, F_{\mu\nu}F^{\mu\nu} - \frac{\alpha_B Z_3}{2}\,(\partial_\mu A^\mu)^2 + \ldots \\
&= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{\alpha}{2}(\partial_\mu A^\mu)^2 - \frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} + \ldots \ ,
\end{aligned} \tag{2.68}$$

with $\alpha = Z_3\,\alpha_B$. The terms $+\ldots$ involve the couplings to the matter fields and enter only in the *loop*-corrections to the photon propagator. We see that the free propagator for the renormalized field A_μ now is the same $\Delta_{\mu\nu}$ with the same $\xi(\alpha)$ as before, while the quadratic "counterterm"

[15] Except for $\alpha \to \infty$ one has $\xi \neq 1$.

$-\frac{1}{4}(Z_3-1)F_{\mu\nu}F^{\mu\nu}$ gives a purely transverse contribution to the one-particle irreducible propagator $\Pi^*_{\mu\nu}$. Hence it contributes a piece $1-Z_3$ to $\pi(q^2)$ and we conclude

$$\pi(q^2) = 1 - Z_3 + \pi_{\rm loop}(q^2) \ . \tag{2.69}$$

The residue condition (2.67) then determines Z_3 in terms of $\pi_{\rm loop}$ as

$$Z_3 = 1 + \pi_{\rm loop}(0) \ . \tag{2.70}$$

We have shown how all the renormalization constants $Z,\ Z_2$ and now Z_3 are determined by the renormalization conditions in terms of the loop contributions to the various one-particle irreducible functions. In practice though, if one wants to compute these one-particle irreducible functions one does not need to explicitly determine the Z's, as we have seen above for Π^* (for the scalar) or for Σ^* for the fermions. Similarly, for the photon, inserting the value (2.70) of Z_3 into (2.69) simply gives

$$\boxed{\pi(q^2) = \pi_{\rm loop}(q^2) - \pi_{\rm loop}(0) \ ,} \tag{2.71}$$

which clearly satisfies (2.67), and we see (again) that the renormalized $\pi(q^2)$ is given by its loop-contribution, with its first Taylor coefficient subtracted.

3 One-loop radiative corrections in ϕ^4 and QED

We will first work out the one-loop radiative corrections in some detail for QED and at the end of this section quickly consider those of scalar ϕ^4 theory.

3.1 Setup

Recall that the renormalized fields are those that have correctly normalized residues of their propagators close to their poles. They are related to the bare fields which appear in the original Lagrangian by field renormalization factors. The renormalized masses are defined as the positions of the poles of the complete propagators (poles at $q^2 = -m^2$) and are related to the bare masses appearing in the Lagrangian. Finally, coupling constants are also renormalized. For electrodynamics with charged Dirac fermions ψ

$$\begin{aligned} \psi &= Z_2^{-1/2}\psi_B \quad , \qquad m = m_B + \delta m \\ A^\mu &= Z_3^{-1/2}A_B^\mu \quad , \qquad e = Z_3^{1/2}e_B \ , \end{aligned} \tag{3.1}$$

while for scalars

$$\phi = Z^{-1/2}\phi_B \quad , \quad m^2 = m_B^2 + \delta m^2 \ . \tag{3.2}$$

The original Lagrangian is always written in terms of the bare fields and bare masses, with the same numerical coefficients as for the free fields, plus the usual interactions with bare coupling constants. For spinor QED we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^B F_B^{\mu\nu} - \bar\psi_B(\not\partial + m_B)\psi_B - ie_B A_{B\mu}\bar\psi_B\gamma^\mu\psi_B - \frac{\alpha_B}{2}(\partial_\mu A_B^\mu)^2 \ . \tag{3.3}$$

We have included the gauge-fixing term with a bare parameter as discussed above (2.68). Using (3.1) together with $\alpha = Z_3\alpha_B$, $\mathcal{L}$ is rewritten as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \ , \tag{3.4}$$

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar\psi(\not\partial + m)\psi - \frac{\alpha}{2}(\partial_\mu A^\mu)^2 \ , \tag{3.5}$$

$$\mathcal{L}_1 = -ieA_\mu\bar\psi\gamma^\mu\psi \ , \tag{3.6}$$

$$\mathcal{L}_2 = -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} - (Z_2 - 1)\bar\psi(\not\partial + m)\psi + Z_2\delta m\bar\psi\psi - ie(Z_2 - 1)A_\mu\bar\psi\gamma^\mu\psi \, . \tag{3.7}$$

Clearly, $\mathcal{L}_0$ is exactly like the free Lagrangian but now with renormalized fields and masses. Similarly, $\mathcal{L}_1$ is exactly like the original interaction term but now with renormalized fields and couplings, resp. charges. The third term, $\mathcal{L}_2$ is due to the difference between bare and renormalized quantities. Its terms are called the counterterms. If we would take all couplings to zero so that all fields become free, there no longer would be a distinction between bare and renormalized quantities and in this limit $\mathcal{L}_2$ would vanish.

The complete propagators for the renormalized fields are $-iS'(\not{k})$, $-i\Delta'_{\mu\nu}(q)$ and $-i\Delta'(q)$ with

$$\text{for } \psi \quad : \quad S'(\not{k}) \;=\; \left(i\not{k} + m - \Sigma^*(\not{k}) - i\epsilon\right)^{-1} , \tag{3.8}$$

$$\text{for } A_\mu \quad : \quad \Delta'_{\mu\nu} \;=\; (1-\Delta\Pi^*)^{-1}_{\mu\rho}\Delta^\rho{}_\nu \qquad \text{with} \quad \Delta_{\mu\nu}(q) \;=\; \frac{\eta_{\mu\nu} - \xi\frac{q_\mu q_\nu}{q^2}}{q^2 - i\epsilon}$$

$$\Pi^*_{\mu\nu}(q) \;=\; (q^2\eta_{\mu\nu} - q_\mu q_\nu)\pi(q^2) \tag{3.9}$$

$$\text{or } \Delta'_{\mu\nu}(q) \;=\; \frac{\eta_{\mu\nu} - [\xi + (1-\xi)\pi(q^2)]\frac{q_\mu q_\nu}{q^2}}{(1-\pi(q^2))(q^2 - i\epsilon)} , \tag{3.10}$$

$$\tag{3.11}$$

$$\text{for } \phi \quad : \quad \Delta'(q) \;=\; \left(q^2 + m^2 - \Pi^*(q^2) - i\epsilon\right)^{-1} . \tag{3.12}$$

The $\Sigma(\not{k}), \pi(q^2)$ and $\Pi^*(q^2)$ get contributions from the loops (including also counterterms in the loops) and from the counterterms at tree-level:

$$\text{fermions} \;:\; \Sigma^*(\not{k}) \;=\; -(Z_2 - 1)(i\not{k} + m) + Z_2\,\delta m + \Sigma^*_{\rm loop}(\not{k}) , \tag{3.13}$$

$$\text{photons} \;:\; \pi(q^2) \;=\; -(Z_3 - 1) + \pi_{\rm loop}(q^2) , \tag{3.14}$$

$$\text{scalars} \;:\; \Pi^*(q^2) \;=\; -(Z-1)(q^2 + m^2) + Z\,\delta m^2 + \Pi^*_{\rm loop}(q^2) . \tag{3.15}$$

One imposes the renormalization conditions (correct poles and residues)

$$\text{fermions} \;:\; \Sigma^*(im) \;=\; 0 , \qquad \frac{\partial}{\partial\not{k}}\Sigma^*(\not{k})\Big|_{\not{k}=im} \;=\; 0 , \tag{3.16}$$

$$\text{photons} \;:\; \pi(0) \;=\; 0 \qquad \text{and} \quad \pi(q^2) \text{ should have no pole at any } q^2 , \tag{3.17}$$

$$\text{scalars} \;:\; \Pi^*(-m^2) \;=\; 0 , \qquad \frac{\mathrm{d}}{\mathrm{d}q^2}\Pi^*(q^2)\Big|_{q^2=-m^2} \;=\; 0 . \tag{3.18}$$

3.2 Evaluation of one-loop integrals and dimensional regularization

When evaluating one-loop diagrams one typically encounters integrals of the type

$$I_N = \int \mathrm{d}^4k\, \frac{1}{D_1\, D_2 \ldots D_N} \qquad \text{and} \qquad I_N^{\mu_1\ldots\mu_r} = \int \mathrm{d}^4k\, \frac{k^{\mu_1}\ldots k^{\mu_r}}{D_1\, D_2\ldots D_N} , \tag{3.19}$$

where

$$D_i = [(k-p_i)^2 + m_i^2 - i\epsilon] . \tag{3.20}$$

The p_i are combinations of the external momenta and the m_i are the masses appearing in the propagators. Since each D_i contains a k^2 the integrand of $I_N^{\mu_1\ldots\mu_r}$ behaves for large k as $\sim \mathrm{d}^4k\, \frac{k^r}{k^{2N}} \sim k^{3+r-2N}\,\mathrm{d}k$. (This is easy to see once the integral has been continued to Euclidean signature, but the following discussion is equally valid in Minkowski signature.) Hence:

- The integral diverges for $3 + r - 2N > -1$, i.e. $2N - r < 4$.
- The integral diverges logarithmically for $3 + r - 2N = -1$, i.e. $2N - r = 4$.
- The integral converges for $3 + r - 2N < -1$, i.e. $2N - r > 4$.

The 4 comes from the space-time dimension and would have to be replaced by d in d space-time dimensions. Actually, one can formally continue to non-integer and more-over to complex dimensions d where the integral can be evaluated and yields a meromorphic function of d with poles when d is an even integer. This is the basis for *dimensional regularization*.

In dimensional regularization one replaces

$$\int \mathrm{d}^4 k \ (\ldots) \ \rightarrow \ \int \mathrm{d}^d k \ (\ldots) \ . \tag{3.21}$$

As we will see, this makes convergent all our integrals (which we continue to call I_N and $I_N^{\mu_1 \ldots \mu_r}$) for $d \neq 2, 4, \ldots$. The indices $\mu, \nu, \rho, \ldots$ formally become d-dimensional indices. One still has $k^\mu k_\mu = k^2$ but this now is a sum of d terms. In particular, $\eta^{\mu\nu}\eta_{\mu\nu} = \delta^\mu_\mu = d$ or, when dealing with γ-matrices, $\gamma^\mu \gamma_\mu = \delta^\mu_\mu = d$. Also, the usual rules for replacements in tensor integrals have to be accordingly modified, e.g.

$$\int \mathrm{d}^d k \ k^\mu k^\nu \ f(k^2) = \int \mathrm{d}^d k \ \frac{1}{d} \eta^{\mu\nu} k^2 \ f(k^2) \ , \tag{3.22}$$

where the factor $\frac{1}{d}$ can be checked by contracting with $\eta_{\mu\nu}$. Dimensional regularization can be consistently implemented except when the antisymmetric 4-index tensor $\epsilon^{\mu\nu\rho\sigma}$ plays an important role, as in the definition of γ_5 and of chiral fermions.[16]

Feynman's trick : To facilitate the integration, one rewrites the denominators appearing in the integrals (3.19), using the formula

$$\frac{1}{D_1^{a_1} D_2^{a_2} \ldots D_N^{a_N}} = \frac{\Gamma(a_1 + \ldots a_N)}{\Gamma(a_1) \ldots \Gamma(a_N)} \int_0^1 \mathrm{d}x_1 \ldots \int_0^1 \mathrm{d}x_N \delta(1 - \sum x_j) \frac{x_1^{a_1 - 1} \ldots x_N^{a_N - 1}}{[x_1 D_1 + \ldots + x_N D_N]^{\sum a_j}} \ . \tag{3.23}$$

(Recall $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(n) = (n-1)!$, in particular $\Gamma(1) = \Gamma(2) = 1$, $\Gamma(3) = 2$.) Note that the $i\epsilon$ terms in $\sum x_j D_j$ add up to $\sum x_j \ i\epsilon = i\epsilon$. If all $a_j = 1$ eq. (3.23) simplifies:

$$\frac{1}{D_1 D_2 \ldots D_N} = (N-1)! \int_0^1 \mathrm{d}x_1 \ldots \int_0^1 \mathrm{d}x_N \ \frac{\delta(1 - \sum x_j)}{[x_1 D_1 + \ldots + x_N D_N]^N} \ , \tag{3.24}$$

and in particular

$$\boxed{\begin{aligned} \frac{1}{D_1 D_2} &= \int_0^1 \mathrm{d}x \ \frac{1}{[x D_1 + (1-x) D_2]^2} \ , \\ \frac{1}{D_1 D_2 D_3} &= 2 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \ \frac{1}{[x D_1 + y D_2 + (1-x-y) D_3]^3} \ . \end{aligned}} \tag{3.25}$$

Since $D_j = (k - p_j)^2 + m_j^2 - i\epsilon = k^2 - 2kp_j + p_j^2 + m_j^2 - i\epsilon$, one has

$$x_1 D_1 + \ldots + x_N D_N = k^2 - 2kP(x_j) + M^2(x_j) - i\epsilon = (k - P(x_j))^2 + M^2(x_j) - P^2(x_j) - i\epsilon \ , \tag{3.26}$$

[16] This is one way to see why chiral fermions can lead to anomalies: one cannot simply use the gauge invariant dimensional regularization in the presence of chiral fermions.

with

$$\begin{aligned} P(x_j) &= x_1 p_1 + x_2 p_2 + \ldots + (1 - x_1 - \ldots - x_{N-1}) p_N \\ M^2(x_j) &= x_1(p_1^2 + m_1^2) + x_2(p_2^2 + m_2^2) + \ldots + (1 - x_1 - \ldots - x_{N-1})(p_N^2 + m_N^2) \ . \end{aligned} \tag{3.27}$$

Then

$$I_N = \int \mathrm{d}^d k \ (N-1)! \int_0^1 \mathrm{d}x_1 \ldots \int_0^1 \mathrm{d}x_N \ \frac{\delta(1 - \sum x_j)}{[(k - P(x_j))^2 + M^2(x_j) - P^2(x_j) - i\epsilon]^N} \tag{3.28}$$

One interchanges the now convergent k-integration with the x_j-integrations, so that

$$I_N = (N-1)! \int_0^1 \mathrm{d}x_1 \ldots \int_0^1 \mathrm{d}x_N \ \delta(1 - \sum x_j) \ \widetilde{I}_N\big(P(x_j), M^2(x_j)\big) \ , \tag{3.29}$$

with

$$\widetilde{I}_N(P, M) = \int \frac{\mathrm{d}^d k}{[(k-P)^2 + M^2 - P^2 - i\epsilon]^N} = \int \frac{\mathrm{d}^d k'}{[k'^2 + R^2 - i\epsilon]^N} \ , \tag{3.30}$$

where we set $R^2 = M^2 - P^2$ and shifted the integration variables form k to $k' = k - P$. Note that such shifts are justified only because we have a convergent integral. Of course, one has an analogous formula for the $I_N^{\mu_1 \ldots \mu_r}$ with

$$\widetilde{I}_N^{\mu_1 \ldots \mu_r}(P, M) = \int \mathrm{d}^d k \ \frac{k^{\mu_1} \ldots k^{\mu_r}}{[(k-P)^2 + M^2 - P^2 - i\epsilon]^N} = \int \mathrm{d}^d k' \ \frac{(k'+P)^{\mu_1} \ldots (k'+P)^{\mu_r}}{[k'^2 + R^2 - i\epsilon]^N} \ , \tag{3.31}$$

At this point one needs to make a Wick rotation, to be discussed in the next subsection. This results in a factor of i and the four-momentum k^μ then is a Euclidean four-momentum, which we denote by k_E^μ. (Strictly speaking, the P^μ appearing as argument in $\widetilde{I}_N(P, M)$ should then also be continued to a Euclidean P_E^μ.) The last expression in (3.30) for the integral is easily evaluated due to its spherical symmetry. Using $\mathrm{d}^d k_E = \mathrm{d}\Omega_{d-1} k_E^{d-1} \mathrm{d}k_E$ and

$$\int_{S^{d-1}} \mathrm{d}\Omega_{d-1} = \mathrm{vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \ , \tag{3.32}$$

we get

$$\begin{aligned} \widetilde{I}_N(P, M) &= i \, \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_0^\infty \frac{k_E^{d-1} \mathrm{d}k_E}{[{k_E}^2 + R^2]^N} = i \, \frac{\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_0^\infty \frac{x^{\frac{d}{2}-1} \mathrm{d}x}{[x + R^2]^N} \\ &= i \, \frac{\pi^{d/2}}{\Gamma(\frac{d}{2})} (R^2)^{\frac{d}{2} - N} \int_0^\infty \mathrm{d}y \, y^{\frac{d}{2}-1} (1+y)^{-N} \ . \end{aligned} \tag{3.33}$$

The last integral is

$$\int_0^\infty \mathrm{d}y \, y^{\frac{d}{2}-1} (1+y)^{-N} = B\Big(\frac{d}{2}, N - \frac{d}{2}\Big) \equiv \frac{\Gamma(\frac{d}{2}) \, \Gamma(N - \frac{d}{2})}{\Gamma(N)} \ , \tag{3.34}$$

so that finally ($R^2 = M^2 - P^2$)

$$\widetilde{I}_N(P, M) = i \, \pi^{d/2} \frac{\Gamma(N - \frac{d}{2})}{(N-1)!} \big(M^2 - P^2\big)^{\frac{d}{2} - N} \ . \tag{3.35}$$

Inserting this into (3.29) and performing the integrations over the Feynman parameters x_j then yields I_N. Performing explicitly these x_j integrations is often the less obvious part of the story. Let us summarize: so far, we have shown that, using Feynman's trick, the dimensionally regularized momentum integration reduces to

$$\boxed{\int \frac{\mathrm{d}^d k}{[k^2+R^2-i\epsilon]^N} = i\,\pi^{d/2}\frac{\Gamma(N-\frac{d}{2})}{\Gamma(N)}(R^2)^{\frac{d}{2}-N}\ .} \tag{3.36}$$

By differentiating $\widetilde{I}_N(P,M)$ with respect to the P_μ one can generate the integrals with factors of k^μ in the numerator. Indeed, rewriting the integrand in (3.30) as $[k^2-2kP+M^2]^{-N}$ it is obvious that

$$\widetilde{I}_N^{\mu_1\ldots\mu_r}(P,M) = \frac{(N-r-1)!}{2^r(N-1)!}\frac{\partial}{\partial P_{\mu_1}}\cdots\frac{\partial}{\partial P_{\mu_r}}\widetilde{I}_{N-r}(P,M)\ . \tag{3.37}$$

Plugging in the explicit expression (3.35) for $\widetilde{I}_{N-r}(P,M)$ one gets

$$\widetilde{I}_N^{\mu_1\ldots\mu_r}(P,M) = i\,\pi^{d/2}\,\frac{\Gamma(N-r-\frac{d}{2})}{2^r(N-1)!}\frac{\partial}{\partial P_{\mu_1}}\cdots\frac{\partial}{\partial P_{\mu_r}}\big(M^2-P^2\big)^{\frac{d}{2}-N+r} \tag{3.38}$$

Integrating this expression, with $M^2=M^2(x_j)$ and $P^2=P^2(x_j)$, over $\int_0^1\mathrm{d}x_1\ldots\int_0^1\mathrm{d}x_N\ \delta(1-\sum x_j)$ yields the $I_N^{\mu_1\ldots\mu_r}$.

The most commonly encountered cases are $r=1$ with

$$\widetilde{I}_N^{\mu}(P,M) = i\,\pi^{d/2}\,\frac{\Gamma(N-\frac{d}{2})}{(N-1)!}\ P^\mu\ \big(M^2-P^2\big)^{\frac{d}{2}-N}\ , \tag{3.39}$$

and $r=2$ where[17]

$$\widetilde{I}_N^{\mu\nu}(P,M) = i\,\pi^{d/2}\left\{\frac{\Gamma(N-\frac{d}{2}-1)}{2(N-1)!}\ \eta^{\mu\nu}\big(M^2-P^2\big)^{\frac{d}{2}-N+1}+\frac{\Gamma(N-\frac{d}{2})}{(N-1)!}\ P^\mu P^\nu\ \big(M^2-P^2\big)^{\frac{d}{2}-N}\right\}\ . \tag{3.40}$$

There is an alternative, often simpler way, to compute the integrals I_N^μ, $I_N^{\mu\nu}$, etc. First note that a slight generalization of equations (3.33) to (3.36) yields the useful formula

$$\boxed{\int \frac{\mathrm{d}^d k\,(k^2)^s}{(k^2+R^2-i\epsilon)^N} = i\,\pi^{d/2}\frac{\Gamma(\frac{d}{2}+s)\Gamma(N-\frac{d}{2}-s)}{\Gamma(\frac{d}{2})\Gamma(N)}(R^2)^{\frac{d}{2}+s-N}\ .} \tag{3.41}$$

Next, it follows from Lorentz (resp. Euclidean rotational) invariance (cf. (3.22)) that

$$\boxed{\int \mathrm{d}^d k\,\frac{k^\mu}{(k^2+R^2-i\epsilon)^N}=0\quad ,\quad \int \mathrm{d}^d k\,\frac{k^\mu k^\nu}{(k^2+R^2-i\epsilon)^N} = \frac{1}{d}\,\eta^{\mu\nu}\int\frac{\mathrm{d}^d k\ k^2}{(k^2+R^2-i\epsilon)^N}\ .} \tag{3.42}$$

[17] In the Euclidean, one should replace $\eta^{\mu\nu}$ by $\delta^{\mu\nu}$.

One then has from (3.31), e.g. for $r=2$

$$\widetilde{I}_N^{\mu\nu}(P,M) = \int \mathrm{d}^d k\, \frac{(k^\mu+P^\mu)(k^\nu+P^\nu)}{(k^2+R^2)^N} = \frac{\eta^{\mu\nu}}{d}\int \frac{\mathrm{d}^d k\, k^2}{(k^2+R^2)^N} + P^\mu P^\nu \int \frac{\mathrm{d}^d k}{(k^2+R^2)^N}$$
$$= i\,\pi^{d/2}\Big\{\eta^{\mu\nu}\frac{\Gamma(N-\frac{d}{2}-1)}{2(N-1)!}(R^2)^{\frac{d}{2}+1-N} + P^\mu P^\nu \frac{\Gamma(N-\frac{d}{2})}{(N-1)!}(R^2)^{\frac{d}{2}-N}\Big\}\ , \tag{3.43}$$

which, of course, coincides with (3.40).

Finally one needs the expansions of the various Γ-factors as $d\to 4$. We always let

$$\boxed{d = 4-\epsilon\ .} \tag{3.44}$$

Then

$$\boxed{\begin{aligned} \Gamma\Big(2-\frac{d}{2}\Big) &= \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\ , \\ \Gamma\Big(1-\frac{d}{2}\Big) &= \frac{2}{2-d}\Gamma\Big(2-\frac{d}{2}\Big) = -\frac{2}{\epsilon} + \gamma - 1 + \mathcal{O}(\epsilon)\ , \end{aligned}} \tag{3.45}$$

where $\gamma \simeq 0.5772\ldots$ is Euler's constant.

3.3 Wick rotation

In the previous subsection we had to evaluate integrals like

$$\widetilde{I}_N(P,M) = \int \frac{\mathrm{d}^d k}{[k^2+R^2-i\epsilon]^N}\ , \tag{3.46}$$

where $R^2 = M^2 - P^2$. Furthermore, $\mathrm{d}^d k = \mathrm{d}k_0\, \mathrm{d}^{d-1}\vec{k}$ and $k^2 = \vec{k}^2 - k_0^2$. Thus the integral explicitly is

$$\widetilde{I}_N(P,M) = \int d^{d-1}\vec{k}\int_{-\infty}^{\infty}\mathrm{d}k_0 \left[\frac{-1}{\Big(k_0 - \Big(\sqrt{\vec{k}^2+R^2} - i\epsilon'\Big)\Big)\Big(k_0 + \Big(\sqrt{\vec{k}^2+R^2} - i\epsilon'\Big)\Big)}\right]^N\ . \tag{3.47}$$

The integrand has poles at $\sqrt{\vec{k}^2+R^2} - i\epsilon'$ and at $-\sqrt{\vec{k}^2+R^2} + i\epsilon'$, as shown on the left of Fig. 6. As it is also clear from this figure, one can deform the k_0-integration contour away from the real axis without crossing any of these poles until on gets the integration contour depicted on the right part of the figure and denoted $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$:

$$\int_{-\infty}^{\infty}\mathrm{d}k_0(\ldots) = \int_{\Gamma_1}\mathrm{d}k_0(\ldots) + \int_{\Gamma_2}\mathrm{d}k_0(\ldots) + \int_{\Gamma_3}\mathrm{d}k_0(\ldots)\ . \tag{3.48}$$

Now with Γ_1 and Γ_3 being "quarter"-circles of radius going to infinity, the corresponding integrals vanish, so that only the integral over Γ_2 remains. But Γ_2 is the imaginary axis in the complex k_0 plane, and if one sets

$$k_0 = i k_0^E\ , \tag{3.49}$$

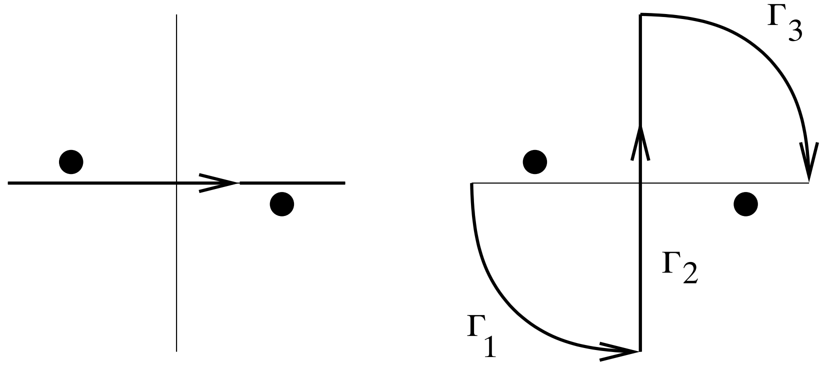


Figure 6: The integration contours for the k_0-integration

then, as k_0^E runs from $-\infty$ to $+\infty$, k_0 runs along the imaginary axis, i.e. along Γ_2. Hence

$$\int_{\Gamma_2} \mathrm{d}k_0 f(k_0) = i \int_{-\infty}^{\infty} \mathrm{d}k_0^E f(ik_0^E) \ . \tag{3.50}$$

Applying this to (3.47) or directly to (3.46) finally gives

$$\begin{aligned} \widetilde{I}_N(P,M) &= i \int d^{d-1}\vec{k} \int_{-\infty}^{\infty} \mathrm{d}k_0^E \left[\frac{-1}{\left(ik_0^E - \left(\sqrt{\vec{k}^2+R^2}\right)\right)\left(ik_0^E + \left(\sqrt{\vec{k}^2+R^2} - i\epsilon'\right)\right)} \right]^N \\ &= i \int \frac{\mathrm{d}^d k_E}{[k_E^2+R^2]^N} \ , \end{aligned} \tag{3.51}$$

where, of course, $k_E^2 = (k_0^E)^2 + \vec{k}^2$.

What happens if the integrand contains some expression involving $k_\mu p^\mu = \vec{k}\cdot\vec{p} - k_0 p_0$? Obviously, this becomes $\vec{k}\cdot\vec{p} - ik_0^E p_0 = \vec{k}\cdot\vec{p} + k_0^E p_0^E \equiv k_\mu^E\, p_\mu^E$ if we also let $p_0 = ip_0^E$. Finally, consider

$$\widetilde{I}_N^{\mu\nu}(P,M) = \int \frac{\mathrm{d}^d k\ k^\mu\, k^\nu}{[k^2+R^2-i\epsilon]^N} \ . \tag{3.52}$$

Doing the Wick rotation yields

$$\widetilde{I}_N^{jk}(P,M) = i \int \frac{\mathrm{d}^d k_E\ k_E^j\, k_E^j}{[{k_E}^2+R^2]^N} \ , \quad \widetilde{I}_N^{00}(P,M) = i \int \frac{\mathrm{d}^d k_E\ (-)k_E^0\, k_E^0}{[{k_E}^2+R^2]^N} \quad \widetilde{I}_N^{0j}(P,M) = i \int \frac{\mathrm{d}^d k_E\ (-i)k_E^0\, k_E^j}{[{k_E}^2+R^2]^N} \ . \tag{3.53}$$

Due to the spherical symmetry of the Euclidean integral one has

$$\int \frac{\mathrm{d}^d k_E\ k_E^\mu\, k_E^\nu}{[{k_E}^2+R^2]^N} = \frac{1}{d}\ \delta^{\mu\nu} \int \frac{\mathrm{d}^d k_E\ k_E^2}{[{k_E}^2+R^2]^N} \ , \tag{3.54}$$

which together with (3.53) gives

$$\widetilde{I}_{N,d}^{\mu\nu}(P,M) = \frac{1}{d}\ \eta^{\mu\nu}\ i \int \frac{\mathrm{d}^d k_E\ k_E^2}{[{k_E}^2+R^2]^N} \ . \tag{3.55}$$

This is consistent with what would have been obtained by first replacing $k^\mu\, k^\mu$ by $\frac{1}{d}\eta^{\mu\nu}k^2$, and then doing the Wick rotation. One can summarize all this as the following

Recipe for the Wick rotation: do the continuation $k_0 = i\, k_0^E$ in the integrand, replace similarly all external momenta that may appear in scalar products by their Euclidean counterparts, replace $\mathrm{d}^d k$ by $\mathrm{d}^d k_E$ (with all components of k_E^μ real), and multiply the whole integral by a factor of i. Integrands involving $k^\mu\, k^\nu \times f(k^2)$ can be replaced by $\frac{1}{d}\eta^{\mu\nu}\, k^2 \times f(k^2)$, independently of the Wick rotation.

3.4 Vacuum polarization

The computation of the one-loop vacuum-polarization diagram shown in Fig. 7 is straighforward. To order e^2 this is the only contribution to $\Pi^{*\mu\nu}_{\rm loop}(q)$, while at order e^4 one would get 2-loop diagrams, as well as one-loop diagrams with counterterms inserted, as shown in Fig. 8.

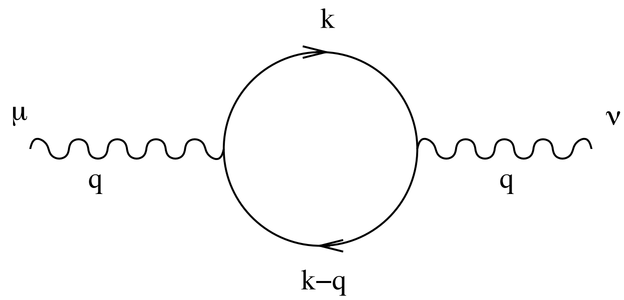


Figure 7: The one-loop vacuum polarization diagram. Of course, although we have drawn the external photon propagators, they are not to be included in $\Pi^*_{\mu\nu}$.

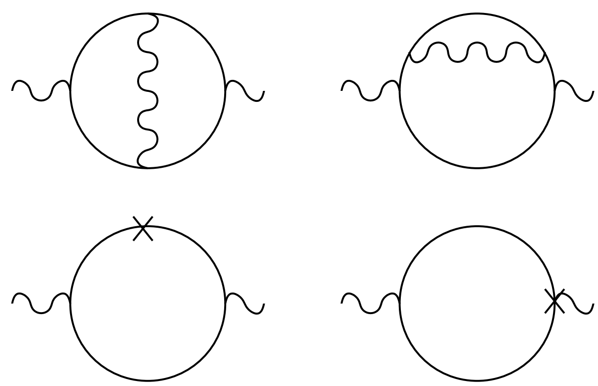


Figure 8: Loop contributions to the vacuum polarization at order e^4. The two upper diagrams are two-loop diagrams, while the two lower diagrams are one-loop diagrams with $\mathcal{O}(e^2)$ counterterms inserted

Now, applying the Feynman rules to the order e^2 diagram of Fig. 7 we get

$$i(2\pi)^4 \Pi^{*\mu\nu}_{{\rm loop},e^2}(q) = -\int \mathrm{d}^4 k\, {\rm tr}\left\{ \frac{(-i)}{(2\pi)^4} \frac{-i\not{k}+m}{k^2+m^2-i\epsilon}(2\pi)^4 e\gamma^\mu \frac{(-i)}{(2\pi)^4}\frac{-i(\not{k}-\not{q})+m}{(k-q)^2+m^2-i\epsilon}(2\pi)^4 e\gamma^\nu \right\}\ , \tag{3.56}$$

where the overall minus sign is due to the fermion loop. Simplifying a bit gives

$$\Pi^{*\mu\nu}_{{\rm loop},e^2}(q) = \frac{-i\, e^2}{(2\pi)^4}\int \mathrm{d}^4 k \frac{{\rm tr}\ [(-i\not{k}+m)\gamma^\mu(-i(\not{k}-\not{q})+m)\gamma^\nu]}{[k^2+m^2-i\epsilon][(k-q)^2+m^2-i\epsilon]}\ . \tag{3.57}$$

This integral clearly diverges for large $|k|$ where the integrand behaves like $\sim |k|^{-2}$ while the integration measure goes like $|k|^3 \mathrm{d}|k|$. If we simply "cut off" the large $|k|$ values et $|k|_{\max} = \Lambda$ we expect the resulting integral to be dominated by a term $\sim \int^\Lambda |k|^3 \mathrm{d}|k| \times |k|^{-2} \sim \Lambda^2$. If this is the case one says the integral is quadratically divergent. We have seen that gauge invariance requires $\Pi^{*\mu\nu}(q) = (q^2\eta^{\mu\nu} - q^\mu q^\nu)\pi(q^2)$ and hence also $\Pi^{*\mu\nu}_{\text{loop},e^2}(q) = (q^2\eta^{\mu\nu} - q^\mu q^\nu)\pi_{\text{loop},e^2}(q^2)$. As long as our regularization procedure does not destroy gauge invariance this must still be true for the regularized integrals. In order to manifestly preserve gauge invariance we will use dimensional regularization. Before doing the computation, let us argue a bit more what we should expect. It is not difficult to see that taking a derivative of $\Pi^{*\mu\nu}_{\text{loop},e^2}(q)$ with respect to q^ρ results in an integrand behaving as $|k|^{-3}$ for large $|k|$ and hence an integral that is less divergent, i.e. behaves as $\sim \Lambda$ rather than Λ^2. Taking one more derivative with respect to q^σ results in an integral that is only logarithmically divergent, and taking a third derivative with respect to a q^λ gives a convergent integral. This means that the diverging part of the integral is annihilated by 3 derivatives with respect to the external momentum. Thus the diverging part of $\Pi^{*\mu\nu}_{\text{loop},e^2}(q)$ must be a polynomial of second degree in the q, and by gauge invariance:

$$\Pi^{*\mu\nu}_{\text{loop},e^2,\text{div}}(q) = (q^2\eta^{\mu\nu} - q^\mu q^\nu)\pi_{\text{loop},e^2,\text{div}} \quad , \quad \frac{\partial}{\partial q^\sigma}\pi_{\text{loop},e^2,\text{div}} = 0 \ . \tag{3.58}$$

Hence, although π_{loop,e^2} will be a non-trivial function of q^2, its diverging part (the coefficient of the poles in ϵ) will be constant. From (3.58) we see that we can e.g. extract this divergent part of π_{loop,e^2} by taking two derivatives of (3.57) and setting $q = 0$. The resulting integral clearly is logarithmically divergent. One often says that gauge invariance has reduced the degree of divergence of the vacuum polarization from 2 (quadratic) to 0 (logarithmic).

Let us now do the computation. As already mentioned, we choose dimensional regularization as explained above. Applying also the Feynman trick yields

$$\Pi^{*\mu\nu}_{\text{loop},e^2}(q) = \frac{-i\,e^2}{(2\pi)^4}\int_0^1 \mathrm{d}x \int \mathrm{d}^d k \frac{\text{tr}\,[\ldots]}{[(k-xq)^2 + m^2 + x(1-x)q^2 - i\epsilon]^2} \ . \tag{3.59}$$

The Dirac trace can still be evaluated as in 4 dimensions[18] so that

$$\text{tr}\,[\ldots] = 4\left[-k^\mu(k-q)^\nu - k^\nu(k-q)^\mu + k(k-q)\eta^{\mu\nu} + m^2\eta^{\mu\nu}\right] \ . \tag{3.60}$$

Shifting the integration variable $k - xq \to k$ this becomes

$$\Pi^{*\mu\nu}_{\text{loop},e^2}(q) = \frac{-4i\,e^2}{(2\pi)^4}\int_0^1 \mathrm{d}x \int \mathrm{d}^d k \Big[-(k+xq)^\mu(k-(1-x)q)^\nu - (k+xq)^\nu(k-(1-x)q)^\mu + (k+xq)(k-(1-x)q)\eta^{\mu\nu} + m^2\eta^{\mu\nu}\Big] \frac{1}{[k^2 + m^2 + x(1-x)q^2]^2} \ . \tag{3.61}$$

Now one can do the Wick rotation. As explained in the previous subsection, in order for this to make sense one must also continue the external q^μ to a Euclidean q_E^μ. Furthermore, due to rotational symmetry,

[18] All one needs is that $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}$ and the only issue concerns $\text{tr}\,\mathbf{1}$ which can be chosen to be $2^{d/2} = 4\times 2^{-\epsilon/2}$ or just 4. The ambiguity consisting in an overall $2^{-\epsilon/2}$ is of the same type as the ambiguity in choosing to continue also the factors $\frac{1}{(2\pi)^4}$ to $\frac{1}{(2\pi)^d}$. In the end this only changes the renormalization constants Z_i by finite amounts and does not affect the renormalized Π^* or Σ^*.

terms linear in k^μ don't contribute to the integral, while terms $k^\mu k^\nu$ can be replaced by $\frac{1}{d}\eta^{\mu\nu}k^2$ which become $\frac{1}{d}\eta^{\mu\nu}k_E^2$ after the continuation. Hence

$$\Pi^{*\mu\nu}_{\text{loop},e^2}(q_E) = \frac{-4i\,e^2}{(2\pi)^4}\int_0^1 \mathrm{d}x\ i\int \mathrm{d}^d k_E \frac{\big[m^2 - x(1-x)q_E^2 + (1-\frac{2}{d})k_E^2\big]\eta^{\mu\nu} + 2x(1-x)q_E^\mu q_E^\nu}{\big[k_E^2 + m^2 + x(1-x)q_E^2\big]^2}\ . \tag{3.62}$$

Using the formula (3.41) for $s=0$ and $s=1$ with $R^2 = m^2 + x(1-x)q_E^2$ (and taking into account that here we already have done the Wick rotation[19]) yields

$$\begin{aligned}\Pi^{*\mu\nu}_{\text{loop},e^2}(q_E) \ &= \ \frac{4e^2}{(2\pi)^4}\int_0^1 \mathrm{d}x \Bigg\{ \Big[\big[m^2 - x(1-x)q_E^2\big]\eta^{\mu\nu} + 2x(1-x)q_E^\mu q_E^\nu\Big]\pi^{d/2}\,\Gamma\Big(2-\frac{d}{2}\Big)(R^2)^{\frac{d}{2}-2} \\ &\qquad + \Big(1-\frac{2}{d}\Big)\,\eta^{\mu\nu}\,\pi^{d/2}\,\frac{\Gamma(\frac{d}{2}+1)\Gamma(1-\frac{d}{2})}{\Gamma(\frac{d}{2})}(R^2)^{\frac{d}{2}-1}\Bigg\}\ . \end{aligned} \tag{3.63}$$

Observing that $(1-\frac{2}{d})\frac{\Gamma(\frac{d}{2}+1)\Gamma(1-\frac{d}{2})}{\Gamma(\frac{d}{2})} = (1-\frac{2}{d})\,\frac{d}{2}\,\Gamma(1-\frac{d}{2}) = (\frac{d}{2}-1)\Gamma(1-\frac{d}{2}) = -\Gamma(2-\frac{d}{2})$ we get

$$\begin{aligned}\Pi^{*\mu\nu}_{\text{loop},e^2}(q_E) \ &= \frac{4e^2}{(2\pi)^4}\Gamma\Big(2-\frac{d}{2}\Big)\,\pi^{d/2}\int_0^1 \mathrm{d}x\ \Big[\underbrace{\big[m^2 - x(1-x)q_E^2 - R^2\big]}_{-2x(1-x)q_E^2}\eta^{\mu\nu} + 2x(1-x)q_E^\mu q_E^\nu\Big](R^2)^{\frac{d}{2}-2} \\ &= \frac{8e^2}{(2\pi)^4}\Gamma\Big(2-\frac{d}{2}\Big)\,\pi^{d/2}\left(q_E^\mu q_E^\nu - q_E^2\eta^{\mu\nu}\right)\int_0^1 \mathrm{d}x\, x(1-x)\,\big[m^2 + x(1-x)q_E^2\big]^{\frac{d}{2}-2}\ . \end{aligned} \tag{3.64}$$

Recall that $\Pi^{*\mu\nu}(q) = \big(q^2\eta^{\mu\nu} - q^\mu q^\nu\big)\,\pi(q^2)$, so that we read from (3.64), upon changing back the argument from q_E to q,

$$\pi_{\text{loop},e^2}(q^2) = -\frac{8e^2}{(2\pi)^4}\Gamma\Big(2-\frac{d}{2}\Big)\,\pi^{d/2}\int_0^1 \mathrm{d}x\, x(1-x)\,\big[m^2 + x(1-x)q^2\big]^{\frac{d}{2}-2}\ . \tag{3.65}$$

Note that, when doing the dimensional regularization, one has various options. In addition to continuing $\mathrm{d}^4k \to \mathrm{d}^dk$ one can also continue $\frac{1}{(2\pi)^4} \to \frac{1}{(2\pi)^d}$ and, as already mentioned, $\mathrm{tr}\,\mathbf{1} = 4 \to 2^{\frac{d}{2}}$. Furthermore, in d dimensions the coupling e no longer would be dimensionless. In order to avoid this, one replaces e by $\widetilde{\mu}^{2-\frac{d}{2}}\,e$ where $\widetilde{\mu}$ is some mass scale and then e remains dimensionless. All this results in an additional factor $(C\widetilde{\mu})^{4-d}$ where C possibly includes the additional factors 2π or $1/\sqrt{2}$, so that

$$\pi_{\text{loop},e^2}(q^2) = -\frac{e^2}{2\pi^2}\Gamma\Big(2-\frac{d}{2}\Big)\,\pi^{d/2-2}\,C^{4-d}\int_0^1 \mathrm{d}x\, x(1-x)\left[\frac{m^2 + x(1-x)q^2}{\widetilde{\mu}^2}\right]^{\frac{d}{2}-2}\ . \tag{3.66}$$

Next one sets $d = 4-\epsilon$ and expands the result in ϵ. Recall from (3.45) that $\Gamma(2-\frac{d}{2}) = \Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$, and also $a^\epsilon = e^{\epsilon\log a} = 1 + \epsilon\log a + \mathcal{O}(\epsilon^2)$, so that

$$\begin{aligned}\Gamma\Big(2-\frac{d}{2}\Big)\,\pi^{d/2-2}\,C^{4-d}[\ldots]^{\frac{d}{2}-2} \ &= \ \Big(\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\Big)\Big(1 - \frac{\epsilon}{2}\log\pi - \frac{\epsilon}{2}\log[\ldots] + \epsilon\log C + \mathcal{O}(\epsilon^2)\Big) \\ &= \ \frac{2}{\epsilon} - \gamma - \log\pi + 2\log C - \log[\ldots] + \mathcal{O}(\epsilon)\ . \end{aligned} \tag{3.67}$$

Since $\int_0^1 \mathrm{d}x\, x(1-x) = \frac{1}{6}$, we finally get

$$\pi_{\text{loop},e^2}(q^2) = -\frac{e^2}{6\pi^2}\left(\frac{1}{\epsilon} - \frac{\gamma}{2} + \log\frac{C}{\sqrt{\pi}} - 3\int_0^1 \mathrm{d}x\, x(1-x)\log\left[\frac{m^2 + x(1-x)q^2}{\widetilde{\mu}^2}\right] + \mathcal{O}(\epsilon)\right)\ . \tag{3.68}$$

[19] Indeed in eq. (3.41) the l.h.s. is still Minkowskian, so in using (3.41) for our present Euclidean integral one has to omit the i on the r.h.s. of (3.41).

Recall from (2.69) that π also contains the contribution from the counterterm $1 - Z_3$, and hence π_{e^2} includes these contributions up to order e^2:

$$\pi_{e^2}(q^2) = (1 - Z_3)_{e^2} + \pi_{\text{loop},e^2}(q^2) \ . \tag{3.69}$$

The renormalization condition (2.70) then gives

$$(Z_3 - 1)_{e^2} = \pi_{\text{loop},e^2}(0) = -\frac{e^2}{6\pi^2}\left(\frac{1}{\epsilon} - \frac{\gamma}{2} + \log\frac{C}{\sqrt{\pi}} - \frac{1}{2}\log\left[\frac{m^2}{\widetilde{\mu}^2}\right] + \mathcal{O}(\epsilon)\right) \ . \tag{3.70}$$

For later reference we note that the divergent part simply is

$$\boxed{(Z_3 - 1)_{e^2}\Big|_{\text{div}} = -\frac{e^2}{6\pi^2}\frac{1}{\epsilon} \ .} \tag{3.71}$$

Finally (cf (2.71)),

$$\boxed{\begin{aligned}\pi_{e^2}(q^2) &= \pi_{\text{loop},e^2}(q^2) - \pi_{\text{loop},e^2}(0) \\ &= \frac{e^2}{2\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\log\left[1 + x(1-x)\frac{q^2}{m^2}\right] + \mathcal{O}(\epsilon) \ .\end{aligned}} \tag{3.72}$$

This now has a finite limit as $\epsilon \to 0$, so that one can remove the regularization and simply set $\epsilon = 0$. Note that this renormalized $\pi_{e^2}(q^2)$ does not depend on $\widetilde{\mu}$ nor on the arbitrariness of the continuation which showed up through the constant C. We also note that $\pi_{e^2}(q^2)$ is a monotonuous function of q^2.

Note that $\pi_{e^2}(q^2)$ is positive for $q^2 > 0$ and negative for $-4m^2 < q^2 < 0$, while it develops an imaginary part for $q^2 < -4m^2$. This imaginary part translates the possibility that a photon with such q can yield an on-shell electron-positron pair. More precisely, if viewed as a function of the complex variable q^2, the function $\pi_{e^2}(q^2)$, and thus also the full photon propagator, has a branch cut along the negative real axis for $q^2 < -4m^2$. Thus the intermediate two-particle $e^+ e^-$ physical state yields a branch cut singularity at the corresponding values of q^2, in agreement with the discussion in section 2.1.

3.5 Electron self energy

The electron self-energy diagram shown in Fig. 9 gives after dimensional regularization

$$i(2\pi)^4\, \Sigma^*_{\text{loop},e^2}(\not p) = \int \mathrm{d}^d k\, \frac{(-i)}{(2\pi)^4}\frac{\eta_{\rho\sigma}}{k^2 - i\epsilon}\, (2\pi)^4 e\gamma^\rho\, \frac{(-i)}{(2\pi)^4}\frac{\big(-i(\not p - \not k) + m\big)}{(p-k)^2 + m^2 - i\epsilon}\, (2\pi)^4 e\gamma^\sigma \ , \tag{3.73}$$

or after simplifying:

$$\begin{aligned}\Sigma^*_{\text{loop},e^2}(\not p) &= ie^2\int \frac{\mathrm{d}^d k}{(2\pi)^4}\, \frac{1}{k^2 - i\epsilon}\frac{\gamma^\rho\big(-i(\not p - \not k) + m\big)\gamma_\rho}{(p-k)^2 + m^2 - i\epsilon} \\ &= ie^2\int \frac{\mathrm{d}^d k}{(2\pi)^4}\, \frac{1}{k^2 - i\epsilon}\frac{(d-2)i(\not p - \not k) + d\, m}{(p-k)^2 + m^2 - i\epsilon}\end{aligned} \tag{3.74}$$

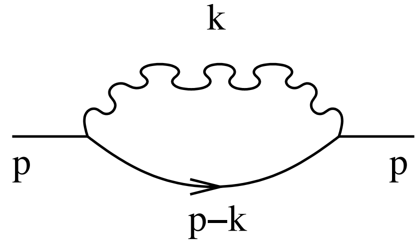


Figure 9: The electron self-energy diagram. As before, the external propagators are not to be included in Σ^*.

Introducing the Feynman parameter and shifting the integration variables gives

$$\begin{aligned}\Sigma^*_{\text{loop},e^2}(\not p) &= ie^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^4}\, \frac{(d-2)i(\not p - \not k) + d\, m}{[(k-xp)^2 + x(1-x)p^2 + xm^2 - i\epsilon]^2} \\ &= ie^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k'}{(2\pi)^4}\, \frac{(d-2)i\left((1-x)\not p - \not k'\right) + d\, m}{[k'^2 + x(1-x)p^2 + xm^2 - i\epsilon]^2} \\ &= ie^2 \int_0^1 \mathrm{d}x\, [(d-2)\,(1-x)i\not p + d\, m] \int \frac{\mathrm{d}^d k}{(2\pi)^4}\, \frac{1}{[k^2 + x(1-x)p^2 + xm^2 - i\epsilon]^2} \\ &= -\frac{e^2}{(2\pi)^4}\, \pi^{d/2}\, \Gamma\Big(2-\frac{d}{2}\Big) \int_0^1 \mathrm{d}x\, [(d-2)\,(1-x)i\not p + d\, m]\, \big[xm^2 + x(1-x)p^2\big]^{\frac{d}{2}-2} \ . \end{aligned} \tag{3.75}$$

As for the vacuum polarization, we introduce $(C\widetilde{\mu})^{4-d}$ to keep e dimensionless and to allow for the other options in doing the dimensional continuation. Expanding in $\epsilon = 4-d$ as above in eq. (3.67) gives

$$\begin{aligned}\Sigma^*_{\text{loop},e^2}(\not p) &= -\frac{e^2}{16\pi^2} \int_0^1 \mathrm{d}x \left(\frac{2}{\epsilon} - \gamma + \log\frac{C^2}{\pi} - \log\frac{xm^2 + x(1-x)p^2}{\widetilde{\mu}^2}\right) [(d-2)\,(1-x)i\not p + d\, m] \\ &= -\frac{e^2}{16\pi^2} \Bigg\{ \left(\frac{2}{\epsilon} - \gamma + \log\frac{C^2}{\pi}\right) \left[\frac{d-2}{2} i\not p + d\, m\right] \\ &\qquad - \int_0^1 \mathrm{d}x\, [(d-2)\,(1-x)i\not p + d\, m] \log\frac{xm^2 + x(1-x)p^2}{\widetilde{\mu}^2} \Bigg\} \ . \end{aligned} \tag{3.76}$$

Now we can drop the $\mathcal{O}(\epsilon)$ terms to get

$$\begin{aligned}\Sigma^*_{\text{loop},e^2}(\not p) \ &= \ -\frac{e^2}{16\pi^2} \Bigg\{ \frac{2}{\epsilon}(i\not p + 4m) - i\not p - 2m + \left(\log\frac{C^2}{\pi} - \gamma\right)(i\not p + 4m) \\ &\qquad - \int_0^1 \mathrm{d}x\, [2(1-x)i\not p + 4m] \log\frac{xm^2 + x(1-x)p^2}{\widetilde{\mu}^2} \Bigg\} \ . \end{aligned} \tag{3.77}$$

As discussed in general above, the self-energy $\Sigma^*(\not p)$ also receives contributions from the counterterms, cf. eq. (2.38):

$$\Sigma^*_{e^2}(\not p) = -(Z_2 - 1)_{e^2}(i\not p + m) + (Z_2 \delta m)_{e^2} + \Sigma^*_{\text{loop},e^2}(\not p) \ . \tag{3.78}$$

The renormalization conditions then fix Z_2 and δm according to (2.40) and (2.41) which now read

$$(Z_2 - 1)_{e^2} = -i \frac{\partial}{\partial \not p} \Sigma^*_{\text{loop},e^2}(\not p)\Big|_{\not p = im} \quad , \quad \delta m_{e^2} = -\Sigma^*_{\text{loop},e^2}(im) \ , \tag{3.79}$$

so that

$$\delta m_{e^2} = \frac{e^2}{8\pi^2}\, m \left\{ \frac{3}{\epsilon} - \frac{1+3\gamma}{2} + 3\log\frac{C}{\sqrt{\pi}} - \int_0^1 \mathrm{d}x (1+x)\log\frac{x^2 m^2}{\widetilde{\mu}^2} \right\} , \tag{3.80}$$

and

$$(Z_2-1)_{e^2} = -\frac{e^2}{8\pi^2}\left\{ \frac{1}{\epsilon} - \frac{1+\gamma}{2} + \log\frac{C}{\sqrt{\pi}} - \int_0^1 \mathrm{d}x \left[(1-x)\log\frac{x^2m^2}{\widetilde{\mu}^2} + \frac{2(1-x^2)}{x} \right] \right\} . \tag{3.81}$$

Note that the dx-integral for Z_2 diverges as $x \to 0$. This is actually an *infrared* divergence which occurs for on-shell electrons and is related to the possibility of emitting soft (very low energy) photons. The proper treatment of such infrared divergences would be a chapter by itself. Let us only say that when summing appropriate diagrams corresponding to physically measurable and distinguishable situations such infrared divergences cancel. Note that the UV-divergent part simply is

$$\boxed{(Z_2-1)_{e^2}\Big|_{\rm div} = -\frac{e^2}{8\pi^2}\frac{1}{\epsilon}\ .} \tag{3.82}$$

We finally get

$$\Sigma^*_{e^2}(\not p) = \frac{e^2}{8\pi^2}\int_0^1 \mathrm{d}x \left\{ \Big[(1-x)\, i\not p + 2m \Big] \log\left[\frac{m^2+(1-x)p^2}{x m^2}\right] - \frac{2(1-x^2)}{x}(i\not p + m) \right\} . \tag{3.83}$$

Although there are infrared divergences as $x \to 0$ as just discussed, all ultraviolet divergences ($\frac{1}{\epsilon}$ poles) have cancelled.

3.6 Vertex function

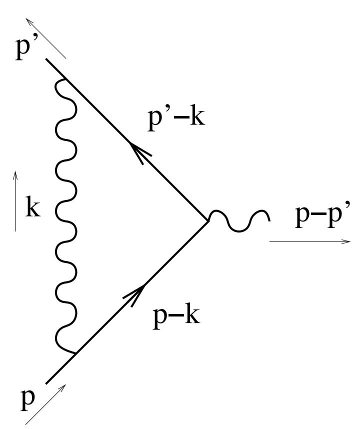


Figure 10: The one-loop vertex function diagram. The external photon and electron propagators are not to be included in $\Gamma^\mu(p',p)$.

The vertex function $\Gamma^\mu(p',p)$ was defined in (2.47). Since we are computing with renormalized fields (and not bare fields) we will automatically compute the renormalized vertex function, and not the bare one. Note that $\Gamma^\mu(p',p)$ is normalized such that its lowest order contribution is just γ^μ, cf.

eq. (2.48), while the tree vertex Feynman diagram would give a $(2\pi)^4 e\gamma^\mu$. Taking this into account and applying the Feynman rules to the one-loop diagram shown in Fig. 10 gives

$$\begin{aligned}\Gamma^\mu_{\text{loop},e^2}(p',p) &= \int \mathrm{d}^4k\ (2\pi)^4 e\gamma^\nu\ \frac{(-i)}{(2\pi)^4}\ \frac{-i(\not p' - \not k)+m}{(p'-k)^2+m^2-i\epsilon}\ \gamma^\mu\ \frac{(-i)}{(2\pi)^4}\frac{-i(\not p - \not k)+m}{(p-k)^2+m^2-i\epsilon}\ \times \\ &\qquad \times (2\pi)^4 e\gamma_\nu\ \frac{(-i)}{(2\pi)^4}\frac{1}{k^2-i\epsilon} \\ &= i\,e^2\ \int \frac{\mathrm{d}^4k}{(2\pi)^4}\ \gamma^\nu\ \frac{-i(\not p' - \not k)+m}{(p'-k)^2+m^2-i\epsilon}\ \gamma^\mu\ \frac{-i(\not p - \not k)+m}{(p-k)^2+m^2-i\epsilon}\ \gamma_\nu\ \frac{1}{k^2-i\epsilon}\ . \qquad (3.84)\end{aligned}$$

For large $|k|$ the integral behaves like $\int \frac{\mathrm{d}^4 k}{k^4}$ which diverges logarithmically. We have to introduce again some regularization, e.g. dimensional regularization as before. Actually, this diagram could also be regularized by simply including an additional factor $\frac{M^2}{k^2+M^2}$ into the photon propagator with M being taken to ∞ in the end. It can be shown that this does not affect the gauge invariance. Of course, there is also a contribution from the counterterm $-ie\,(Z_2-1)A_\mu\bar\psi\gamma^\mu\psi$ in $\mathcal{L}_2$:

$$\Gamma^\mu_{e^2}(p',p) = \Gamma^\mu_{\text{loop},e^2}(p',p) + (Z_2-1)\gamma^\mu\ . \qquad (3.85)$$

We will not do the complete computation of $\Gamma^\mu_{\text{loop},e^2}(p',p)$ here. However, we will extract its divergent piece and show that it is precisely cancelled by the counterterm with the same Z_2 as already determined from the self-energy. We will also extract a certain finite part which gives the first higher-order correction to the magnetic moment of the electron, the famous $g-2$.

3.6.1 Cancellation of the divergent piece

The divergent part of $\Gamma^\mu_{\text{loop},e^2}(p',p)$ arises from the large $|k|$ limit of the integrand. In this limit one can neglect the external momenta and the mass so that with dimensional regularization

$$\begin{aligned}\Gamma^\mu_{\text{loop},e^2}(0,0)\Big|_{m=0} &= i\,e^2\ \int \frac{\mathrm{d}^dk}{(2\pi)^4}\ \gamma^\nu\ \frac{i\not k}{k^2-i\epsilon}\ \gamma^\mu\ \frac{i\not k}{k^2-i\epsilon}\ \gamma_\nu\ \frac{1}{k^2-i\epsilon} \\ &= -i\,e^2\ (d-2)\int \frac{\mathrm{d}^dk}{(2\pi)^4}\ \frac{k^2\gamma^\mu - 2k^\mu k^\nu\gamma_\nu}{(k^2-i\epsilon)^3} \\ &= -i\,e^2\,\gamma^\mu\ \frac{(d-2)^2}{d}\int \frac{\mathrm{d}^dk}{(2\pi)^4}\ \frac{1}{(k^2-i\epsilon)^2}\ . \qquad (3.86)\end{aligned}$$

The trouble with this integral is that for $d<4$ it is UV convergent but IR divergent, and vice versa for $d>4$ (and UV and IR divergent for $d=4$). To avoid the IR divergence, we should have kept the electron mass m. If we are only interested in the UV divergent behavior it is enough to keep m in the denominators so that

$$\begin{aligned}\Gamma^\mu_{\text{loop},e^2}(0,0)\Big|_{\text{div}} &\sim -i\,e^2\,\gamma^\mu\ \frac{(d-2)^2}{d}\int \frac{\mathrm{d}^dk}{(2\pi)^4}\ \frac{1}{(k^2+m^2-i\epsilon)^2} \\ &\sim e^2\,\gamma^\mu\ \frac{(d-2)^2}{d}\ \frac{\pi^{d/2}}{(2\pi)^4}\,\Gamma\Big(2-\frac{d}{2}\Big)\,(m^2)^{\frac{d}{2}-2} \\ &\sim \frac{e^2}{8\pi^2}\,\gamma^\mu\ \frac{1}{\epsilon}\ . \qquad (3.87)\end{aligned}$$

This divergence should be cancelled by the diverging part of $(Z_2-1)\gamma^\mu$. However, Z_2 is already determined from the above computation of the electron self-energy and, hence, it is by no means obvious that this cancellation does indeed take place. Nevertheless, we know from the Ward identity (2.52) that if Σ^* is finite, then $\Gamma^\mu(p,p)$, and in particular $\Gamma^\mu(0,0)$ must also be finite. It follows that the divergences must cancel between (3.87) and $(Z_2-1)\gamma^\mu$. Of course, this is confirmed by the explicit expression (3.82) of Z_2: $(Z_2-1)_{e^2}\Big|_{\rm div}=-\frac{e^2}{8\pi^2}\frac{1}{\epsilon}$. We conclude that the renormalized vertex function is finite (at least to the order we computed).

3.6.2 The magnetic moment of the electron: $g-2$

Historically, the computation of the vertex function has played an important role since it gives the first correction to the magnetic moment of the electron. This magnetic moment μ is usually expressed through the g-factor as $\mu=g\frac{e}{2m}s$, where $s\equiv j=\frac{1}{2}$ is the spin of the electron. The tree-level result which can also be obtained from studying the Dirac equation in a magnetic field is $g=2$. Let us sketch how to obtain the corrections to this result.

First note that one is interested in evaluating $\Gamma^\mu(p',p)$ between on-shell wave-functions $\bar u(p',\sigma')$ and $u(p,\sigma)$ as appropriate when computing e.g. the S-matrix elements between an incoming electron of momentum p and polarization σ and an outgoing one with p' and σ'. The interaction is with an electromagnetic field $A_\mu(q)$ where $q=p'-p$. Now $\Gamma^\mu(p',p)$ can involve various products of γ^μ, $\not p$ and $\not p'$, together with p^μ and p'^μ. Moving all $\not p$ to the right till one can use $\not p u(p,\sigma)=imu(p,\sigma)$ and moving all $\not p'$ to the left until $\bar u(p',\sigma')\not p'=\bar u(p',\sigma')im$, one is left with the general structure

$$\bar u(p',\sigma')\Gamma^\mu(p',p)u(p,\sigma)=\bar u(p',\sigma')\Big\{F(q^2)\,\gamma^\mu-\frac{i}{2m}G(q^2)\,(p+p')^\mu+\frac{1}{2m}H(q^2)\,(p-p')^\mu\Big\}u(p,\sigma)\ , \quad (3.88)$$

where the coefficient functions can depend on the only scalar available, i.e. $q^2=-2m^2-2p\cdot p'$ and, of course, on m. If one contracts this equation with $(p-p')_\mu$, the left-hand-side vanishes by the Ward identity (2.54), while the right-hand-side equals $\frac{q^2}{2m}H(q^2)\bar u u$ and thus

$$H(q^2)=0\ . \quad (3.89)$$

The following identity is valid between the on-shell wave-functions $\bar u(p',\sigma')$ and $u(p,\sigma)$: $i[\gamma^\mu,\gamma^\nu](p'-p)_\nu=i\gamma^\mu(\not p'-\not p)-i(\not p'-\not p)\gamma^\mu=2i(p+p')^\mu+4m\gamma^\mu$. We can use it to rewrite (3.88) as

$$\begin{aligned}\bar u(p',\sigma')\Gamma^\mu(p',p)u(p,\sigma)&=\bar u(p',\sigma')\Big\{\big(F(q^2)+G(q^2)\big)\,\gamma^\mu-\frac{i}{4m}G(q^2)\,[\gamma^\mu,\gamma^\nu](p'-p)_\nu\Big\}u(p,\sigma)\\ &=\bar u(p',\sigma')\Big\{-\frac{i}{2m}\big(F(q^2)+G(q^2)\big)\,(p+p')^\mu\\ &\qquad+\frac{i}{4m}F(q^2)\,[\gamma^\mu,\gamma^\nu](p'-p)_\nu\Big\}u(p,\sigma)\ ,\end{aligned} \quad (3.90)$$

This form is particularly useful when studying the limit where $q=p'-p\to0$. The Ward identity (2.53) states $\bar u(p,\sigma')\Gamma^\mu(p,p)u(p,\sigma)=\bar u(p,\sigma')\gamma^\mu u(p,\sigma)$, so that

$$F(0)+G(0)=1\ . \quad (3.91)$$

Indeed, the vertex function captures the quantum corrections to the coupling of the electron to the electromagnetic field. It contributes $ie\,\bar u(p',\sigma')\Gamma^\mu(p',p)u(p,\sigma)A_\mu(q)$ to an "effective" $-\mathcal{L}_{\rm int}=\mathcal{H}_{\rm int}$. For $p=p'$ and in the rest frame of the electron this is just $+eA_0=(-e)A^0$, stating that $-e$ is indeed the charge of the electron one can measure. To determine the magnetic moment, consider the second rewriting in (3.90). The term $\sim (p+p')^\mu$ is blind to the spin σ and cannot contribute to the magnetic moment. The second term yields a contribution to the effective interaction Hamiltonian

$$\frac{i}{4m}F(q^2)\,\bar u(p',\sigma')[\gamma^\mu,\gamma^\nu]u(p,\sigma)(p'-p)_\nu\ ieA_\mu(q)=\frac{ie}{8m}F(q^2)\,\bar u(p',\sigma')[\gamma^\mu,\gamma^\nu]F_{\mu\nu}(q)u(p,\sigma)\ , \quad (3.92)$$

where $F_{\mu\nu}(q)=iq_\mu A_\nu(q)-iq_\nu A_\mu(q)$. In an almost static situation (i.e. to first order in q), one has $\bar u(0,\sigma')[\gamma^\mu,\gamma^\nu]F_{\mu\nu}(0)u(0,\sigma)=8i\vec B\cdot\vec S_{\sigma'\sigma}$ so that the spin-dependent terms in $\mathcal{H}_{\rm int}$ are $-\frac{e}{m}F(0)\vec B\cdot\vec S\equiv -\vec B\cdot\vec\mu$, with the magnetic moment of the electron being

$$\vec\mu\ =\ \frac{e}{m}F(0)\,\vec S\ \equiv\ g\,\frac{e}{2m}\,\vec S\ , \quad (3.93)$$

so that the celebrated g-factor equals

$$g=2\,F(0)=2-2\,G(0)\ , \quad (3.94)$$

where we used (3.91).

It remains to explicitly compute $G(0)$ at one loop. To do so, we only need to keep the part $\sim (p+p')^\mu$ in $\Gamma^\mu(p',p)$, while we can drop the part $\sim\gamma^\mu$. Due to the explicit factor $(p+p')^\mu$ we expect this part to be given by a finite (converging) integral. We start with (3.84) and introduce dimensional regularization and Feynman parameters as usual:

$$\Gamma^\mu_{{\rm loop},e^2}(p',p)$$
$$=-\frac{2i\,e^2}{(2\pi)^4}\int_0^1{\rm d}x\int_0^{1-x}{\rm d}y\int{\rm d}^dk\ \frac{\gamma^\nu\big((\not p'-\not k)+im\big)\gamma^\mu\big((\not p-\not k)+im\big)\gamma_\nu}{\Big\{x\big[(p'-k)^2+m^2\big]+y\big[(p-k)^2+m^2\big]+(1-x-y)k^2-i\epsilon\Big\}^3}. \quad (3.95)$$

The actual computation is a bit lengthy: First one does the γ-matrix algebra in the numerator. The denominator is $\big[(k-xp'-yp)^2+(x+y)^2m^2+xyq^2-i\epsilon\big]^3$ and one shifts $k\to k+xp'+yp$ and drops the terms linear in k. The result still contains many terms, but taking Γ^μ between $\bar u(p')$ and $u(p)$ this can be further simplified as above. Dropping then the terms $\sim\gamma^\mu$ one gets

$$\bar u(p')\Gamma^\mu_{{\rm loop},e^2}(p',p)u(p)\Big|_{p+p'}=\frac{e^2}{2\pi^4}m\int_0^1{\rm d}x\int_o^{1-x}{\rm d}y\ \bar u\Big[(x+y)\big((1-y)p^\mu+(1-x)p'^\mu\big)-yp^\mu-xp'^\mu\Big]u\ I_d(x,y,q^2) \quad (3.96)$$

where $I_d(x,y,q^2)$ is a convergent integral for $d<6$. For $d=4$ it equals

$$I_4(x,y,q^2)=\int{\rm d}^4k\ \big[k^2+(x+y)^2m^2+xyq^2-i\epsilon\big]^{-3}=\frac{i\pi^2}{2}\big[(x+y)^2m^2+xyq^2\big]^{-1}\ . \quad (3.97)$$

Writing $\int_0^1{\rm d}x\int_0^{1-x}{\rm d}y\,f(x,y)=\int_0^1{\rm d}x\int_0^1{\rm d}y\,\theta(1-x-y)f(x,y)=\int_0^1{\rm d}x\int_0^1{\rm d}y\,\theta(1-x-y)f(y,x)$, we see that we can symmetrize the integrand of (3.96) in x and y, and thus in p and p':

$$\bar u(p')\Gamma^\mu_{{\rm loop},e^2}(p',p)u(p)\Big|_{p+p'}=\frac{ie^2}{8\pi^2}\,m\ \bar u(p+p')^\mu u\int_0^1{\rm d}x\int_0^{1-x}{\rm d}y\ \frac{(x+y)(2-x-y)-(x+y)}{(x+y)^2m^2+xyq^2}\ , \quad (3.98)$$

from which we identify

$$G(q^2) = -\frac{e^2}{4\pi^2} m^2 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{(x+y)(2-x-y)-(x+y)}{(x+y)^2 m^2 + xyq^2} \ . \tag{3.99}$$

We only need

$$G(0) = -\frac{e^2}{4\pi^2} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \, \frac{1-x-y}{x+y} = -\frac{e^2}{8\pi^2} \ , \tag{3.100}$$

so that

$$g = 2\left(1+\frac{e^2}{8\pi^2}\right) = 2\left(1+\frac{\alpha}{2\pi}\right) \quad \Leftrightarrow \quad g-2 = \frac{\alpha}{\pi} \ . \tag{3.101}$$

This is the classical result of Schwinger. Since then, the art of measuring and computing $g-2$ has been pushed to an extreme refinement (four loops in QED!) – with an excellent agreement.

3.7 One-loop radiative corrections in scalar ϕ^4

The Ward identity of QED linked the charge or coupling constant renormalization to the photon wave-function renormalization, see eq. (3.1). In general though, such a relation is not expected, and here we will briefly discuss scalar ϕ^4 theory where the coupling constant gets renormalized separately.

We start with (cf. eqs. (2.17), (2.18) and (2.19))

$$\mathcal{L} = -\frac{1}{2}(\partial_\nu \phi_B)^2 - \frac{1}{2} m_B^2 \phi_B^2 - \frac{g_B}{4!}\phi_B^4 \ , \tag{3.102}$$

and let

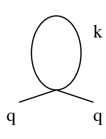
$$\phi_B = \sqrt{Z}\,\phi \ , \quad m_B^2 = m^2 - \delta m^2 \ , \quad g_B = \frac{Z_g}{Z^2}\, g \ , \tag{3.103}$$

so that

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 \ , && (3.104)\\
\mathcal{L}_0 &= -\frac{1}{2}(\partial_\nu\phi)^2 - \frac{1}{2}m^2\phi^2 \ , && (3.105)\\
\mathcal{L}_1 &= -\frac{g}{4!}\,\phi^4 \ , && (3.106)\\
\mathcal{L}_2 &= -\frac{1}{2}(Z-1)(\partial_\nu\phi)^2 - \frac{1}{2}(Z-1)m^2\phi^2 + \frac{1}{2}Z\delta m^2\phi^2 - \frac{g}{4!}(Z_g-1)\phi^4 \ . && (3.107)
\end{aligned}$$

Propagator: Recall that the complete propagator is $\Delta'(q) = \Big(q^2+m^2-\Pi^*(q^2)-i\epsilon\Big)^{-1}$ with $\Pi^*(q^2) = -(Z-1)(q^2+m^2)+Z\delta m^2+\Pi^*_{\rm loop}(q^2) = \Pi^*_{\rm loop}(q^2)-\Pi^*_{\rm loop}(-m^2)-(q^2+m^2)\frac{\mathrm{d}}{\mathrm{d}q^2}\Pi^*_{\rm loop}(q^2)|_{q^2=-m^2}$. Then the one-loop contribution of order g to the one-particle irreducible complete propagator Π^* is given by[20]

$$i(2\pi)^4\Pi^*_{{\rm loop},g}(q^2) =$$



$$= \frac{1}{2}\int \mathrm{d}^d k \frac{(-i)}{(2\pi)^4}\frac{1}{k^2+m^2-i\epsilon}\big(-ig(2\pi)^4\big) = -\frac{i}{2}\,g\int\frac{\mathrm{d}^d k_E}{k_E^2+m^2-i\epsilon} \ , \tag{3.108}$$

[20]There are also one-loop contributions with additional insertions of the counterterms. These are higher order in g.

where the factor $\frac{1}{2}$ comes from the symmetry factor of the diagram. In dimensional regularization we get[21]

$$\begin{aligned}
\Pi^*_{\text{loop},g}(q^2) &= -\frac{g}{2}\widetilde{\mu}^{4-d}\frac{\pi^{d/2}}{(2\pi)^4}\Gamma\Big(1-\frac{d}{2}\Big)\, m^{d-2} \\
&= -\frac{g}{2}\, m^2\, \frac{\pi^2}{(2\pi)^4}\Big(1-\frac{\epsilon}{2}\log\pi\Big)\Big(-\frac{2}{\epsilon}+\gamma-1+\mathcal{O}(\epsilon)\Big)\Big(1-\frac{\epsilon}{2}\log\frac{m^2}{\widetilde{\mu}^2}\Big) \\
&= \frac{g}{16\pi^2}\, m^2\left(\frac{1}{\epsilon}-\frac{\gamma+\log\pi-1}{2}-\frac{1}{2}\log\frac{m^2}{\widetilde{\mu}^2}\right)\ . \qquad (3.109)
\end{aligned}$$

Since this is q-independent, $Z=1$ at this order (cf. (2.27)):

$$Z = 1+\mathcal{O}(g^2)\ . \qquad (3.110)$$

For the mass renormalization we then get from (2.26)

$$\delta m_g^2 = -\Pi^*_{\text{loop},g}(-m^2) = -\frac{g}{16\pi^2}\, m^2\left(\frac{1}{\epsilon}-\frac{\gamma+\log\pi-1}{2}-\frac{1}{2}\log\frac{m^2}{\widetilde{\mu}^2}\right) \qquad (3.111)$$

so that

$$\Pi^*_g(q^2) = 0\ . \qquad (3.112)$$

4-point function: We define $F(q_1,q_2\to q_1',q_2')$ as the amputated four-point function, i.e. with the external (full) propagators removed and normalized such that to lowest order $F=g$. Then the connected two-particle to two-particle scattering S-matrix element is given by

$$S^c(q_1,q_2\to q_1',q_2') = -i(2\pi)^4\delta^{(4)}(q_1+q_2-q_1'-q_2')\frac{1}{(2\pi)^6\sqrt{16E_1E_2E_1'E_2'}}F(q_1,q_2\to q_1',q_2')\ , \qquad (3.113)$$

with all momenta being on-shell. In the present ϕ^4 theory, the amputated 4-point function is just the renormalized 1PI 4-point vertex function $\Gamma^{(4)}$ and, taking into account the normalization of F we have[22]

$$\Gamma^{(4)}(q_1,q_2,-q_1',-q_2') = -(2\pi)^4\delta^{(4)}(q_1+q_2-q_1'-q_2')\, F(q_1,q_2\to q_1',q_2')\ . \qquad (3.114)$$

It follows that, up to this order g^2, the four-point function $-i(2\pi)^4F$ is given by the tree-level vertex $-i(2\pi)^4 g$, the one-loop diagrams shown in Fig. 11, as well as the contribution from the counterterms $-i(2\pi)^4 g(Z_g-1)$. Hence

$$F = (q_1,q_2\to q_1',q_2') = g + F_{\text{loop}}(q_1,q_2\to q_1',q_2') + g(Z_g-1)\ , \qquad (3.115)$$

where in dimensional regularization

$$\begin{aligned}
&-i(2\pi)^4F_{\text{loop},g^2}(q_1,q_2\to q_1',q_2') \\
&\qquad = \frac{1}{2}\left[-i(2\pi)^4 g\right]^2\left[\frac{(-i)}{(2\pi)^4}\right]^2\Bigg\{\int \mathrm{d}^d k\frac{1}{[(k+q_1)+m^2-i\epsilon][(q_2-k)^2+m^2-i\epsilon]} \\
&\qquad\qquad +(q_2\to -q_1')+(q_1\to -q_2')\Bigg\}\ . \qquad (3.116)
\end{aligned}$$

[21]Just as in QED, if we want to keep a dimensionless coupling constant for $d\neq 4$ we must replace the coupling in the d-dimensional Lagrangian by $g\widetilde{\mu}^{4-d}$, where $\widetilde{\mu}$ is some arbitrary mass scale.

[22]Recall that we usually take the arguments of Green functions or vertex functions as incoming four-momenta, i.e. an outgoing q_i appears as $-q_i$.

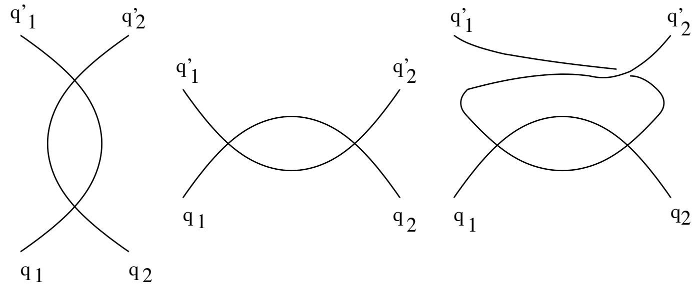


Figure 11: The three one-loop contributions at order g^2 to the 4-point function. Other one-loop contributions also involve counterterm insertions and are of higher order in g.

Introducing the Feynman parameter, as well as the standard notation

$$s = -(q_1 + q_2)^2 \ , \quad t = -(q_1 - q_1')^2 \ , \quad u = -(q_1 - q_2')^2 \ , \tag{3.117}$$

the denominator written explicitly in (3.116) becomes

$$\begin{aligned} [\ldots][\ldots] &= \left[k^2 + 2xq_1 k + xq_1^2 - 2(1-x)q_2 k + (1-x)q_2^2 + m^2 - i\epsilon\right]^2 \\ &= \left[\big(k + xq_1 - (1-x)q_2\big)^2 - \big(xq_1 - (1-x)q_2\big)^2 + xq_1^2 + (1-x)q_2^2 + m^2 - i\epsilon\right]^2 \\ &= \left[\big(k + xq_1 - (1-x)q_2\big)^2 - x(1-x)s + m^2 - i\epsilon\right]^2 . \end{aligned} \tag{3.118}$$

We then get

$$F_{\text{loop},g^2}(q_1, q_2 \to q_1', q_2') = \frac{i\, g^2}{2(2\pi)^4} \int_0^1 \mathrm{d}x \left\{ \int \frac{\mathrm{d}^d k}{[k^2 + m^2 - sx(1-x) - i\epsilon]^2} + (s \to t) + (s \to u) \right\} , \tag{3.119}$$

or after the by now familiar Wick rotation and evaluation of the Euclidean integral:

$$F_{\text{loop},g^2}(q_1, q_2 \to q_1', q_2') = -\frac{g^2}{2(2\pi)^4} \pi^{d/2} \Gamma\Big(2 - \frac{d}{2}\Big) \int_0^1 \mathrm{d}x \left\{ [m^2 - sx(1-x)]^{\frac{d}{2}-2} + (s \to t) + (s \to u) \right\} . \tag{3.120}$$

As already mentioned, if we want to keep a dimensionless coupling constant for $d \neq 4$ we must replace the coupling in the d-dimensional Lagrangian by $g\tilde{\mu}^{4-d}$, where $\tilde{\mu}$ is some arbitrary mass scale. This results in an extra factor $\tilde{\mu}^{4-d}$ accompanying every factor of g. However, we want to normalize F such that its tree-level value is just g, not $g\tilde{\mu}^{4-d}$, so that we just need to include a single factor of $\tilde{\mu}^{4-d} = (\tilde{\mu}^2)^{2-\frac{d}{2}}$ in the r.h.s. of (3.120). This then nicely combines with the terms in the braces to make them dimensionless. Finally, expanding in $\epsilon = 4 - d$ we get

$$\begin{aligned} F_{\text{loop},g^2}(q_1, q_2 \to q_1', q_2') &= -\frac{g^2}{32\pi^2} \left\{ \left(\frac{2}{\epsilon} - \log \pi - \gamma - \int_0^1 \mathrm{d}x \log \frac{m^2 - sx(1-x)}{\tilde{\mu}^2} \right) + (s \to t) + (s \to u) \right\} \\ &= -\frac{3g^2}{32\pi^2} \left(\frac{2}{\epsilon} - \log \pi - \gamma \right) + \frac{g^2}{32\pi^2} \int_0^1 \mathrm{d}x \left\{ \log \frac{m^2 - sx(1-x)}{\tilde{\mu}^2} + (s \to t) + (s \to u) \right\} . \end{aligned} \tag{3.121}$$

The full four-point function up to order g^2 is given by adding the tree-level and the counterterm contributions according to (3.115). Just as the wave-function renormalization factors Z, Z_2 and Z_3 are fixed by requiring the full propagator to satisfy certain normalization conditions, we must impose some condition on the full four-point function to fix Z_g. A rather standard condition is to require that it equals g at the symmetric (off-shell) point $s = t = u = -\frac{4}{3}\mu^2$, where μ is some mass scale (which may or maynot equal the $\tilde{\mu}$ already introduced):

$$F\left(s = t = u = -\frac{4}{3}\mu^2\right) = g \ . \tag{3.122}$$

Using this condition in (3.115) fixes Z_g as

$$g(Z_g - 1) = -F_{\text{loop}}\left(s = t = u = -\frac{4}{3}\mu^2\right) \ . \tag{3.123}$$

Up to the order we computed this gives

$$(Z_g - 1)_{\text{order}\, g} = \frac{3g}{32\pi^2}\left(\frac{2}{\epsilon} - \log\pi - \gamma - \int_0^1 \mathrm{d}x \log\frac{m^2 + \frac{4}{3}\mu^2 x(1-x)}{\tilde{\mu}^2}\right) \ . \tag{3.124}$$

Substituting this back into (3.115) or, equivalently, subtracting (3.123) from (3.115) finally gives

$$F_{g^2}(q_1, q_2 \to q_1', q_2') = g + \frac{g^2}{32\pi^2}\int_0^1 \mathrm{d}x \left\{ \log\frac{m^2 - sx(1-x)}{m^2 + \frac{4}{3}\mu^2 x(1-x)} + (s \to t) + (s \to u)\right\} \ . \tag{3.125}$$

Note that the argument of the logarithm may be negative for $\sqrt{s} \geq 2m$, so that F then has an imaginary part. This is in agreement with the optical theorem (unitarity). Note also that the dependence on the scale $\tilde{\mu}$ has cancelled upon imposing the renormalization condition. However, the latter condition has introduced a physically relevant scale μ: it is the scale where the measurable coupling g is defined. In a massive theory, one could conveniently take $\mu = m$ or $\mu = 0$, but let's stay more general.

PART II :

RENORMALIZATION AND RENORMALIZATION GROUP

4 General renomalization theory

In section 2, we have seen that the notion of renormalization arises in order to keep correctly normalized full propagators, with their poles at the physical masses. This discussion was independent of any divergences arising in loop integrals. In the previous section, we have computed various one-loop two-, three- and four-point functions. The one-loop contributions were divergent and had to be regulated. We have observed that after performing the renormalization, according to the conditions formulated in section 2, the renormalized quantities turned out to be finite (i.e. have finite limits even if the regulator is removed). The purpose of the general renormalization theory is to show that this is no accident but remains valid to all orders in perturbation theory, at least for so-called renormalizable theories as QED or scalar ϕ^4-theory.

4.1 Degree of divergence

The *superficial* degree of divergence D characterizes the behaviour of the momentum integral $\int d^4k_1 \dots d^4k_L[\dots]$ when all $|k_i| \to \infty$ with a common $k \to \infty$. More precicely: if the integral behaves as $\sim \int k^{D-1} dk$ then D is the superficial degree of divergence. An integral with $D \geq 0$ is called superficially divergent, and one with $D < 0$ is called superficially convergent. This does not necessarily mean the integral really diverges or converges, but we will see that the superficial degree of divergence D nevertheless plays an important role. Often, we will talk somewhat loosely about the superficial degree of divergence D of a diagram, meaning the D of the associated integral. To determine D let

- I_f be the number of internal lines of the field f,
- E_f be the number of external lines of the field f,
- N_i the number of vertices of type i with d_i derivatives and n_{if} fields f attached.

Now, for large momenta the propagators behave as $\Delta_f(k) \sim k^{-2+2s_f}$, were

- for scalars: $s_f = 0$,
- for spin 1/2: $s_f = 1/2$,
- for massive spin 1: $s_f = 1$ since $\Delta(q)^{\mu\nu} \sim \frac{1}{q^2}\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{m^2}\right)$
- and for photons or gravitons: $s_f = 0$, since: $\Delta(q)^{\mu\nu} \sim \frac{1}{q^2}\left(\eta^{\mu\nu} - \xi\frac{q^\mu q^\nu}{q^2}\right)$.

It follows that

$$D = 4L + \sum_i N_i d_i + \sum_f I_f(-2+2s_f) \ . \tag{4.1}$$

Now use the relation (cf. (1.45)) $L = I - V + 1 = \sum_f I_f - \sum_i N_i + 1$ to obtain

$$D = 4 + \sum_i N_i(d_i - 4) + \sum_f I_f(2+2s_f) \ . \tag{4.2}$$

Use further $\sum N_i n_{if} = 2I_f + E_f$ to get

$$D = 4 + \sum_i N_i\big((d_i - 4 + \sum_f n_{if}(s_f+1)\big) - \sum_f E_f(s_f+1) \ , \tag{4.3}$$

or

$$D = 4 - \sum_f E_f(s_f+1) - \sum_i N_i \Delta_i \ , \tag{4.4}$$

with

$$\boxed{\Delta_i = 4 - d_i - \sum_f n_{if}(s_f+1) \ .} \tag{4.5}$$

One can repeat this argument in an arbitrary space-time dimension d. One simply has to replace $4L \to d\,L$ in (4.1). The final formula then is modified as

$$D = d - \sum_f E_f\left(s_f + \frac{d-2}{2}\right) - \sum_i N_i\Delta_i \tag{4.6}$$

$$\Delta_i = d - d_i - \sum_f n_{if}\left(s_f + \frac{d-2}{2}\right) \ . \tag{4.7}$$

There is an alternative derivation of (4.4), which works easily as long as only scalars, spin $\frac{1}{2}$ particles, photons or gravitons are involved (no massive spin 1). Define the dimension of a field f as given by the behavior of its propagator: $\langle T(f(x)\, f(y))\rangle \sim \int \mathrm{d}^4k \frac{e^{ik(x-y)}}{k^{2-2s_f}}$ has (mass) dimension $4-(2-2s_f) = 2(1+s_f)$ and hence the field f has (mass) dimension $\mathcal{D}_f = 1+s_f$. (This does not work for massive spin 1 fields that have $s_f = 1$ but (mass) dimension 1, just as photons - it is the explicit appearence of the mass m which messes up the argument.) Then any interaction of type i involving n_{if} fields f and d_i derivatives has dimension $\sum_f n_{if}\mathcal{D}_f + d_i$. Since the Lagrangian must have dimension 4, the coupling constant g_i must have dimension $[g_i] = 4 - \sum_f n_{if}\mathcal{D}_f - d_i$ or

$$[g_i] = \Delta_i \ . \tag{4.8}$$

Now in a diagram with N_i interactions of type i and E_f external lines for fields of type f the corresponding $\langle T(f(x_1)\ldots)\rangle$ has dimension $\sum_f E_f\mathcal{D}_f$. Its Fourier transform then has dimension $\sum_f E_f(\mathcal{D}_f - 4)$. This dimension must arise from the products of an overall $\delta^{(4)}(\sum p_j)$ of dimension -4, propagators for the external lines of total dimension $\sum_f E_f(-2+2s_f) = \sum_f E_f(2\mathcal{D}_f - 4)$, the coupling constants of dimension $\sum_i N_i\Delta_i$ and the momentum integrals whose dimension equals the superficial degree of divergence D of the diagram. Hence $\sum_f E_f(\mathcal{D}_f - 4) = -4 + \sum_f E_f(2\mathcal{D}_f - 4) + \sum_i N_i\Delta_i + D$, or

$$D = 4 - \sum_f E_f \mathcal{D}_f - \sum_i N_i \Delta_i \ , \tag{4.9}$$

in agreement with (4.4). As an example, in Table 2, we consider the various interactions in QED, where one has $\mathcal{D}_\psi = \frac{3}{2}$ and $\mathcal{D}_{A_\mu} = 1$ and one indeed verifies that the Δ_i coincide with the dimensions of the coupling constants.

$-ie\overline{\psi}\not{A}\psi$	$\Delta = 4 - 2 \times \frac{3}{2} - 1 = 0$
$(Z_3 - 1)F_{\mu\nu}F^{\mu\nu}$	$\Delta = 4 - 2 - 2 = 0$
$(Z_2 - 1)\overline{\psi}\not{\partial}\psi$	$\Delta = 4 - 2 \times \frac{3}{2} - 1 = 0$
$[-(Z_2 - 1)m + Z_2\delta m]\overline{\psi}\psi$	$\Delta = 4 - 2 \times \frac{3}{2} = 1$

Table 1: The values of Δ for the various interactions in QED

One can similarly assign $\Delta_a^{\rm comp}$ to composite operators $\mathcal{O}_a(x)$ inserted in a diagram. $\Delta_a^{\rm comp}$ is the same as one would define for the corresponding interaction, and eq. (4.4) would be replaced by $D = 4 - \sum_f E_f\mathcal{D}_f - \sum_i N_i\Delta_i - \sum_a N_a\Delta_a^{\rm comp}$.

The importance of the notion of superficial degree of divergence D resides in the following remarks:

- If $\Delta_i \geq 0$ for all i then only Green's functions with $4 - \sum_f E_f\mathcal{D}_f \geq 0$ can have $D \geq 0$ and, hence, there are only finitely many superficially divergent Green's functions. Note that, unless $\Delta_i > 0,\ \forall i$, these are still infinitely many diagrams (arbitrary N_i for fixed E_f).

- Interactions with $\Delta_i \geq 0$ are called *renormalizable interactions*. Theories with only renormalizable interactions are called *renormalizable theories*. In such theories only finitely many Green's functions are superficially divergent. We will see that in such theories *all* Green's functions can be rendered finite by a finite number of counterterms corresponding to the redefinition of a finite number of physical constants (couplings and masses) and the (re)normalizations of the fields.

- If $\Delta_i < 0$ for some interaction, then an infinite number of Green's functions (arbitrary numbers E_f of external fields) are superficially divergent. In general, one then needs an infinite number of counterterms to make them finite. Such theories are called *non-renomalizable.*

The importance of the notion of superficial divergence is partly due to the following **Theorem** (Weinberg): If $D < 0$ for the complete integration and *any* sub-integration (i.e. holding some linear combination of momenta fixed) then the integral is really convergent.

In particular, at one-loop, there are no sub-integrations and, hence, by this theorem, any one-loop integral with $D < 0$ is convergent. Note also that this theorem does not say anything about the divergence of integrals with $D > 0$. We have seen examples in QED where the integrals are less divergent than expected. The vacuum polarization diagram e.g. had $D = 2$, but gauge invariance allows one to "pull out" two factors of external momentum by writing $\Pi^*_{\mu\nu} = (q^2\eta_{\mu\nu} - q_\mu q_\nu)\pi(q^2)$ with

$\pi(q^2)$ only logarithmically divergent ($D = 0$). More generally, symmetries may result in cancellations between the divergences arising from individual diagrams yielding less divergent or even finite Green's functions.

4.2 Structure of the divergences

Suppose an integral has $D < 0$, i.e. it is superficially convergent. This means that if all k are simultaneously taken to be large the integral converges. Thus the only possibility for this integral not to converge is that it diverges if some combination of momenta is held fixed and the divergence is due to the sub-integration over the other momenta. This must necessarily correspond to a subdiagram. Hence a superficially convergent diagram can only be divergent due to a diverging subdiagram (with $D' \geq 0$), as shown e.g. in Fig. 12. However, such subdiagrams are of lower order in perturbation theory. If they have already been rendered finite by the addition of appropriate conterterms at the lower order in perturbation theory, one no longer has to worry about such diverging subdiagrams any more. Henceforth, we will assume that such diverging subdiagrams have been taken care of and that superficially convergent diagrams are convergent.

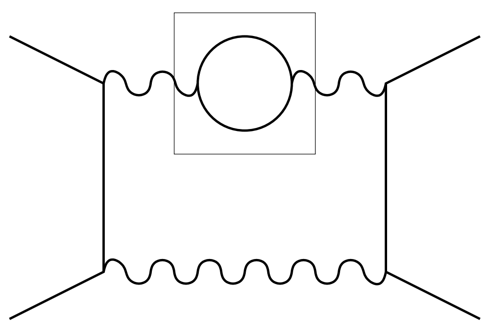


Figure 12: A superficially convergent diagram containing a divergent vacuum-polarization subdiagram.

If an integral $I(p_i)$ depending on external momenta p_i has superficial degree of divergence $D \geq 0$, then by differentiating with respect to the external momenta lowers D by one unit since

$$\frac{\partial}{\partial p_i^\mu} \, \frac{1}{(k+p_i)^2 + m^2 - i\epsilon} = \frac{-2(k+p_i)_\mu}{[(k+p_i)^2 + m^2 - i\epsilon]^2} \ . \tag{4.10}$$

(There is a caveat to this argument to be discussed soon.) Differentiating $D+1$ times results in an integral having degree of divergence -1, i.e. which is superficially convergent. According to the above remarks it is then convergent. Hence

$$\frac{\partial}{\partial p_{i_1}^{\mu_1} \ldots \partial p_{i_{D+1}}^{\mu_{1+1}}} I(p_i) = J(p_i) \ , \tag{4.11}$$

where $J(p_i)$ is convergent. Upon integrating with respect to the external momenta p_i we get

$$I(p_i) = \mathcal{P}_D(p_i) + \widehat{I}(p_i) \ , \tag{4.12}$$

with $\mathcal{P}_D(p_i)$ a polynomial of order D in the external momenta with (a priori) divergent coefficients and $\widehat{I}(p_i)$ a convergent integral. Obviously, the diverging part $\mathcal{P}_D(p_i)$ is entirely contained in the first $D+1$ terms in a Taylor expansion of $I(p_i)$. We may then rephrase this by saying that $I(p_i)$ minus the first $D+1$ terms in a Taylor expansion in the external momenta is finite. We have seen examples of this in our one-loop computations, see e.g. (3.78) and (3.79) for the electron self-energy which one can rewrite as

$$\begin{aligned}\Sigma^*_{\text{loop},e^2}(\not p) &= (Z_2-1)_{e^2}(i\not p+m) - (Z_2\delta m)_{e^2} + \Sigma^*_{e^2}(\not p)\\ &= -i\frac{\partial}{\partial \not p}\Sigma^*_{\text{loop},e^2}(\not p)\Big|_{\not p=im}(i\not p+m) + \Sigma^*_{\text{loop},e^2}(im) + \Sigma^*_{e^2}(\not p)\ .\end{aligned} \tag{4.13}$$

The first two terms indeed constitute a first order polynomial in the external $\not p$ which are the first two terms in a Taylor expansion of $\Sigma^*_{\text{loop},e^2}$ in $\not p$ around $\not p = im$, while the last term $\Sigma^*_{e^2}(\not p)$ is the renormalized finite electron self-energy.

It is very important that the divergent part of the integral $I(p_i)$ is a polynomial in the external momenta since it is precisely such divergences that can be cancelled by *local* counterterms. Indeed, a local counterterm is of the form

$$\prod_f \prod_{i=1}^{E_f} (\partial_{\mu_{i,f}})^{d_{i,f}} f(x) \quad , \quad \sum d_{i,f} = d\ , \tag{4.14}$$

being the product of E_f fields of type f with a total number of d derivatives. Such a counterterm gives a (tree-level) contribution to the diagram with E_f external lines of type f which is a polynomial of order d in the momenta. Thus, to cancel the diverging $\mathcal{P}_D(p_i)$ one can always find a sum of such a local counterterms involving up to D derivatives. Now, such a counterterm will also appear as a new vertex in loop diagrams and it is important to check that it does not render the theory less renormalizable. Indeed, with at most D derivatives, we have $\Delta_{\text{c.t.}} \geq 4 - \sum_f E_f\mathcal{D}_f - D$. But in a renormalizable theory the possible values of D are constrained by $D \leq 4 - \sum_f E_f\mathcal{D}_f$ so that $4 - \sum_f E_f\mathcal{D}_f - D \geq 0$ and $\Delta_{\text{c.t.}} \geq 0$, so the counterterm is part of the finitely many renormalizable interactions. Moreover, if one includes *all* (finitely many) renormalizable interactions in the bare Lagrangian, then all possible counterterms are necessarily of the same form as the terms already present, and they just renormalize the couplings, masses and wave-functions

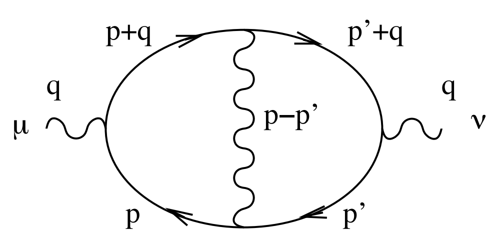


Figure 13: A two-loop diagram with overlapping divergences: two divergent subdiagrams share a common propagator.

There is a caveat in the argument that taking a derivative with respect to the external momenta lowers the superficial degree of divergence by one unit and which has to do with so-called *overlapping* *divergences*. An overlapping divergence occurs if two divergent subdiagrams share a common line (propagator), see e.g. the two-loop cotribution to the vacuum polarization shown in Fig. 13. Then taking a derivative with respect to an external momentum typically results in a *sum* of terms with the large momentum behaviour in one of the sub-loops improved in one term but not in the other, and vice versa for the other sub-loop. To be specific, for the example of Fig. 13 one has (neglecting the electron mass for simplicity)

$$\Pi^{*\text{overlap}}_{\mu\nu}(q) \sim \int \frac{\mathrm{d}^4p\,\mathrm{d}^4p'}{(p-p')^2}\,\mathrm{tr}\left[\frac{1}{\not p'}\gamma_\nu\frac{1}{\not p'+\not q}\gamma_\rho\frac{1}{\not p+\not q}\gamma_\mu\frac{1}{\not p}\gamma^\rho\right] . \tag{4.15}$$

Taking a derivative with respect to the external momenum q gives a sum of two integrals:

$$\frac{\partial}{\partial q^\sigma}\Pi^{*\text{overlap}}_{\mu\nu}(q) \sim \int \frac{\mathrm{d}^4p\,\mathrm{d}^4p'}{(p-p')^2}\,\mathrm{tr}\left[\frac{1}{\not p'}\gamma_\nu\frac{\gamma^\sigma}{(p'+q)^2}\gamma_\rho\frac{1}{\not p+\not q}\gamma_\mu\frac{1}{\not p}\gamma^\rho+\frac{1}{\not p'}\gamma_\nu\frac{1}{\not p'+\not q}\gamma_\rho\frac{\gamma^\sigma}{(p+q)^2}\gamma_\mu\frac{1}{\not p}\gamma^\rho\right] . \tag{4.16}$$

In the first term the large p' behavior is improved but *not* the large p behaviour, while in the second term things are reversed. Of course, the superficial degree of divergence is lowered by one unit, and taking two more derivatives would result in a sum of terms all with a $D' = -1$, but the criteria of the theorem cited above are not satisfied and one cannot conclude that up to a divergent subgraph the integral is convergent. Although this is true for each individual term in the sum, the trouble is that for each term the diverging subgraph is a different one. For example, in a given term in the sum, the divergence may be traced to a divergent p sub-integration corresponding to the divergent subgraph γ_1 in Fig. 14, while in another term the divergence would be due to the p' sub-integration corresponding to the subdiagram γ_2 in Fig. 14.

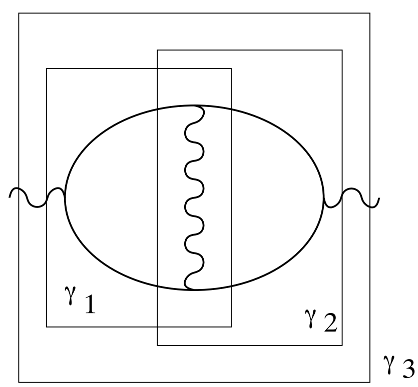


Figure 14: The nested sequences of subdiagrams used in the BPHZ construction: $\gamma_1 \subset \gamma_3$ and $\gamma_2 \subset \gamma_3$.

Although such overlapping divergences are more complicated, the previous discussion shows that one still has a sum of divergences associated with the various subdiagrams. To deal with such overlapping divergences in a systematic way and show that the usual counterterms are exactly sufficient to cancel these divergences, too, was the achievement of Bogoliubov, Parasiuk, Hepp and Zimmermann (BPHZ).

4.3 Bogoliubov-Parasiuk-Hepp-Zimmermann prescription and theorem

One defines a *forest* U as a family of nested (sub)graphs γ_i: if $\gamma_1 \in U$ and $\gamma_2 \in U$ then either $\gamma_1 \subset \gamma_2$ or $\gamma_2 \subset \gamma_1$ or $\gamma_1 \cap \gamma_2 = \emptyset$. For the diagram of Fig. 14 e.g., γ_1 and γ_2 cannot be in the same forest. More generally, overlapping loops cannot be in the same forest. On the other hand, in Fig. 12 the vacuum polarization subdiagram and the whole diagram can be in the same forest. Hence the notion of forest is exactly what one needs to distinguish overlapping subdiagrams (divergences) from non-overlapping ones. Again, in Fig. 14 the list of all forests is:

$$U_0 = \emptyset \ , \quad U_1 = \{\gamma_1\} \ , \quad U_2 = \{\gamma_2\} \ , \quad U_3 = \{\gamma_3\} \ , \quad U_4 = \{\gamma_1, \gamma_3\} \ , \quad U_5 = \{\gamma_2, \gamma_3\} \ . \tag{4.17}$$

For each diagram G one then considers such a family $\mathcal{F}(G)$ of all forests of G. Then we have the *BPHZ prescription*: For each forest $U_i \in \mathcal{F}(G)$ consider the nested sequence $\gamma_a \subset \gamma_b \subset \ldots$ of the (sub)graphs in U_i. Starting with the innermost γ_a, one defines a *subtraction term* $S(U_i)$ by replacing the integrand I_b of each subdiagram γ_b (in the nested sequence) with superficial degree of divergence D_b by the first $D_b + 1$ terms in its Taylor expansion (e.g. around 0) in the momenta flowing into (or out of) this subdiagram. Since at each step one only keeps a polynomial in the corresponding momenta, the final subtraction term is still a polynomial in the external momenta.

To see how this works, consider the diagram of Fig. 13, resp. Fig. 14:

- $U_0 = \emptyset$: no subtraction, $S(U_0) = 0$.
- $U_1 = \{\gamma_1\}$: This subdiagram is just the vertex function $e\Gamma^\mu_{1-\text{loop}}(p'+q, p')$ and we know from our one-loop computations that $e\Gamma^\mu_{1-\text{loop}}(p'+q, p') = eL\gamma^\mu + e\Gamma^\mu_{\text{finite}}(p'+q, p')$, with $\Gamma^\mu_{\text{finite}}(p'+q, p')$ being the finite (renormalized) vertex function and $L = -(Z_2 - 1)_{e^2}$. This subdiagram has $D = 0$ and hence the corresponding subtraction term is just the first term in the Taylor expansion: $S(U_1) = -e[(Z_2 - 1)_{e^2} + c]\gamma^\mu$ with $c = -\Gamma^\mu_{\text{finite}}(0, 0)$ and $(Z_2 - 1)_{e^2}$ given in (3.81). Obviously, this corresponds to a counterterm $\sim A_\mu \bar{\psi}\gamma^\mu\psi$.
- $U_2 = \{\gamma_2\}$: This similarly gives the same subtraction term $S(U_2) = -e[(Z_2 - 1)_{e^2} + c]\gamma^\mu$.
- $U_3 = \{\gamma_3\}$: This is just the full diagram we want to study. It has $D = 2$ and thus the subtraction term must be a second-order polynomial in the q. By gauge invariance this must be of the form $(\eta_{\mu\nu}q^2 - q_\mu q_\nu)C$ with $C = \pi_{\text{overlap}}(q^2 = 0)$. We are not claiming that this is the only divergence of this diagram, but BPHZ tell us that this is the subtraction term for this forest: $S(U_3) = (\eta_{\mu\nu}q^2 - q_\mu q_\nu) \times \pi_{\text{overlap}}(q^2 = 0)$. This corresponds to an order e^4 contribution to Z_3.
- $U_4 = \{\gamma_1, \gamma_3\}$: Here we must first replace the integrand of the γ_1-part by $S(U_1)$ as defined above, so that the integrand for the γ_3 part is the same as for $\Pi^{\mu\nu}_{1-\text{loop}}$ but with $e\gamma^\mu$ in the left vertex replaced by $e[-(Z_2 - 1)_{e^2} - c]\gamma^\mu$. Then the subtraction term is $S(U_4) = (q^2\eta^{\mu\nu} - q^\mu q^\nu)[-(Z_2 - 1)_{e^2} - c](Z_3 - 1)_{e^2}$. This also corresponds to an order e^4 contribution to Z_3.
- $U_5 = \{\gamma_2, \gamma_3\}$: same subtraction term as for U_4.

BPHZ theorem: The original (regularized) diagram *minus* the sum of all subtraction terms is finite. The latter is exactly generated by the (usual) counterterms.

We will not give the proof of this theorem, but just mention that it proceeds by recursion. One assumes that all diagrams at order N are made finite by the counterterms. Then one finds a recursion relation for the large momentum behavior at the next order $N+1$ (including on subspaces $p'_\mu || p_\mu$, etc, as relevant for overlapping divergences). The solution to this recurrence relation uses the sum over forests. It can then be shown that the original diagram minus the sum of all subtraction terms is indeed finite.

4.4 Summary of the renormalization program and proof

- Δ_i is the (mass) dimension of the coupling g_i (provided one only deals with scalars, spin-$\frac{1}{2}$, photons or gravitons). Renormalizable interactions have $\Delta_i \geq 0$, a renormalizable theory only has interactions with $\Delta_i \geq 0$. The superficial degree of divergence (divergence in the region where all $k_i \to \infty$) is $D = 4 - \sum_f E_f \mathcal{D}_f - \sum_i N_i \Delta_i$. In a renormalizable theory only a finite set of E_f gives $D \geq 0$. Diagrams with $D < 0$ are superficially convergent.

- A superficially convergent diagram can be divergent only due to divergent subdiagrams. Then the same counterterms that render finite these subdiagrams also make the whole diagram finite. Thus we only have to deal with superficially divergent diagrams, i.e. only with finitely many E_f if the theory is renormalizable.

- Ignoring overlapping divergences for the moment, a superficially divergent diagram with $D \geq 0$ is made superficially convergent by $D+1$ derivatives with respect to the external momenta. This implies that its diverging part is a polynomial in the external momenta and this can be subtracted by a local counterterm. In a renormalizable theory, even though at each order in perturbation theory there are new diverging diagrams, for fixed E_f, the degree D cannot increase and the structure of the polynomial, i.e. of the counterterms remains the same, only the coefficients (the Z's) get higher and higher order contributions. Since $D \geq 0$ only for finitely many E_f, only finitely many counterterms are needed. Adding these counterterms renders all these diagrams finite to all orders in perturbation theory. On the other hand, in a non-renormalizable theory, at every order in perturbation theory, diagrams with more and more external lines become divergent and one needs an infinite number of counterterms to render them all finite.

- To deal with overlapping divergences, BPHZ define the forests with nested boxes (subdiagrams) γ_a of degree $D_a = D(\gamma_a)$, each forest giving a subtraction term obtained by replacing each subdiagram in the nested sequence by its $D_a + 1$ first terms in a Taylor series in the "external" momenta. This corresponds to the same counterterms as defined above. BPHZ show that this procedure renders also finite these diagrams with overlapping divergences to all orders in perturbation theory. Again, in a renormalizable theory, finitely many counterterms make all diagrams finite.

4.5 The criterion of renormalizability

Renormalizability restricts interactions to those with $\Delta_i = 4 - d_i - \sum_f n_{if}(1+s_f) = 4 - d_i - \sum_f n_{if}\mathcal{D}_f \geq 0$ only. For a finite number of fields these are only finitely many interactions. Of course, there are also other restrictions on the possible interactions like Lorentz invariance or other symmetries one might want to preserve.

If the regulator preserves a given symmetry then the finite and the diverging parts of any Green's function (not necessarily of individual diagrams) at any order in perturbation theory must also preserve this symmetry. Hence the counterterms also respect this symmetry.

What if one allowed to include *non*-renormalizable interactions, like e.g. adding a term $\bar{\psi}\gamma_{\mu\nu}\psi F^{\mu\nu}$ to the QED Lagrangian? This interaction indeed has dimension 5 so that $\Delta = -1$ and it is non-renormalizable. For dimensional reasons its coupling constant must be of the form $\frac{\rho}{M}$ with dimensionless ρ and some mass scale M. Such a term would change the magnetic moment of electrons by an additional amount $\sim \rho \frac{m_e}{M}$ which would thus become an adjustable parameter. From experiment we know that this quantity must be extremely small. Said differently, if we take ρ of the same order as e, the mass scale M must be very large compared to m_e.

More generally, a non-renormalizable interaction has $\Delta_i < 0$ and coupling constants of dimension $\Delta_i = -|\Delta_i|$. Write $g_i = \frac{\widetilde{g}_i}{M^{|\Delta_i|}}$. Then non-renormalized diagrams have divergences that behave as $(\widetilde{g}_i)^N \left(\frac{\Lambda}{M}\right)^{|\Delta_i|N}$ at order N in perturbation theory, Λ being some UV cutoff. Also, infinitely many Green's functions are divergent and we need infinitely many counterterms, possibly with an arbitrary number of derivatives. However, once these divergences are cancelled, we have renormalized Green's functions $G(p_i)$, and at order N in perturbation theory (in g_i) they will behave as $(\widetilde{g}_i)^N \left(\frac{p_j}{M}\right)^{|\Delta_i|N}$. Thus, as long as $|p_j| \ll M$ one can neglect the effect of these non-renormalizable interactions.

Theories like QED are presently thought to be only effective theories, in the sense that they provide the effective description of electromagnetic interactions at energies that are low compared to some scale at which new physics could be expected, like e.g. the grand unification scale of 10^{15} GeV or even the Planck scale of 10^{19} GeV. Such an effective theory then has an effective Lagrangian obtained by "integrating out" the very heavy additional fields that are present in such theories. (We will discuss such integrating out a bit in the next section). This necessarily results in the generation of (infinitely) many non-renormalizable interactions in this effective Lagrangian with couplings $\frac{\widetilde{g}_i}{M^{|\Delta_i|}}$, M being e.g. the grand unification scale. From the previous argument it is then clear that at energies well below this scale these additional non-renormalizable interactions are completely irrelevant, and this is why we only "see" the renormalizable interactions. Our "low-energy" world is described by renormalizable theories like QED not because such theories are somehow better behaved, but because these are the only relevant ones at low energies:

Renormalizable interactions are those that are relevant at low energies, while non-renormalizable interactions are irrelevant at low energies.

This is also the teminology in statistical mechanics where one studies infrared physics at scales $|p| \ll \frac{1}{a} \equiv \Lambda$, where a is the lattice spacing.

5 Renormalization group and Callan-Szymanzik equations

5.1 Running coupling constant and β-function: examples

5.1.1 Scalar ϕ^4-theory

Consider the 4-point vertex function F in the scalar ϕ^4-theory we computed in section 3. At tree-level it equals g, independent of the momenta, while up to order g^2 it was given by (cf. (3.125))

$$F_{g^2}(q_1, q_2 \to q_1', q_2') = g + \frac{g^2}{32\pi^2}\int_0^1 \mathrm{d}x \left\{ \log \frac{m^2 - sx(1-x)}{m^2 + \frac{4}{3}\mu^2 x(1-x)} + (s \to t) + (s \to u) \right\} , \tag{5.1}$$

which depends on the momenta or, equivalently, on s, t and u. It also depends on the scale μ used to fix the renormalization condition (3.122). Below, we will consider different scales, so let us call the present scale μ_*. To emphasize the dependence of F on μ_* we will write $F(s,t,u;\mu_*)$ and call the corresponding coupling g_*. Thus

$$F\left(s = t = u = -\frac{4}{3}\mu_*^2\, ;\, \mu_*\right) = g_* \ , \tag{5.2}$$

and eq. (5.1) is rewritten as

$$F(s,t,u;\mu_*) = g_* + \frac{g_*^2}{32\pi^2}\int_0^1 \mathrm{d}x \left\{ \log \frac{m^2 - sx(1-x)}{m^2 + \frac{4}{3}\mu_*^2\, x(1-x)} + (s \to t) + (s \to u) \right\} . \tag{5.3}$$

Recall from section 3, that this is essentially the 1PI 4-point function, since $\Gamma^{(4)}(q_1, q_2, -q_1', -q_2') = -(2\pi)^4 \delta^{(4)}(q_1 + q_2 - q_1' - q_2')\, F(q_1, q_2 \to q_1', q_2')$.

Let us remark that the condition (5.2) can be viewed as defining what we mean by the coupling constant of our theory. It is *one* convenient definition and obviously one might have chosen a different one. For example, one might have defined g_* in terms of the function F and a scale μ_* but at $c_s s = c_t t = c_u u = -\mu_*^2$ with some unequal coefficients c_s, c_t, c_u. One might also have defined the coupling not in terms of F, resp. $\Gamma^{(4)}$, but in terms of the 4-point Green function at some convenient off-shell point. We will come back to this later-on, for the time being we keep the definition (5.2).

To make the discussion of the following paragraph more intuitive, suppose $m \neq 0$ and that one has chosen $\mu_* = m$. Then, as long as the momenta remain of the same order of magnitude as m, the argument of the logarithm is of order 1, and $\int_0^1 dx\{\ldots\} = \mathcal{O}(1)$, so that the one-loop correction is small with respect to the tree contribution as long as $\frac{g}{32\pi^2} \ll 1$. If, on the other hand, one is interested in extreme high-energy scattering with s, $|t|$, $|u| \gg m^2$, then the logarithms will be large: $\log \frac{1 - \frac{s}{m^2} x(1-x)}{1 + \frac{4}{3}x(1-x)} \simeq \log \frac{-s}{m^2} - \log \frac{x(1-x)}{1 + \frac{4}{3}x(1-x)}$. To fix the ideas, suppose m is of the order of 1 MeV and $\sqrt{s}$ of the order of 100 MeV. Then $\log \frac{s}{m^2} \simeq 10$ which is large but is easily compensated by the $\frac{1}{32\pi^2} \simeq 3 \times 10^{-3}$, and the one-loop contribution will certainly be small with respect to the tree-level result if g is small. However, as one considers higher and higher energy scales, the log will increase further. For $\sqrt{s}$, $\sqrt{|t|}$, $\sqrt{|u|} \simeq 10$ TeV e.g. the sum of the 3 logarithms is about 100. Eventually, perturbation theory will break down as one goes to very high energies. Actually, like for most quantum field theories, the perturbation series in g has a vanishing radius of convergence. Nevertheless, the series is asymptotic, meaning that one gets a good approximation, if g is small enough, by computing the first few orders in g. The approximation will be better for smaller g or, at fixed g, if large logarithms like the ones discussed do not appear. In this sense, to improve perturbation theory one has to avoid such large logarithms. This can be done as follows.

If one is *only* interested in the high-energy scattering, one might define the coupling constant g_* right away at a scale μ_* comparable to $\sqrt{s}$, $\sqrt{|t|}$, $\sqrt{|u|}$. Then no large logarithms will ever appear[23]. In general, however, one wants to be able to compute at different scales and most often the original definition of the coupling g_* is at low or zero momentum, like the definition of the elementary charge e in QED. The solution is to define a different coupling $g(\mu)$ for every scale μ as the value of $F(s,t,u;\mu_*)$ at $s=t=u=-\frac{4}{3}\mu^2$:

$$g(\mu) = F\left(s=t=u=-\frac{4}{3}\mu^2\,;\,\mu_*\right) = g_* + \frac{3g_*^2}{32\pi^2}\int_0^1 \mathrm{d}x\ \log\frac{m^2+\frac{4}{3}\mu^2 x(1-x)}{m^2+\frac{4}{3}\mu_*^2\,x(1-x)}\ . \tag{5.4}$$

Again, if $\mu \gg \mu_*$ the logarithm will be large, but this can be easily avoided by first defining $g(\mu_1)$ at some μ_1 just slightly larger than μ_*, and then use this g_{μ_1} to compute a g_{μ_2} at a scale μ_2 slightly larger than μ_1, etc. More precisely, introduce intermediate $\mu_0=\mu_*, \mu_1, \ldots \mu_{N-1}, \mu_N=\mu$ and *define*

$$g(\mu_{n+1}) = g(\mu_n) + \frac{3g(\mu_n)^2}{32\pi^2}\int_0^1 \mathrm{d}x\ \log\frac{m^2+\frac{4}{3}\mu_{n+1}^2 x(1-x)}{m^2+\frac{4}{3}\mu_n^2\,x(1-x)}\ . \tag{5.5}$$

Iterating this relation allows us to go from $g(\mu_*)=g_*$ to $g(\mu_N)=g(\mu)$ while keeping the logarithms small at every step.[24] Thus eq. (5.5) provides the desired relation between the $g(\mu)$ at different scales. Let us insist that no large logarithms will appear when computing scattering amplitudes at typical s, t, u of order μ^2 if one uses this coupling $g(\mu)$ and, in this sense, $g(\mu)$ is the natural coupling constant at this scale. Obviously, as μ increases, $g(\mu)$ also increases.

It is much more convenient to turn this relation into a differential equation by considering $\mu_n=\mu$ and $\mu_{n+1}=\mu+\delta\mu$ with infinitesimal $\delta\mu$. We get

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g(\mu) = \frac{3}{16\pi^2}\,g(\mu)^2\int_0^1 \mathrm{d}x\ \frac{\frac{4}{3}\mu^2 x(1-x)}{m^2+\frac{4}{3}\mu^2 x(1-x)} \equiv \beta(\mu,m)\ . \tag{5.6}$$

Let us be slightly more precise. What one does is to express $g(\mu')$ in terms of μ', $g(\mu)$ and μ and take the derivative with respect to μ' at fixed μ (and fixed $g(\mu)$) and set $\mu'=\mu$ in the end. With this being understood, one simply writes $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g(\mu)$. In general one defines the functions $\beta(\mu,m)$ and $\beta(\mu)$ as

$$\boxed{\beta(\mu,m) = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g(\mu)\quad,\quad \beta(\mu)=\beta(\mu,0)\ .} \tag{5.7}$$

[23] except in certain kinematical regions like forward scattering

[24] Note that $g(\mu_{n+1})$ differs from $g(\mu_n)$ only by a term of order g^2. Thus, to the order we work, we could just as well replace the $g(\mu_n)$ in front of the integral by $g(\mu_0)=g_*$. If one does so, iterating eq. (5.5) N times exactly yields eq. (5.4).

Obviously, if $\mu \gg m$ one can neglect the effect of the mass m and approximate $\beta(\mu, m)$ by $\beta(\mu)$. Equation (5.6) yields for the ϕ^4-theory:

$$\beta(\mu) = \frac{3}{16\pi^2}\, g(\mu)^2 \equiv \beta_0\ g(\mu)^2\ . \tag{5.8}$$

For the present ϕ^4-theory, the β-function is positive and the differential equation shows that for positive (negative) β-function the coupling increases (decreases) when the scale μ increases. It is easy to solve the differential equation (5.7) with $\beta(\mu, m)$ replaced by (5.8):

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu} g(\mu) = \beta_0 g(\mu)^2 \quad \Rightarrow \quad \frac{1}{g(\mu_1)} - \frac{1}{g(\mu_2)} = \log\left(\frac{\mu_2}{\mu_1}\right)^{\beta_0}\ . \tag{5.9}$$

One sees again that $g(\mu)$ increases when μ increases. A useful rewriting of this solution is as follows:

$$\mu \exp\left(\frac{1}{\beta_0 g(\mu)}\right) \equiv M\ , \quad \text{independent of } \mu. \tag{5.10}$$

The quantity M is called the renormalization group invariant scale, since the change in $g(\mu)$ is exactly such that this quantity does not depend on μ. Being independent of μ, we expect the quantity M to have a physical meaning. Indeed, if $\mu = M$ we have $\frac{1}{\beta_0 g(\mu)} = 0$ and we see that M is the scale where $g(\mu) \to \infty$. Of course, we have only done a one-loop computation and as $g(\mu)$ becomes large we no longer can trust a one-loop result. What one can say is the following: even if one starts at some scale μ_* with a small $g(\mu_*)$, as the scale is increased, the coupling grows and for scales of the order of M the theory enters a strong-coupling regime. We may rewrite eq. (5.9) as

$$g(\mu) = g(\mu_0)\Big[1 - \beta_0 g(\mu_0) \log\frac{\mu}{\mu_0}\Big]^{-1}\ . \tag{5.11}$$

This is valid for any $\mu_0 \gg m$ (since we neglected the mass) and as long as $g(\mu)$ is small, whether $g(\mu_0)\log\frac{\mu}{\mu_0}$ is small or not.

It is useful to note that, in practice, one can compute $\beta(\mu, m)$ directly form (5.4) by taking $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}$ and replacing g_*^2 by $g(\mu)^2$. Note though, that one should *not* take $\mu_*\frac{\mathrm{d}}{\mathrm{d}\mu_*}$ of (5.3) as this would result in the opposite sign. If one traces back our computation one sees that the β-function is the coefficient of the $\log\mu$-term which is minus the coefficient of the $\log\tilde{\mu}$-term in $F_{\rm loop}$ which, in turn is the coefficient of the $\frac{1}{\epsilon}$-pole of the Z_g-factor times g, cf. (3.124). :

$$\beta(\mu)_{1-\rm loop} = \text{coefficient of } \frac{1}{\epsilon} \text{ in } Z_g g. \tag{5.12}$$

We have also seen that, at one loop in the ϕ^4 theory, the wave function renormalization factor is $Z = 1$. Thus, we can just as well say that the one-loop β-function is the coefficient of $\frac{1}{\epsilon}$ of $\frac{Z_g}{Z^2} g = g_B$:

$$\beta(\mu)_{1-\rm loop} = \text{coefficient of } \frac{1}{\epsilon} \text{ in } g_B \text{ when expressed in terms of } g. \tag{5.13}$$

If we go to two loops, however, $Z \neq 1$ and the two expressions for the β-function become different. Also in other theories, like ϕ^3 in 6 dimensions, $Z \neq 1$ already at one loop. Does this mean that in general (5.12) is right and (5.13) is wrong? The answer depends on how exactly one defines the coupling constant g. Obviously, if it is defined in terms of the 1PI vertex function, eq. (5.12) should be correct. More generally, one might define couplings g_n in terms of the renormalized amputated n-point Green functions as is relevant for computing (on-shell) S-matrix elements at typical energy scales μ. On the other hand, 1PI vertex functions *and* full propagators at *off-shell* momenta appear as building blocks of larger Feynman diagrams, as we have extensively discussed in the previous section. Thus one not only has to avoid large logarithms in the 1PI vertex functions but also in the full propagators. It then makes sense to include corrections from the full propagators into the definition of the coupling $g(\mu)$. Since each propagator is linked to two 1PI vertex function, its contributions should be split between the two vertices, assigning half its contribution to each. Again, if one traces the appearance of the $\log\mu$ and $\frac{1}{\epsilon}$ terms, one sees that taking into account half the contribution of a full propagator amounts to picking the coefficient[25] of the $\frac{1}{\epsilon}$-pole in $\frac{1}{\sqrt{Z}}$. In a ϕ^k theory this results in an extra factor of $\frac{1}{Z^{k/2}}$ and with this new definition of the coupling g the β-function would indeed be given by (5.13).

Since the coefficients of the $\frac{1}{\epsilon}$ poles are determined by the divergent parts of the one-loop integrals only, this actually gives a very easy way to get the leading coefficient of the β-function in a large class of theories. We will discuss this in somewhat more detail below.

5.1.2 QED

To see how the effective coupling of QED evolves with the scale of energy, consider how the tree-level coupling of a photon to an electron gets corrected by one-loop effects. Following the above discussion, we think of a complicated Feynman diagram built from 1PI-vertices Γ^μ, complete electron propagators and complete photon propagators, all with off-shell momenta, as shown in Fig. 15. (Note that we use only the electron-electron-photon 1PI vertex explicitly, not the higher vertex functions. Hence the resulting diagram is *not* necessarily a tree diagram.)

Then we need to include half of the corrections coming from the propagators and the entire correction coming from the 1PI-vertex function to define what we mean by the running coupling $e(\mu)$. At the one-loop level, there are the following diagrams to take into account. First, the photon propagator gets corrected by the one-loop vacuum-polarization diagram. We have seen that this modifies $e^2 \to \frac{e^2}{1-\pi(q^2)} \simeq e^2(1+\pi(q^2))$ or $e \to e(1+\frac{1}{2}\pi(q^2))$. Next, the electron propagators get corrected by the self-energy diagrams Σ^* on the fermion lines (there are two of them but each counts half). Finally, there is the one-loop corrected vertex Γ^μ. The latter two are related by the Ward identity. Recall that their divergencies were cancelled by the same counterterm $\sim Z_2 - 1$ and, similarly, one can see that their contributions to the β-function cancel. (Schematically, the vertex diagram contributes the coefficient of $\frac{1}{\epsilon}$ in Z_2, while each of the two electron self-energy diagrams contribute the coefficient of $\frac{1}{\epsilon}$ in $\frac{1}{\sqrt{Z_2}}$, giving a vanishing total contribution.) Thus, the effective

[25] Recall that $Z = 1 + z$ with z of order g at least. Thus $\frac{1}{\sqrt{Z}} \simeq 1 - \frac{z}{2}$.

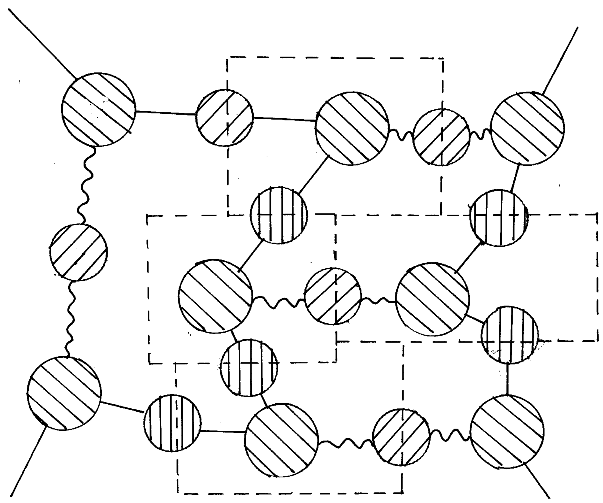


Figure 15: A complicated Feynman diagram in QED is built from the 1PI electron-electron-photon vertex functions, the full electron propagators and the full photon propagators.

coupling is entirely determined by the vacuum polarization diagram:

$$\begin{aligned} e \to e(q^2) &= e\Big(1 + \frac{1}{2}\pi(q^2)\Big) \\ &= e + \frac{e^3}{4\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\log\big[1 + x(1-x)\frac{q^2}{m^2}\big] \ . \end{aligned} \tag{5.14}$$

Repeating the argument done for the coupling in the ϕ^4-theory, we get

$$e(\mu) = e + \frac{e^3}{4\pi^2}\int_0^1 \mathrm{d}x\, x(1-x)\log\big[1 + x(1-x)\frac{\mu^2}{m_e^2}\big] \ , \tag{5.15}$$

and

$$\beta(\mu, m) = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\, e(\mu) = \frac{e^3}{2\pi^2}\int_0^1 \mathrm{d}x\, \frac{x^2(1-x)^2}{\frac{m^2}{\mu^2} + x(1-x)} \ , \tag{5.16}$$

which yields the β-function of QED

$$\boxed{\beta_e(\mu) = \frac{e^3}{12\pi^2} \ .} \tag{5.17}$$

Note that in QED we have $e_B = Z_3^{-1/2} e$ with $Z_3 = 1 - \frac{e^2}{6\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right)$ so that $e_B = e + \frac{e^3}{12\pi^2}\left(\frac{1}{\epsilon} + \text{finite}\right)$ and again

$$\beta_e(\mu)_{1-\text{loop}} = \text{coefficient of } \frac{1}{\epsilon} \text{ in } e_B \text{ when expressed in terms of } e. \tag{5.18}$$

The β-function is again positive and e grows with the energy scale. Sometimes, this is written in terms of the running of the fine-structure constant $\alpha = \frac{e^2}{4\pi}$ as $\beta_\alpha = \frac{\mathrm{d}}{\mathrm{d}\log\mu}\alpha = \frac{e}{2\pi}\beta_e(\mu)$:

$$\beta_\alpha(\mu) = \frac{2}{3\pi}\,\alpha^2 \ , \tag{5.19}$$

which is of the same form as (5.8) with $\beta_0 = \frac{2}{3\pi}$. The β-function (5.17) or (5.19) governs the running of $e(\mu)$ or $\alpha(\mu)$ for values of μ much larger than m_e, say $\mu_* = 10\, m_e$. To get the running for $0 \le \mu \le \mu_*$, one should use the explicit expression (5.15) instead. The latter is perturbatively valid in this region since $\frac{e^2}{4\pi^2} \log \frac{\mu_*^2}{m_e^2} \lesssim \frac{\alpha}{\pi} \log 100 \simeq 0.01$. As discussed in connection with the Ward identity, the renormalized charge of the electron is defined at zero momentum, hence $e_R = e(0)$ and $\alpha_R = \alpha(0) \equiv \alpha \simeq \frac{1}{137}$. Thus $(\int_0^1 \mathrm{d}x\, x(1-x) \log x(1-x) = -\frac{5}{18})$

$$
\begin{aligned}
\alpha(\mu_*) &= \alpha + \frac{2\alpha^2}{\pi} \int_0^1 \mathrm{d}x\, x(1-x) \log \big[1 + x(1-x) \frac{\mu_*^2}{m_e^2}\big] \\
&\simeq \alpha + \frac{2\alpha^2}{\pi} \int_0^1 \mathrm{d}x\, x(1-x) \big[\log x(1-x) + \log \frac{\mu_*^2}{m_e^2}\big] \\
&= \alpha + \frac{2\alpha^2}{3\pi} \big[-\frac{5}{6} + \log \frac{\mu_*}{m_e}\big] \simeq \alpha \Big[1 - \frac{2}{3\pi} \alpha \big(\log \frac{\mu_*}{m_e} - \frac{5}{6}\big)\Big]^{-1} .
\end{aligned}
\tag{5.20}
$$

For $\mu > \mu_*$, one can safely use the solution of $\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \alpha(\mu) = \beta_\alpha(\mu)$ with initial value given by (5.20) and β_α given by (5.19):

$$
\alpha(\mu) = \alpha(\mu_*) \Big[1 - \frac{2}{3\pi} \alpha(\mu_*) \log \frac{\mu}{\mu_*}\Big]^{-1} .
\tag{5.21}
$$

Combining (5.20) and (5.21) yields

$$
\boxed{\alpha(\mu) = \alpha \Big[1 - \frac{2}{3\pi} \alpha \big(\log \frac{\mu}{m_e} - \frac{5}{6}\big)\Big]^{-1} .}
\tag{5.22}
$$

This equation is valid even when $\alpha \log \frac{\mu}{m_e}$ is of order 1. However, just as for the scalar ϕ^4 theory, it shows that the coupling becomes strong as $\mu \simeq M = m_e \exp\left(\frac{3\pi}{2\alpha}\right) \simeq 10^{280} m_e$. Of more experimental relevance is the value of $\alpha(\mu)$ at present day collider energies. For $\mu = 100\,\mathrm{GeV}$ e.g. (LEP) one gets $\alpha(\mu) \simeq \frac{1}{134.6}$. This is the result when taking into account the electron field only. Including the effects of the two other lepton families (as well as the quarks) one gets $\alpha(100\,\mathrm{GeV}) \simeq \frac{1}{128}$.

5.2 Running coupling constant and β-functions: general discussion

Having seen the running of the coupling and the corresponding β-functions in two explicit examples, we will now try to make some more general statements.

5.2.1 Several mass scales

We have seen in the example of QED with only electrons and positrons that the running of $e(\mu)$ or $\alpha(\mu)$ was well determined by the β-function for scales μ above a few electron masses. Below, one had to use the exact form (5.15) or its approximation (5.20). However, in the real world, there are many more charged particles of higher masses that also contribute to the β-function. Let us consider the case of two well-separated mass scales, $0 < m_1 \ll m_2$, the generalization to more than two mass scales will be obvious.

Suppose the one-loop computation yields (cf. (5.4) with $\mu_*=0$ or (5.15) written for $\alpha(\mu)$ instead of $e(\mu)$)

$$g(\mu)=g_*+g_*^2\int \mathrm{d}x\,\Big\{f_1(x)\log\big[1+h_1(x)\frac{\mu^2}{m_1^2}\big]+f_2(x)\log\big[1+h_2(x)\frac{\mu^2}{m_2^2}\big]\Big\}\ , \tag{5.23}$$

where x could stand for multiple Feynman parameters x_i, and the$f_j(x)$ and $h_j(x)$ are some polynomials. Let, much as before

$$\beta_{1,2}(\mu,m_1,m_2)=\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g(\mu)=g_*^2\int \mathrm{d}x\,\Big\{f_1(x)\frac{2h_1(x)\mu^2}{m_1^2+h_1(x)\mu^2}+f_2(x)\frac{2h_2(x)\mu^2}{m_2^2+h_2(x)\mu^2}\Big\}\ , \tag{5.24}$$

Then, as long as $\mu\ll m_1\ll m_2$, one sees from (5.23) that $g(\mu)\simeq g_*$. As μ becomes comparable to m_1 one still has $\mu\ll m_2$, and one can neglect the second logarithm. Actually, as long as μ is considerably smaller than m_2 (exactly how much smaller depends on h_2 and f_2/f_1), one can continue to neglect the second term in the braces: in this region, the heavy particle does not contribute to the running of $g(\mu)$. If, in this region, the first logarithm becomes large for $m_1\ll\mu\ll m_2$, one must use the differential equation with $\beta(\mu,m_1,m_2)$ to evolve $g(\mu)$. However, for $m_1\ll\mu\ll m_2$, one can neglect the second term in the braces in (5.24), while in the first term one can set $m_1=0$. Hence, in this region, the running is effectively governed by

$$\beta_1(\mu)=\beta_{1,2}(\mu,0,\infty)=g_*^2\int \mathrm{d}x\,2f_1(x)\ . \tag{5.25}$$

As μ becomes of the order of m_2, one has to use an expression similar to (5.23) with $\mu_*\neq 0$ to evolve $g(\mu)$ from some $\mu_*<m_2$ to some $\mu>m_2$ beyond which it is reasonable to neglect the effect of m_2 and one can use

$$\beta_{1+2}(\mu)=\beta_{1,2}(\mu,0,0)=g_*^2\int \mathrm{d}x\,\big(2f_1(x)+2f_2(x)\big) \tag{5.26}$$

to evolve $g(\mu)$ further to even larger scales μ. It is often enough to simply evolve $g(\mu)$ with $\beta_1(\mu)$ from m_1 to m_2 and then with $\beta_{1+2}(\mu)$ above m_2.

The general lesson to remember is that at a given scale μ the running of the coupling is determined by the β-function due to loops of only those particles that have masses below μ while particle with masses much larger than μ do not influence the running.

5.2.2 Relation between the one-loop β-function and the counterterms

We have already seen in the ϕ^4-theory and in QED that, with the appropriate definition of the running coupling constant, the one-loop β-function coincided with the coefficient of the $\frac{1}{\epsilon}$ pole in the expression of the bare coupling in terms of the renormalized one. We will now establish this relation more generally.

According to the above discussion, in general, we define the one-loop corrected n-point coupling as given by the value of an appropriate proper n-point vertex function $\Gamma^{(n)}$, plus half of the propagator corrections (as shown in Fig. 15 for the example of QED), all computed up to one loop and evaluated at some conveniently chosen momenta $p_j(\mu)$:

$$g(\mu)=\Gamma^{(n)}\big(p_j(\mu)\big)+\frac{1}{2}\sum_{j=1}^{n}g\,\Delta_j(p_j)\,\Pi_j^*\big(p_j(\mu)\big)\ . \tag{5.27}$$

As before, $\Gamma^{(n)}$ and Π^* are the renormalized vertex function and 1PI propagator ("self-energy"). It will be useful here to separate the contributions from the one-loop diagrams and those of the counterterms:

$$\begin{aligned} \Gamma^{(n)}(p_j) &= g + (Z_g - 1)g + \Gamma^{(n)}_{1-\text{loop}}(p_j) \\ \Pi^*_j(p_j) &= -(Z_j - 1)\,\Delta_j(p_j)^{-1} + Z_j\,\delta m_j^2 + \Pi^*_{j,1-\text{loop}}(p_j) \ . \end{aligned} \tag{5.28}$$

We will *restrict ourselves to the cases where the one-loop contribution* $\Gamma^{(n)}_{1-\text{loop}}$ *diverges logarithmically* and the original coupling g is dimensionless in 4 dimensions. This includes most of the interesting theories. Moreover, since we want to compute $\beta(\mu) = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}g(\mu)\big|_{m_j=0}$, i.e. we are interested in the region where $|p_j^2| \gg m_i^2$, we can drop all terms involving the masses right away (except maybe to regulate IR divergences). In particular, $\delta m_j^2 \sim m_j^2$, so these terms will also be dropped. Thus

$$g(\mu) = g + (Z_g - 1)g + \Gamma^{(n)}_{1-\text{loop}}\big(p_j(\mu)\big) + \frac{1}{2}\sum_{j=1}^{n}\left[-g\,(Z_j - 1) + g\Delta_j(p_j)\,\Pi^*_{j,1-\text{loop}}\big(p_j(\mu)\big)\right] \ . \tag{5.29}$$

The β-function will be given by the coefficient of $\log\mu$ in this expression. We will show shortly that each $\log\mu$ in a one-loop contribution is accompanied by a $-\frac{1}{\epsilon}$ with the same coefficient, and that the finite renormalization conditions require that this is also the coefficient of $+\frac{1}{\epsilon}$ in the corresponding combination of the counterterms. Hence:

$$\beta_{1-\text{loop}}(\mu) = \text{coefficient of } \ \frac{1}{\epsilon} \ \text{ in } \ \left[g(Z_g - 1) - \frac{1}{2}\sum_{j=1}^{n} g(Z_j - 1)\right] \ . \tag{5.30}$$

Note that, since $Z_g - 1$ and $Z_j - 1$ are always at least $\mathcal{O}(g)$, one has to the order we are interested in

$$g_B \equiv g\frac{Z_g}{\Pi_j\sqrt{Z_j}} = g\frac{1 + (Z_g - 1)}{\Pi_j\sqrt{1 + (Z_j - 1)}} = g\Big[1 + (Z_g - 1) - \frac{1}{2}\sum_j (Z_j - 1)\Big] \ , \tag{5.31}$$

and, hence

$$\boxed{\beta_{1-\text{loop}}(\mu) = \text{coefficient of } \ \frac{1}{\epsilon} \ \text{ in } g_B \text{ when expressed in terms of } g.} \tag{5.32}$$

Let us now give the argument: Using dimensional regularization and Feynman parameters, the one-loop contribution to the 1PI vertex function is of the form

$$\Gamma^{(n)}_{1-\text{loop}}(p_j) = -c_n\,g^n\left(\frac{2}{\epsilon} + a\right)\int\prod_j \mathrm{d}x_j\,f(x_j)\left(\frac{R(p_j, x_j)}{\widetilde{\mu}^2}\right)^{-\epsilon/2} \ , \tag{5.33}$$

where $R(p_j, x_j)$ is some quadratic form in the momenta and masses and $\widetilde{\mu}$ is some scale introduced to keep g dimensionless after dimensionally continuing to $d = 4 - \epsilon$. We let

$$b_0 = 2c_n\int\prod_j \mathrm{d}x_j\,f(x_j) \ , \tag{5.34}$$

so that

$$\Gamma^{(n)}_{1-\text{loop}}(p_j) = -\frac{b_0}{\epsilon}\,g^n - \frac{a}{2}\,b_0\,g^n + c_n\,g^n\int\prod_j \mathrm{d}x_j\,f(x_j)\log\left(\frac{R(p_j, x_j)}{\widetilde{\mu}^2}\right) \ . \tag{5.35}$$

As before, we impose a renormalization condition that $\Gamma^{(n)}$ should equal g for some fixed $p_j^* \equiv p_j(\mu_*)$. Combining (5.28) and (5.35) fixes Z_g as

$$(Z_g - 1)g = \frac{b_0}{\epsilon}\, g^n + \frac{a}{2}\, b_0\, g^n - c_n\, g^n \int \prod_j \mathrm{d}x_j\, f(x_j) \log\left(\frac{R(p_j(\mu_*), x_j)}{\widetilde{\mu}^2}\right) . \tag{5.36}$$

The (renormalized) n-point vertex function then reads

$$\Gamma^{(n)}(p_j) = g + c_n\, g^n \int \prod_j \mathrm{d}x_j\, f(x_j) \log\left(\frac{R(p_j, x_j)}{R(p_j(\mu_*), x_j)}\right) . \tag{5.37}$$

We need the coefficient of $\log\mu$ in this expression when $p_j = p_j(\mu)$, for μ sufficiently large so that we can neglect all masses. Now, $R(p_j(\mu), x_j)$ necessarily is of the form

$$R(p_j(\mu), x_j) = r(x_j)\mu^2 + \sum_{r,s} p_{rs}(x_j) m_r m_s \quad \Rightarrow \quad \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \log R(p_j(\mu), x_j)\Big|_{m_r=0} = 2\ , \tag{5.38}$$

Thus the coefficient of $\log\mu$ in $\Gamma^{(n)}(p_j(\mu))$ is $2c_n\, g^n \int \prod_j \mathrm{d}x_j\, f(x_j) = b_0 g^n$ which is also the coefficient of $\frac{1}{\epsilon}$ in $(Z_g - 1)g$, cf. (5.36). The same argument can be repeated to show that the coefficient of $\log\mu$ in $\Delta_j\big(p_j(\mu)\big)\Pi_j^*\big(p_j(\mu)\big)$ is the same as the coefficient of $\frac{1}{\epsilon}$ in $-(Z_j - 1)$. This completes the argument.

5.2.3 Scheme independence of the first two coefficients of the β-function

We have just seen that the result for the β-function does not depend much on the details of the renormalization conditions. One might ask what happens if one uses a different regularization or makes coupling constant redefinitions. We will show the following result:
As long as one defines the coupling such that

$$\beta(g) = b_0 g^2 + b_1 g^3 + b_2 g^4 + \dots\ , \tag{5.39}$$

the first two coefficients b_0 and b_1 are universal, i.e. are unchanged by changing the renormalization scheme or redefining the coupling by higher-order terms.

To prove this, assume that in a different scheme with a different coupling $\widetilde{g}$ one finds

$$\widetilde{\beta}(\widetilde{g}) = \widetilde{b}_0 \widetilde{g}^2 + \widetilde{b}_1 \widetilde{g}^3 + \widetilde{b}_2 \widetilde{g}^4 + \dots\ . \tag{5.40}$$

Now, the bare coupling is scheme independent and must equal g and $\widetilde{g}$ to lowest order: $g_b = g + a_1 g^2 + \mathcal{O}(g^3)$ and $g_b = \widetilde{g} + \widetilde{a}_1 \widetilde{g}^2 + \mathcal{O}(g^3)$. Then $\widetilde{g} = g + a g^2 + \mathcal{O}(g^3)$ with $a = a_1 - \widetilde{a}_1$. Hence $\frac{\mathrm{d}\widetilde{g}}{\mathrm{d}g} = 1 + 2ag + \mathcal{O}(g^2)$. Also, $\widetilde{g}_\mu$ should be a function of g_μ only and not of $\frac{\mu}{m}$ e.g. It follows

$$\begin{aligned}
\widetilde{\beta}(\widetilde{g}_\mu) &= \mu \frac{\mathrm{d}}{\mathrm{d}\mu}\widetilde{g}_\mu = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_\mu\, \frac{\mathrm{d}\widetilde{g}}{\mathrm{d}g} = \beta(g)\frac{\mathrm{d}\widetilde{g}}{\mathrm{d}g} = \Big(b_0 g^2 + b_1 g^3 + \dots\Big)\Big(1 + 2ag + \dots\Big) \\
&= b_0 g^2 + \Big(b_1 + 2ab_0\Big) g^3 + \dots = b_0\Big(\widetilde{g} - a\widetilde{g}^2 + \dots\Big)^2 + \Big(b_1 + 2ab_0\Big)\widetilde{g}^3 + \dots \\
&= b_0 \widetilde{g}^2 + b_1 \widetilde{g}^3 + \dots\ .
\end{aligned} \tag{5.41}$$

Hence $\widetilde{b}_0 = b_0$ and $\widetilde{b}_1 = b_1$, as was claimed.

5.3 β-functions and asymptotic behaviors of the coupling

Let us now study the different asymptotic behaviours of the running coupling constant, given the different possible forms of the β-function. Here we will be mainly interested in the UV asymptotics, i.e. the behaviour as the scale μ becomes very large. In a massless theory, however, the limit where μ becomes very small (IR behaviour) is also of interest, in particular in the description of critical phenomena in statistical physics.

The β-function being defined as $\beta(g(\mu)) = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} g(\mu) = \frac{\mathrm{d}}{\mathrm{d}\log\mu} g(\mu)$, this gives a differential equation for $g(\mu)$ if $\beta(g)$ is known. This differential equation is integrated as

$$\boxed{\int_{g(\mu_1)}^{g(\mu_2)} \frac{\mathrm{d}g}{\beta(g)} = \log \frac{\mu_2}{\mu_1}\ .} \tag{5.42}$$

5.3.1 case a : the coupling diverges at a finite scale M

Suppose $\beta(g) > 0$. Then the coupling $g(\mu)$ increases as the scale μ is increased. Suppose furthermore that $\beta(g)$ increases fast enough with g so that the integral $\int_{g(\mu_1)}^{\infty} \frac{\mathrm{d}g}{\beta(g)}$ converges. Let the value of this integral be $\log \frac{M}{\mu_1}$. Comparing with (5.42) we see that at the scale M the coupling diverges: $g(M) = \infty$. This is shown in the left part of Fig. 16. Explicitly, M is given by

$$M = \mu \exp\left(\int_{g(\mu)}^{\infty} \frac{\mathrm{d}g}{\beta(g)} \right)\ . \tag{5.43}$$

Of course, the running of $g(\mu)$ is precisely such that the r.h.s. does not depend on μ. For this reason, M is called the renormalization group invariant mass. Note that, even if we start with a massless theory, the renormalization group equation (5.42) asserts that there is a well-defined mass scale in this theory! Equation (5.43) allows to trade $g(\mu)$ for M and vice versa. This is sometimes referred to as dimensional transmutation.

We have seen that this would be the behaviour in scalar ϕ^4-theory or QED if we could trust the one-loop result $\beta(g) = b_0 g^2$ with $b_0 > 0$ ($g = \alpha$ for QED) since then $\int_{g(\mu)}^{\infty} \frac{\mathrm{d}g}{\beta(g)} = \frac{1}{b_0 g(\mu)}$ and $M = \mu \exp\left(\frac{1}{b_0 g(\mu)}\right)$. We have also remarked that for QED this M is extremely large and certainly well beyond the energy scales at which QED has to be embedded in a larger and/or more fundamental theory. Of course, as already emphasized, unless one knows $\beta(g)$ exactly, perturbation theory must break down as μ gets closer to M and $g(\mu)$ becomes large. In any case, as μ becomes of the order of M, the theory enters a strong-coupling regime.

5.3.2 case b : the coupling continues to grow with the scale

Suppose $\beta(g) > 0$ but $\beta(g)$ increases slowly enough with g so that the integral $\int_{g(\mu_1)}^{g} \frac{\mathrm{d}g'}{\beta(g')}$ diverges as $g \to \infty$. In this case, (5.42) shows that $g(\mu) \to \infty$ as $\mu \to \infty$, but for any finite μ the coupling $g(\mu)$ remains finite. This situation is depicted in the right part of Fig. 16.

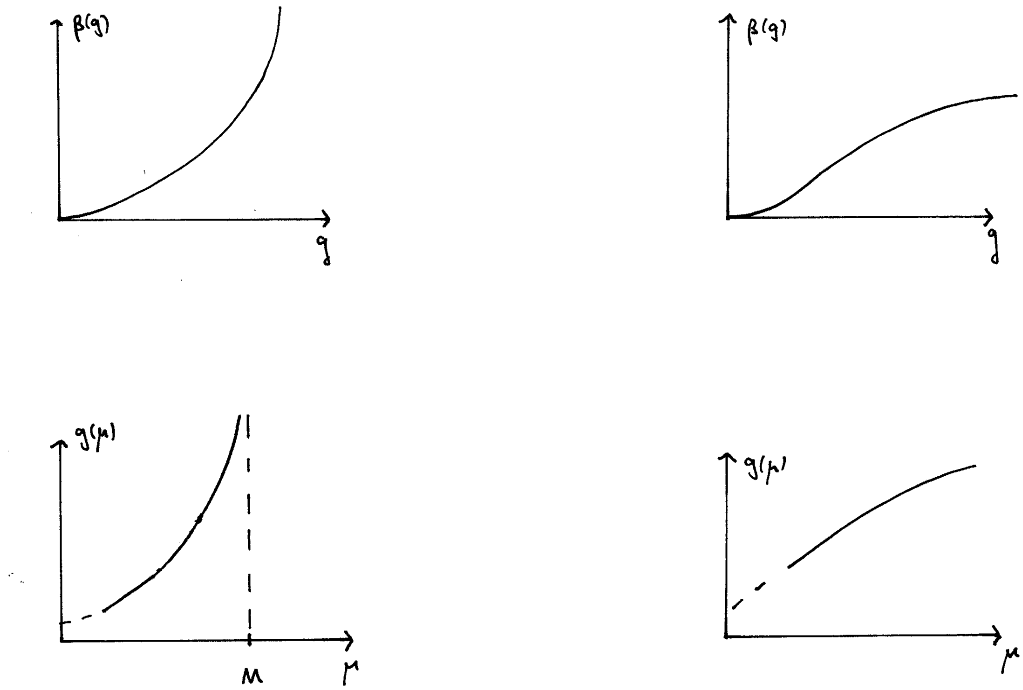


Figure 16: Shown are the β-functions and corresponding scale dependence of the couplings for case a (left) and case b (right).

5.3.3 case c : existence of a UV fixed point

Suppose that $\beta(g)$ starts out positive for small g and has a zero at some finite g_* with $\beta'(g_*) < 0$ (see the left part of Fig. 17):

$$\beta(g_*) = 0 \quad , \quad \beta'(g_*) \equiv -a < 0 \ . \tag{5.44}$$

Then, if one starts with some initial $g(\mu_1) < g_*$, the β-function is positive and the coupling will increase as μ is increased. For some large enough μ_2, $g(\mu_2)$ will become close to g_* and we can approximate $\beta(g) \simeq -a(g - g_*)$. Equation (5.42) then gives

$$\log \frac{\mu}{\mu_2} = \int_{g(\mu_2)}^{g(\mu)} \frac{\mathrm{d}g}{\beta(g)} \simeq -a \log \frac{g(\mu) - g_*}{g(\mu_2) - g_*} \quad \Rightarrow \quad g(\mu) - g_* \simeq (g(\mu_2) - g_*) \left(\frac{\mu_2}{\mu}\right)^a \ . \tag{5.45}$$

We see that, as $\mu \to \infty$, the coupling $g(\mu)$ is driven to g_*. For this reason, g_* is called a UV fixed point:

$$\boxed{g_* > 0 \text{ is a UV fixed point if } \beta(g_*) = 0 \text{ and } \beta'(g_*) < 0.} \tag{5.46}$$

So far, we have considered positive $\beta(g)$ for small enough g. This resulted in a coupling that increased as μ was increased.

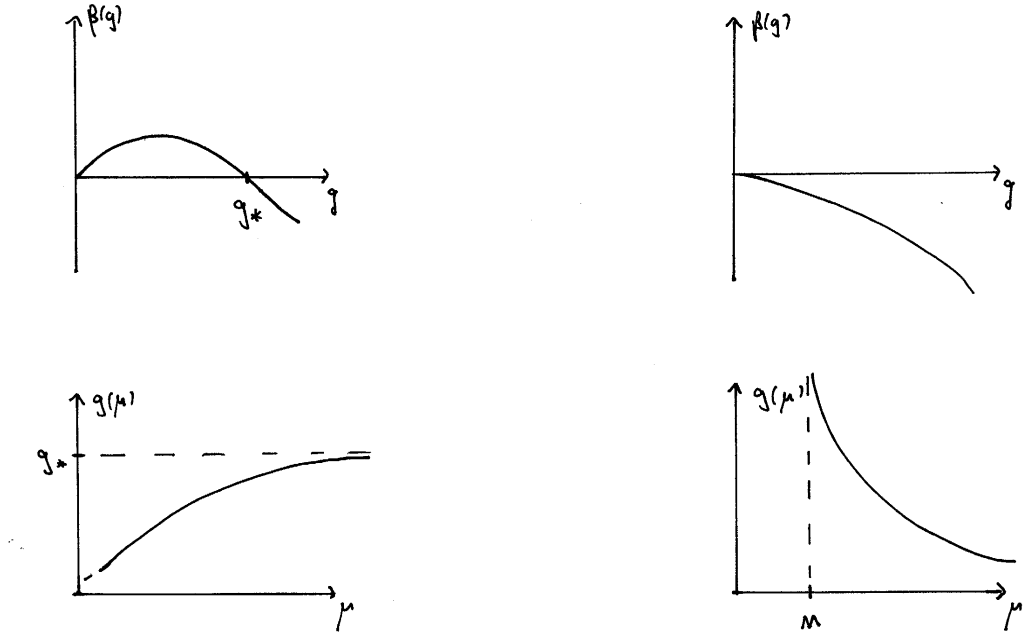


Figure 17: Shown are the β-functions and corresponding scale dependence of the couplings for case c (left) and case d (right).

5.3.4 case d : asymptotic freedom

Now suppose that $\beta(g) < 0$ at least for $0 < g < g_*$. Then, as long as we start with a $g(\mu_1) < g_*$, the coupling $g(\mu)$ will decrease as μ is increased (see the right part of Fig. 17).

To be more specific, suppose $\int_g^{g(\mu_1)} \frac{\mathrm{d}g'}{\beta(g')}$ diverges as $g \to 0$. This will be always realized in perturbation theory since $\beta(g) \sim g^n + \mathcal{O}(g^{n+1})$ with $n \geq 0$. Then (5.42) tells us that as $\mu \to \infty$ one has $g(\mu) \to 0$: the theory becomes a free theory in the UV limit. This is called asymptotic freedom.

To see this in more detail, suppose (with $b_0 < 0$)

$$\beta(g) = -|b_0| g^n + \mathcal{O}(g^{n+1}) \ , \quad n \geq 2 \ . \tag{5.47}$$

Neglecting the higher order contributions to β (since g becomes small this will be more and more justified as μ becomes large!), we have

$$\log \frac{\mu_1}{\mu} = -\frac{1}{|b_0|} \int_{g(\mu)}^{g(\mu_1)} \frac{\mathrm{d}g}{g^n} = \frac{1}{(n-1)|b_0|} \left(\frac{1}{g(\mu_1)^{n-1}} - \frac{1}{g(\mu)^{n-1}} \right) \ . \tag{5.48}$$

One sees that as $\mu \to \infty$ one must have $g(\mu) \to 0$. Indeed

$$g(\mu) = g(\mu_1) \left[1 + (n-1)|b_0| g(\mu_1)^{n-1} \log \frac{\mu}{\mu_1} \right]^{-\frac{1}{n-1}} \ . \tag{5.49}$$

On the other hand, as μ is decreased, the coupling $g(\mu)$ increases. It is easy to see at which scale the coupling must become strong. Again, we see from (5.48) that the combination

$$M = \mu \exp\left(-\frac{1}{(n-1)|b_0| g(\mu)^{n-1}}\right) = \mu \exp\left(\frac{1}{(n-1) b_0 g(\mu)^{n-1}}\right) \tag{5.50}$$

does not depend on μ (at least, as long as one can trust the lowest order result for $\beta(g)$). It is again a renormalization group invariant scale. As $\mu \to M$ the argument of the exponent must vanish and hence $g(M) \to \infty$. Of course, perturbation theory breaks down before μ reaches M, but this shows that M is the scale where the coupling becomes large. The difference with case a considered before is that now this strong coupling region is approached from large values of μ as μ is decreased.

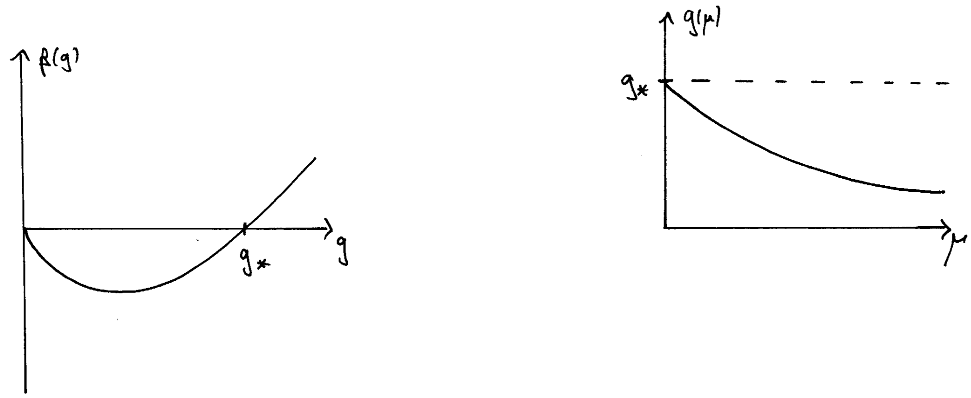


Figure 18: Shown are the β-function and corresponding scale dependence of the coupling for case e (IR fixed point).

5.3.5 case e : IR fixed point

Finally, consider a situation where the β-function starts out negative for small g and then has a zero for some g_*, necessarily with $\beta'(g_*) = a > 0$. In this case, if at some initial scale μ_1 one has $g(\mu_1) < g_*$, the β-function is negative. This means that $g(\mu)$ decreases as μ is increased, and $g(\mu)$ increases as μ is decreased, just as for case d, above. However, as μ gets smaller, and $g(\mu)$ gets larger, the β-function becomes less negative and as $g(\mu)$ approaches g_* it can be well approximated by its linearized form (just as for case c): $\beta(g) \simeq a(g - g_*)$. Suppose that $g(\mu_2)$ is close enough to g_* so that the linearized approximation is good enough. Then

$$\log \frac{\mu_2}{\mu} \simeq a \log \frac{g(\mu_2) - g_*}{g(\mu) - g_*} , \tag{5.51}$$

and $g(\mu) \to g_*$ as $\mu \to 0$, i.e. g_* is an IR fixed point:

$$\boxed{g_* > 0 \text{ is an IR fixed point if } \beta(g_*) = 0 \text{ and } \beta'(g_*) > 0.} \tag{5.52}$$

Such IR fixed points are of particular interest in the study of critical phenomena in statistical physics.

5.4 Callan-Symanzik equation for a massless theory

I most textbooks, when presenting the Callan-Symanzik equation, the discussion is somewhat simplified by switching from dimensional regularization to an explicit UV cut-off Λ. This has the advantage that one does not have to deal with the extra mass scale $\widetilde{\mu}$ introduced in dimensional regularization (to keep the coupling g dimensionless) and which is different from the renormalization group scale μ. Thus with an explicit UV cut-off one only has to deal with Λ and μ, while in dimensional regularization one has to deal with ϵ, μ and $\widetilde{\mu}$. We will nevertheless derive the Callan-Symanzik equations entirely within the framework of dimensional regularization. Indeed, the presence of $\widetilde{\mu}$ presents only a slight complication of the discussion.

5.4.1 Renormalization conditions at scale μ

We have seen that one can define a running coupling constant $g(\mu)$ as the value of (a certain combination of 1PI propagators and) appropriate vertex function evaluated at the scale μ, cf eq. (5.27). Of course, this equation (5.27) can be viewed as imposing a renormalization condition on the vertex function at the scale μ. It is then most convenient to also impose the renormalization condition on the full propagator at scale μ and accordingly require for the corresponding 1PI propagator that

$$\frac{\partial}{\partial p^2}\Pi^*(p^2)\Big|_{p^2=\mu^2} = 0 \ , \tag{5.53}$$

while still keeping the condition that the pole be at the physical mass:

$$\Pi^*(-m^2) = 0 \ . \tag{5.54}$$

(Note that $p^2 = \mu^2$ corresponds to a space-like momentum.) These two equations are for scalars, but their generalization to Dirac fields or the electromagnetic field is obvious as can be seen on the examples to follow.

We will now restrict ourselves to the massless case. This will also include the case where $\mu, p_j \gg m$ and m can be neglected, just as in our computation of $\beta(\mu) = \beta(\mu, m = 0)$. Let's look at the example of massless QED. Then one has for the electron self energy (cf. (3.77) and (3.78))

$$\Sigma^*_{e^2}(\not{p}) = -(Z_2 - 1)_{e^2}\, i\not{p} + (Z_2\delta m)_{e^2} - \frac{e^2}{16\pi^2}\, i\not{p}\Big[\frac{2}{\epsilon} + \log\frac{C^2}{\pi} - \gamma + 1 - \log\frac{p^2}{\widetilde{\mu}^2}\Big] \ , \tag{5.55}$$

where $\widetilde{\mu}$ was some fixed mass scale introduced to keep the coupling e dimensionless in $d = 4 - \epsilon$. For $m = 0$ the condition $\Sigma^*(im) = 0$ simply yields

$$\delta m = 0 \ . \tag{5.56}$$

This was to be expected for a massless theory.[26] The normalization condition (5.53) now reads $\frac{\partial}{\partial \not{p}}\Sigma^*(\not{p})\Big|_{\not{p}=i\mu} = 0$ and yields

$$(Z_2 - 1)_{e^2} = -\frac{e^2}{8\pi^2}\Big[\frac{1}{\epsilon} + \log\frac{C}{\sqrt{\pi}} - \frac{1+\gamma}{2} - \log\frac{\mu}{\widetilde{\mu}}\Big] \ , \tag{5.57}$$

[26] Although expected, it is generally not true if we use a different regularization.

and

$$\Sigma^*_{e^2}(\not p) = \frac{e^2}{16\pi^2}\, i\not p \left[\log\frac{p^2}{\mu^2} - 2\right] . \tag{5.58}$$

In particular, this is now free from infrared divergences!

Similarly, for the vacuum polarization (photon propagator) one gets from (3.68) and (3.69), now with $m = 0$

$$\pi_{e^2}(q^2) = -(Z_3 - 1)_{e^2} - \frac{e^2}{6\pi^2}\left[\frac{1}{\epsilon} + \log\frac{C}{\sqrt{\pi}} - \frac{\gamma}{2} + \frac{5}{6} - \frac{1}{2}\log\frac{q^2}{\widetilde{\mu}^2}\right] . \tag{5.59}$$

Our new condition (5.53) can be easily seen to translate into $\pi(\mu^2) = 0$, i.e. $(Z_3 - 1) = \pi_{\rm loop}(\mu^2)$. One gets

$$(Z_3 - 1)_{e^2} = -\frac{e^2}{6\pi^2}\left[\frac{1}{\epsilon} + \log\frac{C}{\sqrt{\pi}} - \frac{\gamma}{2} + \frac{5}{6} - \log\frac{\mu}{\widetilde{\mu}}\right] , \tag{5.60}$$

and

$$\pi_{e^2}(q^2) = \frac{e^2}{12\pi^2}\log\frac{q^2}{\mu^2} . \tag{5.61}$$

What is the general lesson we learn from these examples? First, we note the obvious fact that the 1PI functions are simpler in the massless case than in the massive one. Second, we see that the renormalized 1PI functions only depend on the coupling e, or rather $e(\mu)$, and explicitly on μ via the dimensionless combination $\frac{p^2}{\mu^2}$ or $\frac{q^2}{\mu^2}$. In addition, they may depend polynomially on the momenta. They have a well-defined finite limit as $\epsilon \to 0$ and they do not depend on $\widetilde{\mu}$. Third, the Z-factors depend on ϵ (they have poles $\sim \frac{1}{\epsilon}$), on the coupling $e(\mu)$ and on the dimensionless ratio $\frac{\mu}{\widetilde{\mu}}$.

Finally, we must formalize a bit more the definition of the β-function of the preceding subsection. We will assume that the coupling is dimensionless as appropriate for a renormalizable coupling. As explained above, the running coupling should be defined in terms of the appropriate vertex function and half the sum of the corresponding 1PI propagators evaluated at some reference momenta $p_j(\mu) \equiv \theta_j \mu$. Let us call F the corresponding combination of vertex function and 1PI propagators. Obviously, at tree level we simply have $F = g$. Beyond tree level, F will depend on g as well as on the momenta p_j. Note that for dimensionless g the function F also is dimensionless and can depend on the p_j only through dimensionless ratios like p_j/μ or $p_j/\widetilde{\mu}$. Just as in the examples of the 1PI propagators above, the loop-contributions will depend on g and $p_j/\widetilde{\mu}$, but there are also the counterterm contributions. If we write the renormalization condition for the coupling at scale μ_0 (more precisely for $p_j = p_j(\mu_0) \equiv \theta_j \mu_0$) then the counterterms are fixed such that the dependence on $\widetilde{\mu}$ will cancel out and

$$F \equiv F\Big(g(\mu_0), \frac{p_j}{\mu_0}\Big) \quad \text{with} \quad F\Big(g(\mu_0), \theta_j\Big) = g(\mu_0) . \tag{5.62}$$

This then allows us to define (at least for μ close to μ_0)

$$g(\mu) = F\Big(g(\mu_0), \theta_j \frac{\mu}{\mu_0}\Big) . \tag{5.63}$$

The β-function is defined in terms of this running coupling $g(\mu)$ as

$$\beta(g(\mu_0)) = \mu\frac{{\rm d}}{{\rm d}\mu} g(\mu)\Big|_{\mu=\mu_0} = \mu\frac{{\rm d}}{{\rm d}\mu} F\Big(g(\mu_0), \theta_j\frac{\mu}{\mu_0}\Big)\Big|_{\mu=\mu_0} . \tag{5.64}$$

Note that β is a function of g only with no *explicit* μ dependence. The definition (5.64) implies that the β-function is obtained by taking $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}$ of $g(\mu)$ keeping $g(\mu_0)$ fixed. Now, for finite regularization, i.e. $\epsilon \neq 0$, one can express $g(\mu_0)$ in terms of the bare coupling g_B, ϵ and μ_0. Thus one sees that one can also write

$$\boxed{\beta(g(\mu)) = \mu\frac{\partial}{\partial\mu}\Big|_{g_B,\epsilon} g(\mu)\ .} \tag{5.65}$$

5.4.2 Callan-Symanzik equations

It is now relatively easy to establish the Callan-Symanzik equations for proper vertices $\Gamma^{(n)}$ or n-point Green functions. They express that the choice of renormalization scale μ is arbitrary. Indeed, the bare action, expressed in terms of bare fields and bare couplings (as well as bare masses if we considered a massive theory), equals the renormalized action which is expressed in terms of renormalized fields and couplings (and masses). Thus when computing normalized Green functions $\widehat{G}_{(n)}$ (cf. (1.37)), i.e. excluding vacuum bubbles, of bare fields or of renormalized fields one uses the same functional integral with the same action, the only difference being the explicit Z-factors. Any difference in the measures due to the different normalizations of the fields cancels in the normalized Green-functions. Thus (cf. (1.38))

$$\widehat{G}^{l_1\ldots l_n}_{B\,(n)}(p_1,\ldots,p_n) = \left[\prod_{r=1}^n \sqrt{Z_{l_r}}\right] \widehat{G}^{l_1\ldots l_n}_{(n)}(p_1,\ldots,p_n)\ , \tag{5.66}$$

and thus for amputated Green functions

$$\widehat{G}^{l_1\ldots l_n}_{B\,(n,\mathrm{amp})}(p_1,\ldots,p_n) = \left[\prod_{r=1}^n Z_{l_r}^{-1/2}\right] \widehat{G}^{l_1\ldots l_n}_{(n,\mathrm{amp})}(p_1,\ldots,p_n)\ , \tag{5.67}$$

As for the fields, a subscript B indicates the bare quantity, while we have dropped the subscript R for the renormalized ones. Often, the amputated Green functions coincide with the 1PI vertex function and, obviously, the latter must satisfy the same relation as the former:

$$\Gamma^{(n)}_{B,l_1\ldots l_n}(p_1,\ldots,p_n) = \left[\prod_{r=1}^n Z_{l_r}^{-1/2}\right] \Gamma^{(n)}_{l_1\ldots l_n}(p_1,\ldots,p_n)\ . \tag{5.68}$$

One can also derive this relation more formally by working with the generating functionals: $e^{iW[J]} \sim \int \mathcal{D}\phi\, e^{iS[\phi]+i\int J\phi}$ and $e^{iW_B[J_B]} \sim \int \mathcal{D}\phi_B\, e^{iS_B[\phi_B]+i\int J_B\phi_B}$. Observing that $S[\phi] = S_B[\phi_B]$ and $\phi_B = \sqrt{Z}\phi$ we find $W_B[J_B] = W[\sqrt{Z}J_B]$ up to an irrelevant additive constant. Upon Legendre transforming $W[J]$ and $W_B[J_B]$ one finds

$$\Gamma[\varphi] = \Gamma_B[\sqrt{Z}\varphi]\ . \tag{5.69}$$

Expanding in powers of φ yields the relations (5.68).

To simplify the notation, we consider just one type of field and suppress all indices. However, we will explicitly indicate the other quantities on which the vertex functions depend:

$$\Gamma^{(n)}_B(p_j, g_B, \epsilon) = \left[Z\Big(g(\mu), \frac{\mu}{\widetilde{\mu}}, \epsilon\Big)\right]^{-n/2} \Gamma^{(n)}\big(p_j, g(\mu), \mu\big)\ . \tag{5.70}$$

This is to be thought as the asymptotics for small ϵ where the renormalized $\Gamma^{(n)}$ are ϵ-independent. On the left-hand side, nothing depends on μ. More precisely, if we take $\mu\frac{\partial}{\partial\mu}$, holding g_B (and ϵ) fixed, we get zero. Thus

$$0 = \left[\mu\frac{\partial}{\partial\mu}\Big|_{g_B,\epsilon} - \frac{n}{2}\left(\mu\frac{\partial}{\partial\mu}\Big|_{g_B,\epsilon}\log Z\right)\right]\Gamma^{(n)}\big(p_j, g(\mu), \mu\big) \ . \tag{5.71}$$

One has

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}\Big|_{g_B,\epsilon}\Gamma^{(n)}\big(p_j, g(\mu), \mu\big) &= \left[\mu\frac{\partial}{\partial\mu}\Big|_{g(\mu)} + \left(\mu\frac{\partial}{\partial\mu}\Big|_{g_B,\epsilon} g(\mu)\right)\frac{\partial}{\partial g(\mu)}\right]\Gamma^{(n)}\big(p_j, g(\mu), \mu\big) \\ &= \left[\mu\frac{\partial}{\partial\mu}\Big|_{g(\mu)} + \beta(g(\mu))\frac{\partial}{\partial g(\mu)}\Big|_{\mu}\right]\Gamma^{(n)}\big(p_j, g(\mu), \mu\big) \ ,\end{aligned} \tag{5.72}$$

where we have used (5.65). Similarly, we have

$$\eta \equiv \mu\frac{\partial}{\partial\mu}\Big|_{g_B,\epsilon}\log Z\Big(g(\mu), \frac{\mu}{\widetilde{\mu}}, \epsilon\Big) = \left[\mu\frac{\partial}{\partial\mu}\Big|_{g(\mu),\epsilon} + \beta(g(\mu))\frac{\partial}{\partial g(\mu)}\Big|_{\mu,\epsilon}\right]\log Z\Big(g(\mu), \frac{\mu}{\widetilde{\mu}}, \epsilon\Big) \ . \tag{5.73}$$

A priori, η can depend on the same arguments as Z, but looking at the above one-loop examples of Z_2 or Z_3, we see that the corresponding η_2 and η_3 are functions only of the renormalized coupling $g(\mu)$. We will shortly see that, in general, η can only be a function of $g(\mu)$, just as is the case for $\beta(\mu)$. Combining the last three equations, we get

$$\boxed{0 = \left[\mu\frac{\partial}{\partial\mu}\Big|_{g(\mu)} + \beta\big(g(\mu)\big)\ \frac{\partial}{\partial g(\mu)}\Big|_{\mu} - \frac{n}{2}\ \eta\big(g(\mu)\big)\right]\Gamma^{(n)}\big(p_j, g(\mu), \mu\big) \ .} \tag{5.74}$$

This is the Callan-Symanzik equation for the n-point vertex function. It is now clear that η cannot depend on ϵ or $\widetilde{\mu}$ since nothing else in this equation depends on these quantities. Since the explicit μ-dependence of Z is only via the combination $\frac{\mu}{\widetilde{\mu}}$ it follows, since η does not depend on $\widetilde{\mu}$ it cannot have any explicit μ-dependence either. Thus η is a function of the dimensionless coupling $g(\mu)$ only:

$$\boxed{\eta\big(g(\mu)\big) = \mu\frac{\partial}{\partial\mu}\Big|_{g_B,\epsilon}\log Z \ .} \tag{5.75}$$

The generalization of (5.74) to the case of a vertex function involving several fields is obvious. Each type of field comes with its own Z_r factor and corresponding η_r-function, so that the Callan-Symanzik equation becomes

$$0 = \left[\mu\frac{\partial}{\partial\mu}\Big|_{g(\mu)} + \beta\big(g(\mu)\big)\ \frac{\partial}{\partial g(\mu)}\Big|_{\mu} - \frac{1}{2}\ \sum_{j=1}^{n}\eta_j\big(g(\mu)\big)\right]\Gamma^{(n)}_{l_1,\ldots l_n}\big(p_j, g(\mu), \mu\big) \ . \tag{5.76}$$

Note that there is an analogous equation for the n-point Green function $\widehat{G}_{(n)}(p_j, g(\mu), \mu)$ with simply the sign in front of the η_j being $+$ rather than $-$.

As an example, let us work out the Callan-Symanzik equation for the electron-electron-photon vertex function Γ^μ of QED (in the limit where one can neglect the electron mass, i.e. $\mu \gg m_e$), including the contributions to β, η_2 and η_3 up to one loop. One has from (5.60), $\log Z_3 = -\frac{e(\mu)^2}{6\pi^2}\left[\widetilde{C} - \log\frac{\mu}{\widetilde{\mu}}\right]$. Then $\mu\frac{\partial}{\partial\mu}\Big|_{g(\mu),\epsilon}\log Z_3 = \frac{e(\mu)^2}{6\pi^2}$ while $\beta(g(\mu))\frac{\partial}{\partial g(\mu)}\Big|_{\mu,\epsilon}\log Z_3$ gives a contribution $\sim e(\mu)^4$ comparable to a two-loop contribution. Hence, at leading order

$$\eta_3 = \frac{e(\mu)^2}{6\pi^2} + \mathcal{O}\big(e(\mu)^4\big) \ . \tag{5.77}$$

Similarly, from (5.57)

$$\eta_2 = \frac{e(\mu)^2}{8\pi^2} + \mathcal{O}\big(e(\mu)^4\big) \ . \tag{5.78}$$

Thus $2\eta_2 + \eta_3 = \frac{5e(\mu)^2}{12\pi^2}$. Finally, recall the one-loop β-function of QED, eq. (5.17), $\beta(e(\mu)) = \frac{e(\mu)^3}{12\pi^2}$, so that

$$0 = \left[\mu\frac{\partial}{\partial\mu}\Big|_{e(\mu)} + \frac{e(\mu)^3}{12\pi^2}\ \frac{\partial}{\partial e(\mu)}\Big|_{\mu} - \frac{5\,e(\mu)^2}{24\pi^2}\right]\Gamma^\mu\big(p_j, e(\mu), \mu\big) \ . \tag{5.79}$$

5.4.3 Solving the Callan-Symanzik equations

To solve (5.74) or (5.76) one first needs to find the running coupling constant as the solution $\overline{g}(\mu)$ of the first order ordinary differential equation with some initial condition

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\overline{g}(\mu) = \beta\big(\overline{g}(\mu)\big) \quad , \quad \overline{g}(\mu_0) = g_0 \ . \tag{5.80}$$

We have already seen that the solution simply is given by $\int_{g_0}^{\overline{g}(\mu)} \frac{\mathrm{d}g}{\beta(g)} = \log\frac{\mu}{\mu_0}$. With this $\overline{g}(\mu)$ the partial differential equations (5.74) or (5.76) are turned into ordinary ones:

$$0 = \left[\mu\frac{\mathrm{d}}{\mathrm{d}\mu} - \frac{1}{2}\ \sum_{j=1}^{n}\eta_j\big(\overline{g}(\mu)\big)\right]\Gamma^{(n)}_{l_1,\ldots l_n}\big(p_j, \overline{g}(\mu), \mu\big) \ . \tag{5.81}$$

This shows that

$$\exp\left[-\frac{1}{2}\ \sum_{j=1}^{n}\int_{\mu_0}^{\mu}\frac{\mathrm{d}\mu'}{\mu'}\ \eta_j\big(\overline{g}(\mu')\big)\right]\Gamma^{(n)}_{l_1,\ldots l_n}\big(p_j, \overline{g}(\mu), \mu\big) = \gamma^{(n)}_{l_1,\ldots l_n}(p_j) \tag{5.82}$$

does not depend on μ. Obviously, $\gamma^{(n)}_{l_1,\ldots l_n}(p_j) = \Gamma^{(n)}_{l_1,\ldots l_n}(p_j, g_0, \mu_0)$. Hence, the Callan-Symanzik equation (5.76) is solved by

$$\boxed{\Gamma^{(n)}_{l_1,\ldots l_n}\big(p_j, \overline{g}(\mu), \mu\big) = \exp\left[\frac{1}{2}\ \sum_{j=1}^{n}\int_{\mu_0}^{\mu}\frac{\mathrm{d}\mu'}{\mu'}\ \eta_j\big(\overline{g}(\mu')\big)\right]\Gamma^{(n)}_{l_1,\ldots l_n}\big(p_j, g_0, \mu_0\big) \ .} \tag{5.83}$$

This is a powerful result. Suppose we have a good approximation for $\Gamma^{(n)}$ for some $\overline{g}(\mu_0)$, like in an asymptotically free theory for $\mu_0 \to \infty$. One can then use (5.83) to reliably compute $\Gamma^{(n)}$ with coupling $\overline{g}(\mu)$ at scale μ, as long as one can trust the approximations made when computing β and η_j.

One can make further statements if one uses "dimensional analysis". Every vertex function or Green function has some well-defined "engineering dimension" which is its mass dimension. We say that some function $\Gamma^{(n)}$ has engineering dimension Δ_n if $\Gamma^{(n)}/\mu^{\Delta_n}$ is a dimensionless quantity. In 4 dimensions, a scalar field has dimension 1 and a scalar n-point Green function $G_{(n)}(x_1, \ldots x_n)$ has $\Delta = n$, its Fourier transform has dimension $n - 4n$. The vertex function $\Gamma^{(n)}$ is obtained by multiplying with n inverse propagators which add a total of $2n$ to the dimension. Hence $\Gamma^{(n)}$ has scaling dimension $-n$. If one writes $\Gamma^{(n)}(p_j) = \widehat{\Gamma}^{(n)}(p_j)\, \delta^{(4)}(\sum p_j)$, as we did e.g. in QED for Γ^μ, we see that the engineering dimension of $\widehat{\Gamma}^{(n)}$ is $4 - n$. More generally, let the engineering dimension of $\Gamma^{(n)}$ be Δ_n. Then

$$\Gamma^{(n)}\Big(p_j, g(\mu), \mu\Big) = \mu^{\Delta_n}\ \Gamma^{(n)}\Big(\frac{p_j}{\mu}, g(\mu), 1\Big)\ , \tag{5.84}$$

since the dimensionless quantity $\Gamma^{(n)}/\mu^{\Delta_n}$ can only depend on the dimensionless combinations $\frac{p_j}{\mu}$ and on $g(\mu)$.

5.4.4 Infrared fixed point and critical exponents / large momentum behavior in asymptotic free theories

Suppose one has an infrared fixed point at $g = g_*$. Recall that this occurs if $\beta(g_*) = 0$ and $\beta'(g_*) = a > 0$. Then as $\mu \to 0$ one has $g(\mu) \to g_*$. Let $\eta_j(g_*) = \eta_j^*$. The integral $\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'}\ \eta_j\big(\overline{g}(\mu')\big)$ for $\mu \to 0$ is dominated by the small values of μ' and one can approximate $\eta_j\big(\overline{g}(\mu')\big) \simeq \eta_j^*$. Hence $\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'}\ \eta_j\big(\overline{g}(\mu')\big) \simeq c_j(\mu_0) + \eta_j^* \log\frac{\mu}{\mu_0}$. Combining (5.83) and (5.84) yields

$$\Gamma^{(n)}_{l_1,\ldots l_n}(p_j, g_0, \mu_0) = \exp\left[-\frac{1}{2}\sum_{j=1}^{n}\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\ \eta_j\big(\overline{g}(\mu')\big)\right]\mu^{\Delta_n}\ \Gamma^{(n)}\Big(\frac{p_j}{\mu}, \overline{g}(\mu), 1\Big)\ . \tag{5.85}$$

We now let $\mu = \lambda\mu_1$ and $p_j = \lambda q_j$ with fixed μ_1 and q_j while letting $\lambda \to 0$. Then $\mu \to 0$ and $\overline{g}(\mu) \to g_*$ so that

$$\begin{aligned}\Gamma^{(n)}_{l_1,\ldots l_n}(\lambda q_j, g_0, \mu_0) &\simeq \exp\left[-\frac{1}{2}\sum_{j=1}^{n}\left(c_j(\mu_0) + \eta_j^*\log\frac{\lambda\mu_1}{\mu_0}\right)\right]\lambda^{\Delta_n}\mu_1^{\Delta_n}\ \Gamma^{(n)}\Big(\frac{q_j}{\mu_1}, g_*, 1\Big)\\ &= \lambda^{\Delta_n - \frac{1}{2}\sum_j \eta_j^*}\ \exp\left[-\frac{1}{2}\sum_{j=1}^{n}\left(c_j(\mu_0) + \eta_j^*\log\frac{\mu_1}{\mu_0}\right)\right]\mu_1^{\Delta_n}\ \Gamma^{(n)}\Big(\frac{q_j}{\mu_1}, g_*, 1\Big)\ .\end{aligned} \tag{5.86}$$

Thus as $\lambda \to 0$, the $\Gamma^{(n)}$ scale as $\lambda^{\Delta_n - \frac{1}{2}\sum_j \eta_j^*}$. The naive scaling exponent Δ_n has been corrected by $-\frac{1}{2}\sum_j \eta_j^*$.

Recall that for Green functions one has to switch the sign in front of the η_j. For the two-point function there is only a single η involved. Finally recall that the momentum-space propagator is the two-point functions without the $\delta^{(4)}$ so that $\Delta = -2$ for scalars and $\Delta = -1$ for Dirac propagators. Then e.g. for a scalar propagator

$$\Delta'(\lambda q, g_0, \mu_0) \simeq \lambda^{-2+\eta_*}\ \times \left[e^{c(\mu_0)}\left(\frac{\mu_1}{\mu_0}\right)^{\eta_*}\mu_1^{-2}\ \Delta'\Big(\frac{q_j}{\mu_1}, g_*, 1\Big)\right] \qquad \text{as}\ \ \lambda \to 0\ . \tag{5.87}$$

This infrared scaling behavior $\sim \lambda^{2+\eta_*}$ is as if the field had dimension $1+\eta_*/2$. This is why $\eta_*/2$ is called the anomalous dimension of the field. When the critical point of the (four-dimensional) Ising model is described by the massless ϕ^4 theory, η_* is called a critical exponent.

The previous arguments are easily adapted to the large momentum behavior of vertex or correlation functions in asymptotic free theories. One now considers the large λ asymptotics. Then $\overline{g}(\lambda\mu_1) \to 0$ and it is enough to compute $\Gamma^{(n)}\Big(\frac{q_j}{\mu_1}, \overline{g}(\lambda\mu_1), 1\Big)$ to the lowest non-trivial order in perturbation theory (i.e. tree-level if non-vanishing). Similarly η_j^* is to be replaced by the one-loop result evaluated at the (small) coupling $\overline{g}(\lambda\mu_1)$.

5.5 Callan-Symanzik equations for a massive theory

5.5.1 Operator insertions and renormalization of local operators

It is often useful to consider composite operators like e.g. $\mathcal{O}(x) = \phi^2(x)$ in a scalar theory. (As always, ϕ denotes the renormalized field.) One should think of this operator as being obtained from $\phi(y)\phi(x)$ in the limit where $y \to x$. One can then compute correlation functions (Green functions) of $\mathcal{O}(x)$ with the elementary fields $\phi(x_i)$ by first computing $\langle T\big[\phi(y)\phi(x)\phi(x_1)\ldots\phi(x_n)\big]\rangle_{\rm vac}$ and then letting $y \to x$. Of course, this limit is singular as one already sees for $n=0$. Even in a free theory one has $\lim_{y\to x}\langle T\big[\phi(y)\phi(x)\big]\rangle_{\rm vac} = -i\ \lim_{y\to x}\int \frac{{\rm d}^4 p}{(2\pi)^4}\frac{e^{ip(y-x)}}{p^2+m^2-i\epsilon} = -i\int \frac{{\rm d}^4 p}{(2\pi)^4}\frac{1}{p^2+m^2-i\epsilon}$. This can be interpreted as a "vertex" with two lines attached that are joined by a propagator, i.e. a one-loop diagram.

To get finite Green functions in the presence of such operators, one should work with the corresponding renormalized operators $\mathcal{O}_R(x)$:

$$\mathcal{O}(x) = Z_{\mathcal{O}}\,\mathcal{O}_R(x) = \mathcal{O}_R(x) + (Z_{\mathcal{O}}-1)\mathcal{O}_R(x)\ . \tag{5.88}$$

The second term is interpreted, as usual, as a counterterm. One can then show that with appropriately chosen $Z_{\mathcal{O}}$ the Green functions involving $\mathcal{O}_R$ are finite.

In the example of $\mathcal{O}=\phi^2$ we actually already know this counterterm. If we start with a massive theory and consider all terms in the action that involve the mass terms as a small perturbation, we see that inserting $-m^2\phi^2 + Z\delta m^2\phi^2 - (Z-1)m^2\phi^2$ in any correlator must give a finite result. Thus

$$\mathcal{O}_{\phi^2,R}(x) = \Big[1-\frac{1}{m^2}\,Z\,\delta m^2 + (Z-1)\Big]\,\mathcal{O}_{\phi^2}(x) = Z\Big[1-\frac{\delta m^2}{m^2}\Big]\,\mathcal{O}_{\phi^2}(x) \quad\Rightarrow\quad Z_{\phi^2} = \frac{1}{Z}\Big[1-\frac{\delta m^2}{m^2}\Big]^{-1}. \tag{5.89}$$

Of course, this is just such that

$$m^2\,\mathcal{O}_{\phi^2,R}(x) = m_B^2\,Z\,\mathcal{O}_{\phi^2}(x) = m_B^2\,\phi_B^2(x)\ . \tag{5.90}$$

In the ϕ^4-theory, eqs (3.110) and (3.111) yield, up to order g : $Z_{\phi^2} = 1+\frac{\delta m^2}{m^2} = 1-\frac{g}{16\pi^2}\Big(\frac{1}{\epsilon} - \frac{\gamma+\log\pi-1}{2} - \frac{1}{2}\log\frac{m^2}{\tilde\mu^2}\Big)$. Note that this is in a massive theory. If we are in a massless theory, $\delta m^2=0$, as already observed above, and $Z_{\phi^2}=1$ at this order.

As another example, consider $\mathcal{O}_{\phi^4}(x) = \phi^4(x)$. Again, we know from the renormalized perturbation theory of ϕ^4-theory that insertions of $g_B\phi_B^4(x)$ in correlation functions lead to finite expressions.

Recall that $g_B = \frac{Z_g}{Z^2} g$ and $\phi_B = \sqrt{Z}\phi$, hence $g_B \phi_B^4(x) = g Z_g \phi^4(x)$. Since g is finite, we see that the operator which will yield finite results when inserted into correlators is

$$\mathcal{O}_{\phi^4,R}(x) = Z_g\, \phi^4(x) \quad \Rightarrow \quad Z_{\phi^4} = Z_g^{-1} \ . \tag{5.91}$$

One sometimes defines $m_B^2 = \frac{Z_m}{Z} m^2$ so that $Z_m = Z\left[1 - \frac{\delta m^2}{m^2}\right]$. From eq. (5.89) one sees that then $Z_{\phi^2} = Z_m^{-1}$, in complete analogy with (5.91)

Note that our definition (5.88) of $Z_\mathcal{O}$ expresses a relation between the renormalized operator $\mathcal{O}_R(x)$ and the operator formed from the elementary renormalized fields. One could further express the latter in terms of the bare fields, e.g. $\phi^n(x) = Z^{-n/2}\phi_B^n(x)$, and define the bare composite operators in terms of the bare fields only, e.g. $\mathcal{O}_B(x) = \phi_B^n(x)$. Thus

$$\mathcal{O}_B(x) = \Big(\prod_l \sqrt{Z_l}\Big)\, \mathcal{O}(x) = \Big(Z_\mathcal{O} \prod_l \sqrt{Z_l}\Big) \mathcal{O}_R(x) \equiv \widetilde{Z}_\mathcal{O}\, \mathcal{O}_R(x) \ . \tag{5.92}$$

5.5.2 Callan-Symanzik equations in the presence of operator insertions

It is now straightforward to derive Callan-Symanzik equations for Green functions or vertex functions that involve insertions of local operators as just discussed. Again one writes that the bare functions with the bare operators $\mathcal{O}_B$ inserted into the correlation functions of the bare fields ϕ_B cannot depend on the renormalization group scale μ. This is then translated into a differential equation for the correlation functions of renormalized fields with renormalized operators inserted.

To simplify the notation, we will only consider one type of field, denoted ϕ, and one type of operator, denoted $\mathcal{O}$. We let

$$\begin{aligned} G_R^{(n,l)}(x_1,\ldots,x_n;y_1,\ldots y_l) &= \langle T\Big[\phi(x_1)\ldots\phi(x_n)\mathcal{O}_R(y_1)\ldots\mathcal{O}_R(y_l)\Big]\rangle_{\rm vac} \quad , \\ G_B^{(n,l)}(x_1,\ldots,x_n;y_1,\ldots y_l) &= \langle T\Big[\phi_B(x_1)\ldots\phi_B(x_n)\mathcal{O}_B(y_1)\ldots\mathcal{O}_B(y_l)\Big]\rangle_{\rm vac} \ . \end{aligned} \tag{5.93}$$

They are related by

$$G_B^{(n,l)} = Z^{n/2}\ \widetilde{Z}_\mathcal{O}^l\ G_R^{(n,l)} \ . \tag{5.94}$$

If several fields and/or operators are present, one has the appropriate products of the Z factors. If one amputates these Green functions by multiplying with n inverse propagators, one gets a similar relation between the bare and renormalized quantities, but with $Z^{n/2}$ replaced by $Z^{-n/2}$. The same relation holds for the 1PI vertex functions with operator insertions:

$$\Gamma_B^{(n,l)} = Z^{-n/2}\ \widetilde{Z}_\mathcal{O}^l\ \Gamma_R^{(n,l)} \ . \tag{5.95}$$

As before (cf. eq. (5.69)), this relation can also be proven more properly by working through the relations between the generating functional, now with an extra source/current for the operator $\mathcal{O}_B = \widetilde{Z}_\mathcal{O}\, \mathcal{O}_R$. Doing the Legendre transform – but only with respect to the elementary fields and not with respect to this extra current, one gets a relation between the generating functional of 1PI diagrams with additional insertions of $\mathcal{O}$. Expanding this relation yields (5.95).

In addition to the definition (5.75) of η, we also define

$$\widetilde{\eta}_{\mathcal{O}}\big(g(\mu)\big) = \mu \frac{\partial}{\partial\mu}\Big|_{g_B,\epsilon} \log \widetilde{Z}_{\mathcal{O}} \ . \tag{5.96}$$

It is then straightforward to generalize (5.74) to

$$0 = \left[\mu\frac{\partial}{\partial\mu}\Big|_{g(\mu)} + \beta\big(g(\mu)\big)\ \frac{\partial}{\partial g(\mu)}\Big|_{\mu} - \frac{n}{2}\ \eta\big(g(\mu)\big) + l\ \widetilde{\eta}_{\mathcal{O}}\big(g(\mu)\big)\right] \Gamma^{(n,l)}\big(p_j, g(\mu), \mu\big) \ . \tag{5.97}$$

The solution of this equation and study of the asymptotic behaviors of the $\Gamma^{(n,l)}$ proceeds just as for $l = 0$ with the obvious replacement $\frac{n}{2}\eta \to \frac{n}{2}\eta - l\,\widetilde{\eta}_{\mathcal{O}}$.

5.5.3 Massive Callan-Symanzik equations

Let us now consider a massive scalar theory and consider the mass term $\int \mathrm{d}^4 z\,\big(-\frac{1}{2}\big) m_B^2 \phi_B^2(z) = -\frac{1}{2}\int \mathrm{d}^4 z\, m^2 \mathcal{O}_{\phi^2,R}(z)$ (cf. (5.90)) as a perturbation. Then any Green function of the massive theory with l insertions of $\mathcal{O}_{\phi^2}$ can be written as

$$G^{(n,l)}_{\text{massive}}(x_1,\ldots x_n; y_1,\ldots y_l) = \sum_{r=0}^{\infty} (-i)^r \frac{(m^2)^r}{2^r\ r!} \int \mathrm{d}^4 z_1 \ldots \mathrm{d}^4 z_r\, G^{(n,l+r)}_{\text{massless}}(x_1,\ldots x_n; y_1,\ldots y_l, z_1,\ldots z_r) \ . \tag{5.98}$$

Each term in the sum on the right-hand side satisfies a Callan-Symanzik equation (5.97) with the last term in the bracket being $(l+r)\widetilde{\eta}_{\phi^2}$. But we can generate exactly this expression for each term in the sum by acting on the sum with $\widetilde{\eta}_{\phi^2}\Big(l + m^2\frac{\partial}{\partial m^2}\Big)$. We find that the left-hand side satisfies the massive Callan-Symanzik equation:

$$\begin{aligned} 0 \ = \ & \Bigg[\mu\frac{\partial}{\partial\mu}\Big|_{g(\mu),m} + \beta\big(g(\mu)\big)\ \frac{\partial}{\partial g(\mu)}\Big|_{\mu,m} + \frac{n}{2}\ \eta\big(g(\mu)\big) \\ & \qquad + \widetilde{\eta}_{\phi^2}\big(g(\mu)\big)\ \Big(l + m^2\frac{\partial}{\partial m^2}\Big|_{\mu,g(\mu)}\Big)\Bigg] G^{(n,l)}_{\text{massive}}\big(p_j, g(\mu), \mu\big) \ . \end{aligned} \tag{5.99}$$

The relation between the bare and renormalized 1PI functions with l insertions of an operator $\mathcal{O}$ was obtained before. In particular, this also applies for the insertions of the mass operator and we get, in analogy with the preceeding Callan-Symanzik equation for the Green functions,

$$\begin{aligned} 0 \ = \ & \Bigg[\mu\frac{\partial}{\partial\mu}\Big|_{g(\mu),m} + \beta\big(g(\mu)\big)\ \frac{\partial}{\partial g(\mu)}\Big|_{\mu,m} - \frac{n}{2}\ \eta\big(g(\mu)\big) \\ & \qquad + \widetilde{\eta}_{\phi^2}\big(g(\mu)\big)\ \Big(l + m^2\frac{\partial}{\partial m^2}\Big|_{\mu,g(\mu)}\Big)\Bigg] \Gamma^{(n,l)}_{\text{massive}}\big(p_j, g(\mu), \mu\big) \ . \end{aligned} \tag{5.100}$$

PART III :

NON-ABELIAN GAUGE THEORIES

6 Non-abelian gauge theories: formulation and quantization

Gauge invariance, and in particular non-abelian gauge invariance, plays a most important role in the formulation of the quantum field theories that seem to describe (almost?) all of the particle physics as it is known today: quantum electrodynamics to begin with and its embedding into the electro-weak theory based on the gauge group $SU(2) \times U(1)$, as well as the theory of strong interactions based on the gauge group $SU(3)$. Here, we will briefly show how to construct (classical) actions that are invariant under non-abelian gauge symmetries. Then we will discuss how to quantize these theories using the functional integral approach. This will involve the issues of gauge-fixing, Faddeev-Popov procedure and the appearance of so-called ghost fields. The gauge-fixed action no longer is gauge invariant but instead has a new symmetry, the BRST-symmetry which we will identify. This BRST symmetry will play a crucial role when showing that these gauge theories are renormalizable.

6.1 Non-abelian gauge transformations and gauge invariant actions

Recall from quantum electrodynamics that the classical action is invariant under the following gauge transformation:

$$\psi_l(x) \to \psi'_l(x) = e^{i\alpha(x) q_l} \psi_l(x) \quad , \quad A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x) = A_\mu(x) + \frac{i}{q_l} e^{i\alpha(x) q_l} \partial_\mu \, e^{-i\alpha(x) q_l} \ . \tag{6.1}$$

This is often called a $U(1)$ gauge invariance because obviously $g(x) = e^{i\alpha(x) q_l} \in U(1)$. There seems to be a different $g(x)$ for every different charge q_l, but note that q_l is the eigenvalue of the charge operator Q when acting on ψ_l. Hence we can write $g(x) = e^{i\alpha(x) Q}$. Using this $g(x)$ we can rewrite the transformations (6.1) as

$$\psi(x) \to \psi'_l(x) = g(x)\psi(x) \quad , \quad QA_\mu(x) \to QA'_\mu(x) = Q\, A_\mu(x) + i g(x) \partial_\mu \, g(x)^{-1} \ . \tag{6.2}$$

Of course, gauge invariant terms then are built from covariant derivatives, i.e.

$$D_\mu \psi = (\partial_\mu - iQA_\mu)\psi \ , \tag{6.3}$$

since this transforms as

$$\begin{aligned} D_\mu \psi \to D'_\mu \psi' &= \Big(\partial_\mu - iQA_\mu + (g\partial_\mu g^{-1})\Big) g\psi = g\Big(\partial_\mu \psi_l + (g^{-1}\partial_\mu g) - iQA_\mu + (\partial_\mu g^{-1} g)\Big)\psi \\ &= g\Big(\partial_\mu - iQA_\mu\Big)\psi = g D_\mu \psi. \end{aligned} \tag{6.4}$$

Indeed, since g is just a phase, $|D_\mu \psi|^2$ obviously is invariant.

We have rewritten the simple $U(1)$ gauge transformation of the electromagnetic (gauge) field A_μ and of the matter fields ψ, as well as the definition of the covariant derivative, in a way that makes their generalization to other gauge groups almost obvious. Let us now consider the case where the abelian $U(1)$ group is replaced by some, generally non-abelian Lie group G, called the gauge group. In general this can be a product of so-called simple groups and $U(1)$ factors, like e.g. $SU(3)\times SU(2)\times U(1)$ for the standard model of electro-weak and strong interactions, or even some more exotic groups like e.g. E_8. For simplicity, one might simply think of $SU(N)$, the group of unitary $N\times N$ matrices with unit determinant. The structure of a Lie group is almost entirely captured by the commutators of its generators: the group elements close to the identity are always of the form $e^{i\theta^a t_a}$ with $\dim G$ small parameters θ^a and $\dim G$ generators t_a. The latter must satisfy closed (Lie algebra) commutation relations

$$[t_\alpha, t_\beta] = i C^\gamma{}_{\alpha\beta} t_\gamma \ , \tag{6.5}$$

with real structure constants $C^\gamma{}_{\alpha\beta}$ which satisfy the Jacobi identity $C^\delta{}_{[\alpha\beta}C^\epsilon{}_{\gamma]\delta}=0$. It may happen that several groups like e.g. $SU(2)$ and $SO(3)$ have the same Lie algebra. In this case the groups are locally identical but not globally. We will be not so much concerned with the group elements themselves but rather with their representations. If we use a specific representation $\mathcal{R}$ we write $t_\alpha^\mathcal{R}$ or $(t_\alpha^\mathcal{R})_k{}^l$ for the $\dim\mathcal{R}\times\dim\mathcal{R}$ matrices of this representation. For compact Lie algebras (i.e. if $\mathrm{tr}\, t_\alpha t_\beta$ is positive-definite), all finite dimensional representations are hermitian. This is the case of most interest in gauge theories and, hence, $(t_\alpha^\mathcal{R})^\dagger = t_\alpha^\mathcal{R}$. Of course, the corresponding representations of the group then are unitary. In this case one can also find a basis for the generators for which the $C^\gamma{}_{\alpha\beta}$ are antisymmetric in all 3 indices and one may then drop the distinction between upper and lower indices. Note that the Jacobi identity implies that the

$$(t_\alpha^{\rm adj})_{\beta\gamma} = -i C_{\alpha\beta\gamma} \tag{6.6}$$

satisfy the algebra (6.5). This special representation is called the adjoint representation.

Consider a collection of matter fields $\psi_l,\ l=1,\ldots r$ transforming in some r-dimensional representation $\mathcal{R}$ of G:

$$\psi_l(x)\to\psi'_l(x) = U^\mathcal{R}(g(x))_l{}^k\ \psi_k(x)\ , \tag{6.7}$$

where $U^\mathcal{R}(g(x))$ is the $r\times r$ matrix associated to the group element $g(x)\in G$ in the representation $\mathcal{R}$. For $G=SU(N)$, the simplest example is the so-called fundamental (or "vector")) representation for which $U(g(x))_i{}^j = g(x)_i{}^j$ with i and j running simply from 1 to N. In the following, we will not write the indices explicitly so that we simply write

$$\psi(x)\to\psi'(x) = U^\mathcal{R}(g(x))\ \psi(x)\ , \tag{6.8}$$

with the understanding that ψ is a $\dim\mathcal{R}$-dimensional column vector of matter fields. We want to construct a covariant derivative D_μ involving ∂_μ and some gauge field A_μ. In the abelian case the relevant combination was $A_\mu Q$ with Q the generator of the $U(1)$ group. Let us try

$$A_\mu = A_\mu^\alpha t_\alpha^\mathcal{R} \equiv A_\mu^\mathcal{R}\ , \tag{6.9}$$

since this can act on the dim $\mathcal{R}$-dimensional column vector of matter fields ψ, and a definition of the covariant derivative as

$$D_\mu^{\mathcal{R}}\,\psi = \left(\partial_\mu - iA_\mu^{\mathcal{R}}\right)\psi\ . \tag{6.10}$$

One often drops the superscript $\mathcal{R}$ on D_μ or A_μ, but one should remember that these quantities take values in the representation $\mathcal{R}$ of the gauge group determined by the matter fields. We want to determine the gauge transformation properties of A_μ in such a way that the covariant derivative simply transforms as

$$D_\mu\,\psi \to D'_\mu\psi' = U^{\mathcal{R}}(g) D_\mu\psi\ . \tag{6.11}$$

First, we look at the transformation of $\partial_\mu\psi$. To further simplify the notation we will simply assume that the representation is the fundamental representation where $D(g(x)) = g(x)$, but one could replace everywhere g by $D^{\mathcal{R}}(g)$). Also we will drop the arguments x, although g, ψ and A_μ all depend on x. One has

$$\partial_\mu\psi \to \partial_\mu\psi' = \partial_\mu(g\psi) = g\partial_\mu\psi + (\partial_\mu g)\psi = g\Big(\partial_\mu + \big(g^{-1}\partial_\mu g\big)\Big)\psi \tag{6.12}$$

On the other hand,

$$-iA_\mu\psi \to -iA'_\mu\psi' = -iA'_\mu g\psi = -ig\Big(g^{-1}A'_\mu g\Big)\psi\ . \tag{6.13}$$

It is then obvious that the covariant derivative will transform covariantly, i.e. as in (6.12) provided A_μ transforms as $-ig^{-1}A'_\mu g + g^{-1}\partial_\mu g = -iA_\mu$, i.e. $A'_\mu = gA_\mu g^{-1} - i\partial_\mu g g^{-1}$. Since $\partial_\mu g^{-1} = -g^{-1}\partial_\mu g g^{-1}$ one has $-\partial_\mu g g^{-1} = g\partial_\mu g^{-1}$ so that one gets

$$A'_\mu = gA_\mu g^{-1} + ig\partial_\mu g^{-1} = g\Big(A_\mu + i\partial_\mu\Big)g^{-1}\ . \tag{6.14}$$

Note that for $G = U(1)$ and $g = e^{i\alpha Q}$ and the replacement $A_\mu \to QA_\mu$ one gets back (6.1). For a general matter representation one simply has

$$\boxed{\begin{array}{lcr} \psi' = U^{\mathcal{R}}(g)\,\psi & , & A_\mu'^{\mathcal{R}} = U^{\mathcal{R}}(g)\Big(A_\mu^{\mathcal{R}} + i\partial_\mu\Big)U^{\mathcal{R}}(g^{-1}) \\ D_\mu^{\mathcal{R}} = \partial_\mu - iA_\mu^{\mathcal{R}} & , & \big(D_\mu^{\mathcal{R}}\psi\big)' = U^{\mathcal{R}}(g)D_\mu^{\mathcal{R}}\,\psi\ . \end{array}} \tag{6.15}$$

For a unitary representation, $(D_\mu\psi)^\dagger D_\mu\psi$ then is obviously invariant. This will be the ingredient to write gauge invariant matter kinetic terms.

It is often enough and simpler to consider only infinitesimal gauge transformations with $g(x) = e^{i\epsilon^\alpha t_\alpha}$ or $U^{\mathcal{R}}(g) = e^{i\epsilon^\alpha(x)t_\alpha^{\mathcal{R}}}$. It is convenient to define $\epsilon^{\mathcal{R}} = \epsilon^\alpha t_\alpha^{\mathcal{R}}$. Then the previous equations yield

$$\boxed{\begin{array}{rcl} \delta\psi = \psi' - \psi & = & i\epsilon^{\mathcal{R}}\psi \\ \delta A_\mu^{\mathcal{R}} = A_\mu'^{\mathcal{R}} - A_\mu^{\mathcal{R}} & = & \partial_\mu\epsilon^{\mathcal{R}} - i\big[A_\mu^{\mathcal{R}}, \epsilon^{\mathcal{R}}\big]\ . \end{array}} \tag{6.16}$$

The latter equation reads in components

$$\delta A_\mu^\alpha = \partial_\mu\epsilon^\alpha + C_{\beta\gamma}{}^\alpha A_\mu^\beta\epsilon^\gamma\ , \tag{6.17}$$

which shows that, of course, the transformation of the gauge field components does not depend on the representation $\mathcal{R}$ of the matter fields. Using the generators of the adjoint representation (6.6) one may also write this as

$$\delta A_\mu^\alpha = (D_\mu^{\rm adj})^\alpha{}_\gamma \, \epsilon^\gamma \ . \tag{6.18}$$

We still need to define the field strength $F_{\mu\nu}$. Since $D_\nu\psi$ transforms exactly as ψ, taking a further covariant derivative D_μ is defined in exactly the same way:

$$D_\mu^{\mathcal{R}} D_\nu^{\mathcal{R}}\psi = (\partial_\mu - iA_\mu^{\mathcal{R}})(\partial_\nu - iA_\nu^{\mathcal{R}})\psi = \partial_\mu\partial_\nu\psi - i\partial_\mu A_\nu^{\mathcal{R}}\psi - iA_\nu^{\mathcal{R}}\partial_\mu\psi - iA_\mu^{\mathcal{R}}\partial_\nu\psi - A_\mu^{\mathcal{R}} A_\nu^{\mathcal{R}}\psi \ . \tag{6.19}$$

Antisymmetrizing in μ and ν eliminates all terms with a derivative of ψ. Hence

$$[D_\mu^{\mathcal{R}}, D_\nu^{\mathcal{R}}]\psi = -iF_{\mu\nu}^{\mathcal{R}}\psi \ , \tag{6.20}$$

with

$$F_{\mu\nu}^{\mathcal{R}} = \partial_\mu A_\nu^{\mathcal{R}} - \partial_\nu A_\mu^{\mathcal{R}} - i[A_\mu^{\mathcal{R}}, A_\nu^{\mathcal{R}}] \ , \tag{6.21}$$

or with $F_{\mu,\nu}^{\mathcal{R}} = F_{\mu\nu}^\alpha t_\alpha^{\mathcal{R}}$:

$$F_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + C_{\beta\gamma}{}^\alpha A_\mu^\beta, A_\nu^\gamma \ . \tag{6.22}$$

It follows from (6.20) that $F_{\mu\nu}$ transforms covariantly:

$$F_{\mu\nu}^{\mathcal{R}} \to F_{\mu\nu}'^{\mathcal{R}} = U^{\mathcal{R}}(g)\, F_{\mu\nu}^{\mathcal{R}}\, U^{\mathcal{R}}(g^{-1}) \qquad \text{or} \qquad \delta F_{\mu\nu}^{\mathcal{R}} = i[\epsilon^{\mathcal{R}}, F_{\mu\nu}^{\mathcal{R}}] \tag{6.23}$$

Obviously then, a gauge invariant scalar density is $\operatorname{tr} F_{\mu\nu}^{\mathcal{R}} F^{\mathcal{R}\mu\nu}$. For compact groups we can choose a basis of generators such that

$$\operatorname{tr} t_\alpha^{\mathcal{R}} t_\beta^{\mathcal{R}} = C_{\mathcal{R}}\, \delta_{\alpha\beta} \quad \Rightarrow \quad \operatorname{tr} F_{\mu\nu}^{\mathcal{R}} F^{\mathcal{R}\mu\nu} = C_{\mathcal{R}} F_{\mu\nu}^\alpha F^{\alpha\mu\nu} \ . \tag{6.24}$$

With the appropriate normalization, this will be the gauge kinetic term, generalizing $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Note that, contrary to the abelian gauge theory of QED, in the non-abelian case, $F_{\mu\nu}^\alpha F^{\alpha\mu\nu}$ contains cubic and quartic terms in the gauge fields. This will lead to cubic and quartic vertices involving only gauge boson lines.

In QED, the coupling constant is given by the electric charge, i.e. the eigenvalue of the charge operator Q. In analogy we may include the Yang-Mills coupling constant $g_{\rm YM}$ in the generators t_α and their representations $t_\alpha^{\mathcal{R}}$. In this case $C_{\mathcal{R}} \sim g_{\rm YM}^2$ and $C_{\beta\gamma}{}^\alpha \sim g_{\rm YM}$. This is what Weinberg does, and we will follow this convention. Alternatively one can use conventionally normalized Lie algebra generators (where the $C_{\mathcal{R}}$ do not include any factors of $g_{\rm YM}$). Then a factor of $g_{\rm YM}$ does appear explicitly in front of the A_μ in the covariant derivative, and similarly in front of the quadratic term in $F_{\mu\nu}$. This explicit factors of $g_{\rm YM}$ can then be removed by rescaling $A_\mu^{\mathcal{R}} \to \frac{1}{g_{\rm YM}} A_\mu^{\mathcal{R}}$. The only appearance of the coupling then is a factor $\frac{1}{g_{\rm YM}}$ in front of every $F_{\mu\nu}$. If the gauge group is simple, there is only a single coupling constant $g_{\rm YM}$. However, in the general case where the gauge group is a product of simple and $U(1)$ factors, every simple or $U(1)$ factor can have its own coupling constant.

The unique Lorentz and gauge invariant Lagrangian quadratic in the field strength then is $\mathcal{L}_{\rm gauge}[F_{\mu\nu}] = -\frac{1}{4}F^\alpha_{\mu\nu}F^{\alpha\mu\nu}$. Of course, there is one more possibility, $\theta_{\alpha\beta}\epsilon^{\mu\nu\rho\sigma}F^\alpha_{\mu\nu}F^\beta_{\rho\sigma}$ but this is a total derivative. Hence the Lagrangian for matter and gauge fields is

$$\mathcal{L}[A_\mu,\psi] = -\frac{1}{4}F^\alpha_{\mu\nu}F^{\alpha\mu\nu} + \mathcal{L}_{\rm matter}[\psi, D_\mu\psi]\ , \tag{6.25}$$

where we collectively denoted all matter fields by ψ. If temporarily ψ denotes spin $\frac{1}{2}$ fields and ϕ scalars, an example of $\mathcal{L}_{\rm matter}$ would be

$$\mathcal{L}_{\rm matter}[\psi, D_\mu\psi,\phi,D_\mu\phi] = -\bar\psi(\gamma^\mu D_\mu + m)\psi - (D_\mu\phi)^\dagger D_\mu\phi - \tilde m^2\phi^\dagger\phi - V(\phi^\dagger\phi)\ . \tag{6.26}$$

We only need the fact that the matter Lagrangian is gauge invariant. This will be the case if $\mathcal{L}_{\rm matter}[\psi,\partial_\mu\psi,\phi,\partial_\mu\phi]$ is invariant under global (rigid) transformation by elements of G.

6.2 Quantization

Just as with QED, direct canonical quantization does not work due to the presence of constraints, in particular one again has $\Pi^\mu_\alpha = F^{\mu 0}_\alpha$ and hence $\Pi^0_\alpha = 0$ which is a primary constraint. Together with a secondary constraint which does not involve A_0 these are first class constraints. As usual, they have to be eliminated by a gauge choice. In the non-abelian case, a convenient choice is the axial gauge

$$A^\alpha_3 = 0\ . \tag{6.27}$$

Then A^α_1 and A^α_2 become canonical variables, while A^α_0 is given by the solution of the secondary constraint. One can then go through the canonical formulation of the functional integral in Hamiltonian form, as we did for QED. To begin with one only integrates over the canonical fields A^α_i $i=1,2$ and their conjugate fields Π^α_i. Proceeding through similar steps as we did for QED[27] , one ends up with a Lagrangian version of the functional integral as (C is a normalization constant)

$$\langle T\Big(\mathcal{O}_a\ldots\mathcal{O}_N\Big)\rangle_{\rm vac} = C\int \mathcal{D}A^\alpha_\mu\,\mathcal{D}\psi_l \prod_{x,\alpha}\delta(A^\alpha_3)\ \mathcal{O}_a\ldots\mathcal{O}_N\ e^{i\int {\rm d}^4x\mathcal{L}[A_\mu,\psi]} \tag{6.28}$$

where one integrates over all 4 components with the manifestly gauge invariant action $\int {\rm d}^4x\mathcal{L}[A_\mu,\psi]$. Of course, the integration over A^α_3 is effectively suppressed by the functional $\delta(A^\alpha_3)$ which imposes our gauge choice. Note that the derivation ensures that this gives a manifestly unitary theory. On the other hand, Lorentz invariance is not manifest.

One is usually interested in computing the vacuum expectation valued of (time-oredered) *gauge invariant* operators $\mathcal{O}_1,\ldots\mathcal{O}_N$, although, at intermediate stages one may also compute and use non-gauge invariant objects as e.g. the gauge field propagator $\langle T\Big(A^\alpha_\mu(x)A^\beta_\nu(y)\Big)\rangle_{\rm vac}$. Note that the measure $\mathcal{D}A^\alpha_\mu$ is gauge invariant: It is easy to show that the Jacobian for the gauge transformation (6.17) equals one.[28] In the absence of (massless) chiral fermions one can also show that the matter

[27] Just as when discussing QED, in this section, we will simply write $\langle(\ldots)\rangle_{\rm vac}$ instead of $\langle{\rm vac,out}|\,(\ldots)\,|{\rm vac,in}\rangle$.

[28] $J_{\alpha\mu x,\beta\nu y} = \frac{\delta A'^{\alpha}_\mu(x)}{\delta A^\beta_\nu(y)} = \delta^{(4)}(x-y)\,\delta^\mu_\nu\left[\delta^\alpha_\beta + C_{\beta\gamma}^{\ \ \alpha}\epsilon^\gamma(x)\right]$, and due to the antisymmetry of the structure constants $C_{\alpha\gamma}^{\ \ \alpha}=0$ and ${\rm Det}J = \exp{\rm Tr}\log J = \exp(\mathcal{O}(\epsilon^2))$.

measure is gauge invariant. However, if chiral fermions are present this is no longer guaranteed. If the matter measure is not invariant, one has a so-called anomaly, and the entire following discussion breaks down. It is thus important that, if present, gauge anomalies cancel between the contributions of the different chiral matter fields.

6.2.1 Faddeev-Popov method

We want to show that one can rewrite the expectation values (6.28) in different equivalent ways, corresponding to different gauge choices. First it is useful to rewrite (6.28) as

$$\langle T\Big(\mathcal{O}_a \ldots \mathcal{O}_N\Big)\rangle_{\rm vac} = C\int \mathcal{D}A_\mu^\alpha \prod_{x,\alpha} \delta(A_3^\alpha)\ G[A_\mu^\alpha] \quad \text{with} \quad G[A_\mu^\alpha] = \int \mathcal{D}\psi_l\, \mathcal{O}_1 \ldots \mathcal{O}_N\ e^{i\int {\rm d}^4x \mathcal{L}[A_\mu,\psi]}\ , \tag{6.29}$$

the important point being that $G[A_\mu^\alpha]$ is gauge invariant. Indeed, with a gauge invariant matter measure and gauge invariant $\mathcal{O}_i$ we have $G[A_\mu'^\alpha] = \int \mathcal{D}\psi_l\, \mathcal{O}_a \ldots \mathcal{O}_N\ e^{i\int {\rm d}^4x \mathcal{L}[A'_\mu,\psi]}$. Upon changing the integration variable from ψ_l to the gauge transformed ψ'_l and using the gauge invariance of the measure this is $G[A_\mu'^\alpha] = \int \mathcal{D}\psi_l\, \mathcal{O}_a \ldots \mathcal{O}_N\ e^{i\int {\rm d}^4x \mathcal{L}[A'_\mu,\psi']} = G[A_\mu^\alpha]$, where we used the gauge invariance of the action in the last step.

The expression of eq. (6.29) is of the general form

$$I_G = C\int \mathcal{D}A_\mu\ B\big[f[A_\mu]\big]\ {\rm Det}\mathcal{F}[A_\mu]\ G[A_\mu] \quad , \qquad \mathcal{F}_{\alpha\beta}(x,y) = \frac{\delta f_\alpha[A'(x)]}{\delta\epsilon^\beta(y)}\Big|_{\epsilon=0}\ , \tag{6.30}$$

where A' denotes the gauge transformed A with parameter ϵ. Indeed, if we let $f^\alpha[A] = A_3^\alpha$ and $B[f] = \prod_{x,\alpha} \delta(f^\alpha(x))$, we have $A_3'^\alpha = \partial_3\epsilon^\alpha$ (since $A_3 = 0$) so that $\mathcal{F}_{\alpha\beta}(x,y) = \delta_{\alpha\beta}\, \frac{\partial}{\partial x^3}\, \delta^{(4)}(x-y)$ and ${\rm Det}\mathcal{F}$ is just an irrelevant constant.

We want to show that I_G does not depend on the choice of the *gauge-fixing function* f or on B. Intuitively, what happens is the following: due to the gauge invariance of $G[A_\mu]$ there are many gauge-equivalent configurations and integrating over all A_μ would result in an infinite factor equal to the "volume" of a "gauge slice". The gauge-fixing condition restricts the functional integration to exactly one configuration among the gauge equivalent ones. There are many different ways to do this and the factor ${\rm Det}\mathcal{F}$ ensures the independence of the specific choice of f or B. This is the functional analogue of the well known fact that $\int {\rm d}x\, \delta(f(x)) f'(x) g(x) = g(x_0)$ does not depend on f, as long as f has a single root x_0.

Faddeev-Popov theorem : For gauge invariant $G[A_\mu]$, the functional integral I_G is independent of the gauge-fixing function f and depends on B only through an irrelevant overall factor.

To prove this theorem we first change variables from A_μ to some $A'_\mu \equiv A_\mu^g$ which we identify with A_μ gauge transformed by some $g(x) \in G$. Using the invariance of the measure and of $G[A_\mu]$ we get

$$I_G = C\int \mathcal{D}A_\mu\ B\big[f[A_\mu^g]\big]\ {\rm Det}\mathcal{F}[A_\mu^g]\ G[A_\mu]\ . \tag{6.31}$$

Since we only introduced g by a change of variables, this cannot depend on g. We multiply both sides by some weight function $\rho[g]$ such that $\int \mathcal{D}g\, \rho[g] = C_0$ is finite:

$$C_0\ I_G = \int \mathcal{D}A_\mu\ G[A_\mu]\, H[A_\mu] \quad , \qquad H[A_\mu] = \int \mathcal{D}g\, \rho[g]\, B\big[f[A_\mu^g]\big]\, {\rm Det}\mathcal{F}[A_\mu^g]\ , \tag{6.32}$$

where now (suppressing the $\alpha, \beta, \ldots$ labels but showing the $x, y, \ldots$, and using the fact that the gauge transformations form a group)

$$\mathcal{F}[A^g_\mu](x,y) = \frac{\delta f[(A^g)'(x)]}{\delta\epsilon(y)}\Big|_{\epsilon=0} = \frac{\delta f[A^{\tilde g(g,\epsilon)}(x)]}{\delta\epsilon(y)}\Big|_{\epsilon=0} = \int \mathrm{d}^4 z\, \frac{\delta f[A^{\tilde g}(x)]}{\delta \tilde g(z)}\Big|_{\tilde g = g} \times \frac{\delta\tilde g(g,\epsilon;z)}{\delta\epsilon(y)}\Big|_{\epsilon=0} . \tag{6.33}$$

We let $\mathcal{G}[g](z,x) = \frac{\delta\tilde g(g,\epsilon;z)}{\delta\epsilon(y)}\Big|_{\epsilon=0}$. Obviously, this only depends on $g(x)$ and one can show that $\rho[g] = \left(\mathrm{Det}\mathcal{G}\right)^{-1}$ yields a reasonable weight function. With this choice, (6.32) yields

$$H[A_\mu] = \int \mathcal{D}g\,\rho[g]\,B\big[f[A^g_\mu]\big]\,\mathrm{Det}\frac{\delta f[A^g_\mu]}{\delta g}\,\mathrm{Det}\mathcal{G} = \int \mathcal{D}g\,B\big[f[A^g_\mu]\big]\,\mathrm{Det}\frac{\delta f[A^g_\mu]}{\delta g} = \int \mathcal{D}f B[f] \equiv C_B\,. \tag{6.34}$$

Obviously, this is independent of the choice of f and depends on B only through the constant C_B. Substituting this result into the first equation (6.32), the same is also true for I_G, i.e I_G/C_B is independent of f and B, which was to be proven.

Since for the special choice $f^\alpha[A_\mu] = A^\alpha_3$ and $B[f] = \prod_{x,\alpha}\delta(f)$ (in this case $C_B = 1$) the integral I_G equals (6.29), the theorem tells us that the vacuum expectation value of a time-ordered product of gauge invariant operators can be evaluated with any choice of gauge-fixing function and function B, provided $\int \mathcal{D}f\, B[f]$ converges and f indeed fixes the gauge:

$$\langle T\Big(\mathcal{O}_a \ldots \mathcal{O}_N\Big)\rangle_{\rm vac} = \frac{C}{C_B}\int \mathcal{D}A_\mu\; B\big[f[A_\mu]\big]\;\mathrm{Det}\mathcal{F}[A_\mu]\int \mathcal{D}\psi_l\,\mathcal{O}_1\ldots\mathcal{O}_N\; e^{i\int \mathrm{d}^4x\mathcal{L}[A_\mu,\psi]}\,, \tag{6.35}$$

6.2.2 Gauge-fixed action, ghosts and Feynman rules

Let us now make a convenient choice of gauge-fixing function f and function B which, in particular, will be manifestly Lorentz invariant:

$$f_\alpha = \partial_\mu A^\mu_\alpha \quad , \qquad B[f] = \exp\left[-\frac{i}{2\xi}\int \mathrm{d}^4x\; f_\alpha(x) f_\alpha(x)\right] . \tag{6.36}$$

As one can see from (6.35), the factor $B[f]$ just contributes an extra term

$$\mathcal{L}_{\rm g.f.} = -\frac{1}{2\xi} f_\alpha f_\alpha \tag{6.37}$$

to the classical Lagrangian. One can then read off the gauge boson propagator $-i\Delta_{\alpha\mu,\beta,n}(p)$ from the quadratic part of $\mathcal{L} + \mathcal{L}_{\rm g.f.}$ and finds

$$\Delta_{\alpha\mu,\beta,\nu}(p) = \delta_{\alpha\beta}\left(\eta_{\mu\nu} + (\xi - 1)\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2 - i\epsilon}\,. \tag{6.38}$$

Except for the extra $\delta_{\alpha\beta}$ this is just like the photon propagator, which should not be surprising, given our choice of f_α.

Next, we must evaluate $\mathcal{F}$ and compute its determinant. Since a fermionic gaussian integral equals the determinant of the quadratic form, it is a convenient trick to rewrite $\mathrm{Det}\mathcal{F}$ as

$$\mathrm{Det}\mathcal{F} = \int \mathcal{D}\omega\mathcal{D}\omega^*\exp\left[i\int \mathrm{d}^4x\,\mathrm{d}^4y\,\omega^*_\alpha(x)\,\mathcal{F}_{\alpha,\beta}(x,y)\,\omega_\beta(y)\right]\,, \tag{6.39}$$

where ω_α and ω^*_α are anticommuting, i.e. fermionic fields. On the other hand, given the nature of $\mathcal{F}$ we will soon see that they cannot be spin $\frac{1}{2}$ fields but must be scalars. Moreover, ω_α and ω^*_α are independent *real* scalars. They do not obey the usual spin statistics relation and hence are named ghosts. However, this is not a problem since we do not want to obtain any Lorentz invariant S-matrix for scattering of these ghost fields. More precisely, ω_α is called a ghost and ω^*_α an anti-ghost. Note that the ghost and anti-ghost carry an index α just like the gauge field A_μ. One sometimes says that they are in the adjoint representation of G, just as A_μ. One has $f_\alpha[A'_\mu] = \partial_\mu\big(\partial^\mu\epsilon_\alpha + C_{\gamma\beta}{}^\alpha A^\gamma_\mu \epsilon^\beta\big)$ and hence

$$\mathcal{F}_{\alpha,\beta}(x,y) = \frac{\delta f_\alpha[A'_\mu](x)}{\delta\epsilon^\beta(y)} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial}{\partial x_\mu}\delta_{\alpha\beta} + C_{\gamma\beta}{}^\alpha A^\gamma_\mu(x)\right)\delta^{(4)}(x-y) \ . \tag{6.40}$$

Using (6.39) we get

$$\text{Det}\mathcal{F} = \int \mathcal{D}\omega\mathcal{D}\omega^* \exp\left[i\int \text{d}^4x\, \mathcal{L}_{\text{ghost}}(x)\right] \quad , \qquad \mathcal{L}_{\text{ghost}} = -\partial_\mu\omega^*_\alpha\partial^\mu\omega_\alpha - C_{\gamma\beta}{}^\alpha\partial_\mu\omega^*_\alpha A^\mu_\gamma\omega_\beta \ . \tag{6.41}$$

Thus the effect of the so-called Faddeev-Popov determinant Det$\mathcal{F}$ is to add the ghost Lagrangian $\mathcal{L}_{\text{ghost}}$ to the classical Lagrangian $\mathcal{L}$ and gauge-fixing Lagrangian $\mathcal{L}_{\text{g.f.}}$:

$$\mathcal{L}_{\text{mod}} = \mathcal{L} + \mathcal{L}_{\text{g.f.}} + \mathcal{L}_{\text{ghost}} \ . \tag{6.42}$$

Note that for an abelian theory like QED, the structure constants $C_{\beta\gamma}{}^\alpha$ vanish and the ghosts do not couple to any of the other fields. Equivalently then, their functional integral only leads to the determinant of $\partial_\mu\partial^\mu$ which is an irrelevant constant. This is why we did not have to bother about the ghosts in QED. In the non-abelian gauge theory, with our choice of gauge-fixing function, the ghosts do couple to the gauge fields and do make important contributions to loop diagrams.[29] Their propagator $\frac{-i}{(2\pi)^4}\Delta_{\alpha\beta}(p)$ can be read off from (6.42) to be the same as for a massless spin-0 fermion:

$$\Delta_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{p^2 - i\epsilon} \ , \tag{6.43}$$

while the ghost - anti-ghost - gauge field vertex involves the structure constant $C_{\beta\gamma}{}^\alpha$ as well as a factor of momentum. More precisely, taking all momenta as incoming (q for the ghost with label β, p for the anti-ghost with label α and k for the gauge boson with labels μ and γ), one reads from (6.41) that this vertex contributes

$$i(2\pi)^4\delta^{(4)}(p+q+k)\, ip_\mu\, C_{\alpha\beta\gamma} \ . \tag{6.44}$$

We already gave the gauge boson propagator. The cubic and quartic gauge boson couplings simply follow from the cubic and quartic terms in $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Again, with all momenta taken as incoming, the vertex for the coupling of three gauge bosons with (p,μ,α), (q,ν,β) and (k,ρ,γ) is

$$i(2\pi)^4\delta^{(4)}(p+q+k)\,(-iC_{\alpha\beta\gamma})\Big[p_\nu\eta_{\mu\rho} - p_\rho\eta_{\mu\nu} + q_\rho\eta_{\nu\mu} - q_\mu\eta_{\nu\rho} + k_\mu\eta_{\rho\nu} - k_\nu\eta_{\rho\mu}\Big] \ , \tag{6.45}$$

[29] Of course, in axial gauge, $A_3 = 0$, one finds $\mathcal{F}_{\alpha\beta}(x,y) = \frac{\delta}{\delta x^3}\delta^{(4)}(x-y)\delta_{\alpha\beta}$ and the ghost Lagrangian simply is $-\partial_3\omega^*_\alpha\omega_\alpha$ so that the ghosts again decouple.

while the vertex for the coupling of four gauge bosons (the fourth one having (l, λ, δ)) is

$$i(2\pi)^4\delta^{(4)}(p+q+k+l)\Big[\ -C_{\epsilon\alpha\beta}C_{\epsilon\gamma\delta}(\eta_{\mu\rho}\eta_{\nu\lambda}-\eta_{\mu\lambda}\eta_{\nu\rho}) \\ -C_{\epsilon\alpha\gamma}C_{\epsilon\delta\beta}(\eta_{\mu\lambda}\eta_{\rho\nu}-\eta_{\mu\nu}\eta_{\lambda\rho}) \\ -C_{\epsilon\alpha\delta}C_{\epsilon\beta\gamma}(\eta_{\mu\nu}\eta_{\rho\lambda}-\eta_{\mu\rho}\eta_{\lambda\nu})\ \Big]\ . \tag{6.46}$$

Note that the Lagrangian preserves the ghost-number which translates into the fact that every vertex with one incoming ghost line also has exactly one outgoing ghost line.

6.2.3 BRST symmetry

One can now start computing Feynman diagrams and generating functionals using the Lagrangian $\mathcal{L}_{\rm mod} = \mathcal{L}+\mathcal{L}_{\rm g.f.}+\mathcal{L}_{\rm ghost}$ of (6.42) and integrating over the matter fields, gauge and ghost fields. Note that $\mathcal{L}_{\rm mod}$ no longer is gauge invariant: this was the whole point about gauge-fixing. All terms in $\mathcal{L}_{\rm mod}$ have $\Delta = 0$ and hence this Lagrangian is *renormalizable by power-counting* as discussed in section 4: there are only finitely many divergent Green's functions and they are made finite by the addition of finitely many counterterms with their coefficients fixed at any given order in perturbation theory. Moreover, these counterterms themselves all have $\Delta_{\rm ct} \geq 0$ and do not upset the renormalizability.

However, we want more than this: the counterterms should be of the form of the initial terms in the Lagrangian. In particular, one should still be able to interpret $\mathcal{L}_{\rm mod} + \mathcal{L}_{\rm ct}$ as arising from the gauge-fixing of some gauge invariant Lagrangian, but now with renormalized parameters. We must find out what is the remnant of the gauge invariance of the original classical Lagrangian. This turns out to be the BRST-symmetry.

Let us first do one more rewriting of our Lagrangian. Introducing an auxiliary field h_α we can rewrite the gauge-fixing term as $\frac{\xi}{2}h_\alpha h_\alpha + h_\alpha f_\alpha$ since doing the gaussian integration over h_α reproduces $-\frac{1}{2\xi}f_\alpha f_\alpha$. Thus, our starting point is

$$\mathcal{L}_{\rm new} = \mathcal{L} + \frac{\xi}{2}h_\alpha h_\alpha + h_\alpha f_\alpha + \omega^*_\alpha \rho_\alpha\ , \tag{6.47}$$

where

$$\rho_\alpha(x) = \partial_\mu\partial^\mu\omega_\alpha(x) + C_{\beta\gamma}{}^\alpha\partial_\mu(A^\mu_\beta(x)\omega_\gamma(x)) = \int {\rm d}^4y\, \mathcal{F}_{\alpha\beta}(x,y)\omega_\beta(y)\ . \tag{6.48}$$

The BRST symmetry is defined to act on the "ordinary" fields, i.e. the matter and gauge fields, just like an ordinary infinitesimal gauge transformation but with the (real) parameter ϵ_α replaced by the (real) ghost field ω_α. It is thus a fermionic symmetry that increases the ghost-number by one unit. To begin with, we simply set $\epsilon_\alpha(x) = \theta\omega_\alpha(x)$ with θ an anticommuting parameter ($\theta\omega = -\omega\theta, \theta A_\mu = A_\mu\theta$, etc, and $\theta\theta = 0$). Then

$$\begin{aligned} \delta_\theta A^\alpha_\mu &= \theta\left(\partial_\mu\omega^\alpha + C_{\beta\gamma}{}^\alpha A^\beta_\mu\omega^\gamma\right) \\ \delta_\theta\psi &= i\,\theta\,\omega^\alpha t^{\mathcal{R}}_\alpha\psi \\ \delta_\theta\psi^\dagger &= -i\theta\,\omega^\alpha\psi^\dagger t^{\mathcal{R}}_\alpha = i\theta\,\psi^\dagger\, t^{\mathcal{R}}_\alpha\,\omega^\alpha\ . \end{aligned} \tag{6.49}$$

This is conveniently rewritten by defining a fermionic operator s such that for any functional F we simply let

$$\delta_\theta F = \theta\, sF \ . \tag{6.50}$$

Consistency with the fermionic character requires

$$s(FG) = (sF)G \pm F\, sG \ , \tag{6.51}$$

with a minus sign if F is fermionic (contains an odd number of anticommuting fields). We write furthermore $\omega^{\mathcal{R}} = \omega^\alpha t_\alpha^{\mathcal{R}}$, so that we can reformulate (6.49) as $s\, A_\mu^{\mathcal{R}} = D_\mu \omega^{\mathcal{R}}$ and $s\,\psi = i\omega^{\mathcal{R}}\psi$ or, dropping the superscript $\mathcal{R}$, simply

$$\boxed{\begin{array}{c} s\, A_\mu = \partial_\mu \omega - i[A_\mu, \omega] \\ s\,\psi = i\,\omega\,\psi \quad , \quad s\,\psi^\dagger = i\psi^\dagger\omega \ . \end{array}} \tag{6.52}$$

Note that $s\psi^\dagger$ is not the hermitian conjugate of $s\psi$, but rather $(s\psi)^\dagger = (i\omega\psi)^\dagger = -i\psi^\dagger\omega = -s\psi^\dagger$. More generally, one sees that $(sF)^\dagger = \mp sF^\dagger$ with a minus sign if F is fermionic. Let us complete (6.52) by the rules how s acts on the ghosts, chosen in such a way that s is a nilpotent operation, i.e. $s^2 = 0$ on all fields. Choosing

$$\boxed{s\,\omega = i\omega\,\omega \quad , \quad s\,\omega^* = -h \quad , \quad s\,h = 0 \ ,} \tag{6.53}$$

where h is the auxiliary field introduced above, we have

$$\begin{aligned} s^2 A_\mu &= s\big(\partial_\mu\omega - i[A_\mu,\omega]\big) = \partial_\mu(s\omega) - i(sA_\mu)\omega - iA_\mu s\omega + i(s\omega)A_\mu - i\omega s A_\mu \\ &= i\partial_\mu(\omega\omega) - i(\partial_\mu\omega - iA_\mu\omega + i\omega A_\mu)\omega - iA_\mu i\omega\omega - \omega\omega A_\mu - i\omega(\partial_\mu\omega - i\omega A_\mu\omega + i\omega A_\mu) = 0 \ , \\ s^2\psi &= s(i\omega\psi) = i(s\omega)\psi - i\omega\, s\psi = -\omega\omega\psi + \omega\omega\psi = 0 \ , \\ s^2\omega &= s(i\omega\omega) = i(s\omega)\omega - i\omega(s\omega) = -\omega\omega\omega + \omega\omega\omega = 0 \ , \\ s^2\omega^* &= s(-h) = 0 \ , \end{aligned} \tag{6.54}$$

which shows that the BRST operator s is nilpotent:

$$\boxed{s^2 = 0 \ .} \tag{6.55}$$

It is now easy to show that $\mathcal{L}_{\rm new}$ is BRST invariant. First note that on functionals of A_μ and the matter fields ψ only the BRST transformation is simply a gauge transformation with ϵ replaced by ω. Thus any gauge invariant functional of the gauge and matter fields only automatically also is BRST invariant. Hence

$$s\mathcal{L} = 0 \ . \tag{6.56}$$

Next note that $\mathcal{F}\epsilon$ was defined as the gauge variation of f (which only depends on A_μ) and hence

$$\rho_\alpha = s f_\alpha \quad \Rightarrow \quad s\rho = 0 \ . \tag{6.57}$$

It follows that

$$s\mathcal{L}_{\rm new} = s\big(\frac{\xi}{2} h_\alpha h_\alpha + h_\alpha f_\alpha + \omega^*_\alpha \rho_\alpha\big) = h_\alpha s f_\alpha + (s\omega^*_\alpha)\rho_\alpha = h_\alpha\rho_\alpha - h_\alpha\rho_\alpha = 0 \ . \tag{6.58}$$

Actually one has

$$\mathcal{L}_{\rm new} = \mathcal{L} + s\Psi \quad , \quad \Psi = -\omega^*_\alpha f_\alpha - \frac{\xi}{2}\omega^*_\alpha h_\alpha \ , \tag{6.59}$$

with $\mathcal{L}$ being the gauge invariant Lagrangian of the gauge and matter fields only. In this form, BRST invariance is obvious. This also suggests how to obtain more general gauge-fixings: Any $\mathcal{L}_{\rm new}$ of this form with an arbitrary functional Ψ of ghost-number -1 will provide a BRST invariant starting point for quantizing the gauge theory (provided the quadratic term in the gauge fields is non-degenerate so that one can define a propagator).

Below, we will use this BRST invariance to show that all conterterms also must respect this BRST invariance (but with a renormalized coupling constant which we had hidden in the normalization of the Lie algebra generators). Hence

$$\mathcal{L}_{\rm new} + \mathcal{L}_{\rm ct} = \widetilde{\mathcal{L}} + s\widetilde{\Psi} \ , \tag{6.60}$$

respects the same symmetries as the original $\mathcal{L}_{\rm new}$ with a renormalized Yang-Mills coupling constant and yields finite Green's functions.

6.3 BRST cohomology

We already noted that the BRST symmetry is nilpotent, i.e. $s^2 = 0$. Obviously also, it increases the ghost-number by one unit.

At an algebraic level, this is quite similar to the behavior of the exterior derivative $\mathrm{d} = \mathrm{d}x^\mu \partial_\mu$ acting on differential forms[30] of degree p and yielding a differential form of degree $p+1$. As is well known, one has $\mathrm{d}^2 \equiv \mathrm{d}\mathrm{d} = 0$ (which just states that $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$). A p-form $\xi^{(p)}$ is called closed if $\mathrm{d}\xi^{(p)} = 0$. If there exists a (globally well-defined) $(p-1)$-form $\zeta^{(p-1)}$ such that $\xi^{(p)} = \zeta^{(p-1)}$ then $\xi^{(p)}$ is called exact. Since $\mathrm{d}^2 = 0$, obviously every exact form is also closed. It is then interesting to find out which p-forms are closed without being exact. As an example consider a space-time with space being just the two-dimensional sphere with coordinates θ and ϕ defined in the usual way. Then $\mathrm{d}\phi$ is well-defined everywhere except at $\theta = 0$ or $\theta = \pi$ (north and south pole), and the 2-form $\Omega = \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi$ is well-defined everywhere. One has $\mathrm{d}\Omega = 0$, so it is closed. Although one has $\Omega = \mathrm{d}(-\cos\theta\, \mathrm{d}\phi)$ or $\Omega = \mathrm{d}(-\sin\theta\, \mathrm{d}\theta\, \phi)$, neither $\cos\theta\, \mathrm{d}\phi$ nor $\sin\theta\, \mathrm{d}\theta\, \phi$ are well-defined everywhere on the sphere and one finds that $\Omega \neq \mathrm{d}\zeta^{(1)}$. Hence Ω is closed but not exact. Up to multiplication by a constant[31] this is the only closed and non-exact 2-form on the sphere. Actually, Ω

[30] In exterior calculus, one defines an antisymmetric "wedge product" so that $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$. For every rank-$p$ antisymmetric tensor field $\xi_{\mu_1\mu_2\ldots\mu_p}(x)$ one can then define a p-form as $\xi^{(p)} = \frac{1}{p!}\xi_{\mu_1\mu_2\ldots\mu_p}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \mathrm{d}x^{\mu_p}$. In d dimensions, the maximal degree of a form is $p = d$.

[31] For example, $\cos\theta\, \Omega = \cos\theta\, \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi = \mathrm{d}\alpha$ with $a = \frac{1}{2}\sin^2\theta\, \mathrm{d}\phi$ a well-defined one-form. Hence $\cos\theta\, \Omega$ is exact.

is the volume-form on the sphere, and the volume form on any compact manifold is always closed but not exact. For a given (compact) manifold $\mathcal{M}$, the vector space of closed p-forms that are not exact is called the $p^{\rm th}$ de Rham cohomology and is denoted $H^{(p)}(\mathcal{M})$.

Similarly, one defines the classes of BRST-closed functionals and BRST-exact functionals as follows:

$$\boxed{\begin{array}{c}\text{The functional } F[A_\mu,\psi,\omega,\omega^*,h] \text{ is BRST closed if } s\,F=0,\\[2mm] \text{it is BRST exact if there exists a functional } G \text{ such that } F=s\,G.\end{array}} \tag{6.61}$$

Of course, a BRST-closed functional is a BRST invariant functional. Moreover, the ghost-number of a given monomial of the fields is defined as the number of ghost-fields minus the number of anti-ghost fields. The Lagrangian $\mathcal{L}_{\rm new}$ e.g. has ghost-number zero. The BRST operator s always increases the ghost-number by one unit. Since one cannot have cancellations between terms of different ghost-numbers it follows that one can define the space of BRST-closed functionals of a fixed ghost-number and, similarly the space of BRST-exact functionals of a fixed ghost-number. Hence, the ghost-number plays a role analogous to the degree p of the differential forms. In particular, one defines the BRST-cohomology classes at ghost-number n :

$$\boxed{\begin{array}{c}\text{The BRST-cohomology at ghost-number } n \text{ is given by}\\[2mm] \text{the BRST-closed functionals of ghost-number } n\text{, modulo the BRST-exact functionals.}\end{array}} \tag{6.62}$$

We have seen above that the gauge invariant Lagrangian $\mathcal{L}[A,\psi]$ is BRST invariant, i.e. $s\mathcal{L}[A,\psi]=0$, so it is BRST closed. Moreover, it is of ghost-number zero. The gauge-fixing and ghost terms are of the form $s\Psi$ with Ψ of ghost-number -1, i.e. are BRST-exact terms. Since correlation functions of gauge invariant operators do not depend on Ψ (as long as it provides some gauge-fixing), the physics determined by $\mathcal{L}_{\rm new}=\mathcal{L}[A,\psi]+s\Psi$ depends only on the BRST-closed $\mathcal{L}$ and is independent of the BRST-exact terms $s\Psi$: it only depends on the BRST cohomology class of $\mathcal{L}_{\rm new}$. Later-on we will show that the BRST-cohomology at ghost-number zero is precisely given by the gauge invariant functionals of A_μ and ψ (and $\psi^\dagger$) only, i.e. independent of the ghost, anti-ghost and h-fields.

BRST-charge :

One can introduce a fermionic BRST charge operator (acting on a "Hilbert space") by demanding that for any (Heisenberg picture) field operator Φ one has

$$\delta_\theta\Phi(x)=i[\theta Q,\Phi(x)]\ . \tag{6.63}$$

The right-hand-side equals $i\theta[Q,\Phi(x)]_\mp$, i.e. a commutator if Φ is bosonic and an anti-commutator if Φ is fermionic, while the left-hand-side equals $\theta s\Phi(x)$. Hence the BRST-charge operator must have the following (anti)commutation relations with the field operators:

$$s\Phi(x)=i[Q,\Phi(x)]_\mp\ . \tag{6.64}$$

By taking the hermitian conjugate of either (6.63) (θ is real) or (6.64) and comparing with the corresponding relations for $\Phi^\dagger$, one sees that $Q^\dagger = -Q$. This then implies $(\theta Q)^\dagger = Q^\dagger \theta = -Q\theta = \theta Q$ as expected for a symmetry generator. Using the Jacobi identity, one finds that

$$0 = -s\,s\,\Phi = -is[Q,\Phi]_\mp = [Q,[Q,\Phi]_\mp]_\pm = \frac{1}{2}[[Q,Q]_+,\Phi]_- = [Q^2,\Phi]_- \quad \forall\ \Phi\ . \tag{6.65}$$

An operator that commutes with all fields is either the identity or has to vanish. Since Q increases the ghost-number by one, the first option is excluded and one concludes

$$\boxed{Q^2 = 0\ .} \tag{6.66}$$

The "Hilbert space" on which Q acts is some "big" space of states which must include the Hilbert space of physical states, but also states including an arbitrary number of ghost and anti-ghost excitations, as well as non-physical polarization states of the gauge bosons. This is the space which naturally arises upon the Faddeev-Popov quantization of the gauge theory. In particular, this is not a Hilbert space in the strict mathematical sense since the inner product cannot be positive definite: for any state $|\gamma\rangle \neq 0$ the state $Q\,|\gamma\rangle$ has zero norm. One must then characterize the Hilbert space of physical states. This can be conveniently done using this BRST charge as follows. Matrix elements between physical states must be gauge invariant, i.e. independent of the "gauge-fixing functional" Ψ. Hence, under an infinitesimal variation $\Psi(x) \to \Psi(x) + \delta\Psi(x)$, the gauge-fixed Lagrangian changes by $\delta\mathcal{L}_{\rm new} = s\delta\Psi(x)$ and any matrix element changes by

$$\delta\langle\alpha\,|\beta\rangle = i\,\langle\alpha|\,s\,\delta\Psi\,|\beta\rangle = -\,\langle\alpha|\,[Q,\delta\Psi]_+\,|\beta\rangle = -\,\langle\alpha|\,Q\,\delta\Psi\,|\beta\rangle - \langle\alpha|\,\delta\Psi\,Q\,|\beta\rangle\ . \tag{6.67}$$

If $\langle\alpha|$ and $|\beta\rangle$ are physical states this should vanish. Since $\delta\Psi(x)$ is arbitrary one concludes $Q\,|\beta\rangle = 0$ and $\langle\alpha|\,Q = 0$. Hence, physical states $|{\rm phys}\rangle$ must obey

$$Q\,|{\rm phys}\rangle = 0\ . \tag{6.68}$$

Also, changing $|\beta\rangle \to |\beta\rangle + Q\,|\gamma\rangle$ does not change $\langle\alpha\,|\beta\rangle$ if $\langle\alpha|$ is physical since $\langle\alpha|\,Q\,|\gamma\rangle = 0$. Moreover, a physical state should have ghost-number zero. In conclusion, physical states are determined by $Q\,|{\rm phys}\rangle = 0$ subject to the equivalence relation $|{\rm phys}\rangle \simeq |{\rm phys}\rangle + Q\,|\gamma\rangle$ and the condition of having ghost-number zero:

Physical states are given by the ghost-number zero cohomology class of the BRST-operator Q.

(6.69)

Of course, we have only shown that this cohomology class contains the physical Hilbert space, i.e. that the above conditions are necessary. However, one can also show that they are sufficient and thus (6.69) exactly defines the physical Hilbert space. Note that in any covariant quantization, the modes of the fields A_0^a, when acting on the vacuum, generate states of negative norm (since $\eta_{00} = -1$). One can then also show that the space of physical state as defined by (6.69) has a positive definite norm.

Remarks :

- The Faddeev-Popov procedure always leads to a Lagrangian $\mathcal{L}_{\rm new}$ that is bilinear in the ghost and anti-ghost fields. (Recall that the ghosts appeared from expressing the Faddeev-Popov determinant $\mathrm{Det}\,\mathcal{F}$ as an integral over ω and ω^* of $e^{i\int \omega^* \mathcal{F}\omega}$.) Then, calling these fields the bare fields and rewriting $\mathcal{L}_{\rm new}$ in terms of renormalized fields will generate various counterterms of the same form as the original terms contained in $\mathcal{L}_{\rm new}$. In particular, one will only get counterterms that are at most bilinear in the ghost and anti-ghost fields. This turns out to be sufficient with the gauge choice $f_\alpha = \partial_\mu A^\mu_\alpha$, but for more general choices of f_α one might need counterterms that involve two ghost and two anti-ghost fields. Although one does not have to worry about diagrams with 4 external (anti)ghost lines, such diagrams can well appear as divergent subgraphs, requiring a corresponding counterterm.

- One can consider more general gauge-fixing functionals that do not rely on the Faddeev-Popov procedure. As we have seen, all that is needed for BRST invariance is to define some BRST-operation s such that s acts on A_μ and ψ as a gauge transformation with the parameter replaced by a ghost field, and with the action on the ghost and other fields (anti-ghost, h-field, and possibly others) defined such that $s^2 = 0$. Then for any $\widetilde{\Psi}$ of ghost-number -1 and gauge invariant $\mathcal{L}$, a BRST-ivariant Lagrangian is $\widetilde{\mathcal{L}}_{\rm new} = \mathcal{L} + s\widetilde{\Psi}$. Then just as before, there is a corresponding BRST charge Q that defines the physical states as the zero ghost-number cohomology class and matrix elements between physical states (in particular also vacuum expectation values of time-ordered products of gauge invariant operators) do not depend on the choice of $\widetilde{\Psi}$. In particular, they are the same as with the Ψ from the Faddeed-Popov procedure and thus the same as in axial gauge.

- One can prove independently of comparison with any particular gauge, that the space of physical states, defined as the zero ghost-number cohomology of Q, has a positive definite norm, contains no ghosts or anti-ghosts and has a unitary S-matrix.

- Finally let us just mention that this BRST formalism can be rather straightforwardly extended to other local symmetries as appear e.g. in general relativity or in string theory. If the natural formulation of these symmetries is "too large" in the sense that one has introduced too many "gauge" parameters and actually not all symmetries are independent, one has to introduce "ghosts of ghosts". In all these setting, the BRST operator always increases the ghost-number by one unit.

We have already mentioned that the BRST invariant functionals of ghost-number zero are just the gauge invariant functionals of A_μ and the matter fields, up to adding BRST-exact terms. Let us now prove this result.

Cohomology theorem :
The zero ghost-number cohomology consists of gauge invariant functionals of A_μ and ψ (and $\psi^\dagger$) only, i.e. the most general BRST invariant functional of ghost-number zero is of the form $I = I_0[A_\mu, \psi] + s\Psi[A_\mu, \psi, \omega, \omega^*, h]$.

The *proof* is relatively simple: suppose $sI = 0$. Write $I = \sum_{N=0}^{\infty} I_N$ where I_N contains all terms that have a total number of fields ω^* and h equal to N. (Of course, we do not allow negative powers of the fields.) Since $s\,\omega^* = -h$ and $s\,h = 0$, s does not change this total number N and one cannot have any cancellations between the different sI_N. Hence $sI_N = 0$ for all N separately. Introduce $t = \omega^*_\alpha \frac{\delta}{\delta h_\alpha}$. One may similarly write $s = -h_\beta \frac{\delta}{\delta \omega^*_\beta} + \ldots$ where the unwritten terms do not involve ω^* or h. It follows that $st + ts = -\widehat{N}$ where $\widehat{N} = \omega^*_\alpha \frac{\delta}{\omega^*_\alpha} + h_\alpha \frac{\delta}{\delta h_\alpha}$ is such that $\widehat{N} I_N = N I_N$. Thus

$$-N I_N = -\widehat{N} I_N = (s\,t + t\,s) I_N = s\,t\,I_N \ , \tag{6.70}$$

and we conclude that for every $N \neq 0$ one has $I_N = s\left(-\frac{1}{N}\,t\,I_N\right)$, i.e. I_N is BRST-exact. Hence, $I = I_0 + s\,\Psi$ with $\Psi = \sum_{N=1}^{\infty}\left(-\frac{1}{N}\,t\,I_N\right)$. Now, I_0 cotains no h and no ω^* and, having ghost-number zero, it cannot contain any ω either. Thus $I_0 = I_0[A_\mu, \psi]$, as was to be proven. Finally, I_0 cannot be BRST exact. Indeed, it is easy to see that a BRST exact functional of ghost number one must contain at least either an h_α or an ω^*_α.

7 Renormalization of non-abelian gauge theories

7.1 Slavnov-Taylor identities and Zinn-Justin equation

7.1.1 Slavnov-Taylor identities

Recall that the vacuum expectation values of time-ordered products of field operators can be obtained from the generating functional

$$Z[J] = \int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}\omega \mathcal{D}\omega^* \mathcal{D}h \exp\left[i \int \mathrm{d}^4 x \left(\mathcal{L}_{\rm new} + \widetilde{\chi}^n J_n \right) \right] , \tag{7.1}$$

where, to simplify the notation, $\widetilde{\chi}^n$ stands collectively for any of the fields A_μ, ψ, $\overline{\psi}$, ω, ω^* or h. Of course, we do not really need to compute ghost correlation functions to get S-matrix elements between physical states, but they certainly can and do appear in subdiagrams. Also, the form of $\mathcal{L}_{\rm new}$ originally was derived only for time-ordered products of gauge invariant operators, but we can take (7.1) as a definition for the gauge-dependent vacuum expectation values of time-ordered products of the gauge non-invariant field operators. We have seen in the last subsection that the BRST invariance implies that in the end matrix elements between gauge invariant states do not depend on the gauge-fixing functional. We will now derive the implications of BRST invariance of the action (and measure) for the generating functional $Z[J]$ as defined in (7.1). These are the Slavnov-Taylor identities.

As usual, we define $Z[J] = e^{iW[J]}$, as well as $\chi^n = \frac{\delta_R W[J]}{\delta J_n} = \langle \widetilde{\chi}^n_{\rm operator} \rangle$ with $J_{n,\chi}$ being the solution of this equation for given χ^n. Then the Legendre transform is $\Gamma[\chi] = W[J_\chi] - \int \mathrm{d}^4 x\, \chi^n J_{n,\chi}$. Note that, since χ^n can be either bosonic or fermionic (i.e. anticommuting), one must specify whether functional derivatives should act from the left (L) or from the right (R). Indeed, for fermionic χ^n one has $\frac{\delta_L \Gamma}{\delta \chi^n} = -\frac{\delta_R \Gamma}{\delta \chi^n}$.

As shown in general in section 1.4.3, the invariance of $\int \mathcal{L}_{\rm new}$ *and* of the functional integral measure under

$$\delta_\theta \chi^n = \theta\, \Delta^n , \qquad \Delta^n = s\, \chi^n , \tag{7.2}$$

(dropping the tilde on the χ^n) implies the Slavnov-Taylor identity

$$\int \mathrm{d}^4 x\, \langle \Delta^n(x) \rangle_{J_\chi}\, \frac{\delta_L \Gamma}{\delta \chi^n(x)} = 0 . \tag{7.3}$$

Of course, invariance of the functional integral measure under BRST transformations is not guaranteed and needs to be verified. This will be discussed at the end of this subsection. The conclusion will be that the measure is indeed invariant under BRST transformations provided the measure for the matter fields is gauge invariant, i.e. there are no gauge anomalies. For now, we assume that this is the case and the Slavnov-Taylor identity (7.3) does hold. As discussed in sect. 1.4.3, for a *linear* symmetry ($\Delta^n = c^n{}_m \chi^m$) one would have $\langle \Delta^n(x) \rangle_{J_\chi} = c^n{}_m \langle \chi^m(x) \rangle_{J_\chi} = c^n{}_m \chi^m$, and then the Slavnov-Taylor identity just states that Γ is invariant under this symmetry. At present, however, the *BRST symmetry is non-linear*, e.g. $s\psi = i\omega\omega$, etc, and the Slavnov-Taylor identity (7.3) does *not* tell us that the effective action Γ is also BRST invariant.

7.1.2 Zinn-Justin equation

In order to nevertheless exploit the content of the Slavnov-Taylor identity, one uses the trick to also introduce sources for the "composite" fields Δ^n. (This is somewhat similar to what we did when discussing the renormalization of composite operators and the Callan-Symanzik equations for vertex functions $\Gamma^{(n)}$ with l additional insertions of some composite operator like e.g. ϕ^2.) To begin with, one defines

$$Z[J,K] \equiv e^{iW[J,K]} = \int \mathcal{D}\chi^n \exp\left[i \int \mathrm{d}^4x \, (\mathcal{L}_{\rm new} + \chi^n J_n + \Delta^n K_n) \right] , \tag{7.4}$$

The K_n are like additional (position-dependent) coupling constants. In particular,

$$\frac{\delta_R W[J,K]}{\delta K_n(x)} = \langle \Delta^n(x) \rangle_{J,K} \ . \tag{7.5}$$

The effective action $\Gamma[\chi, K]$ is obtained by performing the Legendre transformation with respect to the sources J_n only, while keeping these extra couplings K_n:

$$\Gamma[\chi, K] = W[J_{\chi,K}, K] - \int \mathrm{d}^4x \, \chi^n(x) J^n_{\chi,K}(x) \ , \tag{7.6}$$

where $J^n_{\chi,K}$ is the solution of

$$\frac{\delta_R W[J,K]}{\delta J_n} = \chi^n \tag{7.7}$$

for given χ^n. (For obvious notational reasons, we write $J^n_{\chi,K}$ rather than $(J_n)_{\chi,K}$.) Note that these definitions imply on the one hand, that χ^n and J_n have the same statistics (both bosonic or both fermionic) and Δ^n and K_n also have the same statistics. On the other hand, any field from the first group (χ^n or J_n) has opposite statistics from the corresponding field of the second group (Δ^n or K_n). Through the usual manipulations one finds from (7.6)

$$\frac{\delta_R \Gamma[\chi,K]}{\delta K_n(x)} = \frac{\delta_R W[J,K]}{\delta K_n(x)}\Big|_{J=J_{\chi,K}} + \int \mathrm{d}^4y \frac{\delta_R W[J,K]}{\delta J_m(y)}\Big|_{J=J_{\chi,K}} \frac{\delta_R J^m_{\chi,K}(y)}{\delta K_n(x)} - \int \mathrm{d}^4y \, \chi^m(y) \frac{\delta_R J^m_{\chi,K}(y)}{\delta K_n(x)} \ . \tag{7.8}$$

The last two terms cancel by (7.7) while, by (7.5), the first term is just $\langle \Delta^n(x) \rangle_{J_{\chi,K},K}$. Hence

$$\frac{\delta_R \Gamma[\chi,K]}{\delta K_n(x)} = \langle \Delta^n(x) \rangle_{J_{\chi,K},K} \ . \tag{7.9}$$

It is completely straightforward to generalize the Slavnov-Taylor identity to the case where the additional couplings $\sim \Delta^n K_n$ are present:

$$\int \mathrm{d}^4x \, \langle \Delta^n(x) \rangle_{J_{\chi,K},K} \, \frac{\delta_L \Gamma[\chi,K]}{\delta \chi^n(x)} = 0 \ . \tag{7.10}$$

Using (7.9), we get the Zinn-Justin equation:

$$\boxed{\int \mathrm{d}^4x \, \frac{\delta_R \Gamma[\chi,K]}{\delta K_n(x)} \, \frac{\delta_L \Gamma[\chi,K]}{\delta \chi^n(x)} = 0 \ .} \tag{7.11}$$

7.1.3 Antibracket

The Zinn-Justin equation displays a nice symmetry between the roles of K_n and χ^n. This is further emphasized by introducing the notion of antibracket. The antibracket of two (bosonic) functionals[32] $F[\chi, K]$ and $G[\chi, K]$ depending on two sets of arguments χ^n and K_n having *opposite* statistics is defined as

$$(F,G) = \int \mathrm{d}^4x \left(\frac{\delta_R F[\chi,K]}{\delta\chi^n(x)} \frac{\delta_L G[\chi,K]}{\delta K_n(x)} - \frac{\delta_R F[\chi,K]}{\delta K_n(x)} \frac{\delta_L G[\chi,K]}{\delta\chi^n(x)} \right) . \tag{7.12}$$

This is somewhat similar to the definition of the Poisson bracket but, as we will see, it is symmetric rather than antisymmetric under the exchange of F and G. Since the χ^n always have the opposite statistics from the K_n, one of the two functional derivatives in the definition of the antibracket always yields an anticommuting expression while the other is commuting. Since exchanging a left with a right derivative yields a minus sign for an anticommuting expression, we can flip left and right derivatives in either of the two terms in (7.12) provided we include one extra minus sign. Thus

$$\begin{aligned}
(F,G) &= \int \mathrm{d}^4x \left(\frac{\delta_R F[\chi,K]}{\delta\chi^n(x)} \frac{\delta_L G[\chi,K]}{\delta K_n(x)} + \frac{\delta_L F[\chi,K]}{\delta K_n(x)} \frac{\delta_R G[\chi,K]}{\delta\chi^n(x)} \right) \\
&= \int \mathrm{d}^4x \left(\frac{\delta_R F[\chi,K]}{\delta\chi^n(x)} \frac{\delta_L G[\chi,K]}{\delta K_n(x)} + \frac{\delta_R G[\chi,K]}{\delta\chi^n(x)} \frac{\delta_L F[\chi,K]}{\delta K_n(x)} \right) \\
&= -\int \mathrm{d}^4x \left(\frac{\delta_R G[\chi,K]}{\delta K_n(x)} \frac{\delta_L F[\chi,K]}{\delta\chi^n(x)} + \frac{\delta_R F[\chi,K]}{\delta K_n(x)} \frac{\delta_L G[\chi,K]}{\delta\chi^n(x)} \right) .
\end{aligned} \tag{7.13}$$

This shows that the antibracket is *symmetric* under the interchange of F and G:

$$(F,G) = (G,F) . \tag{7.14}$$

As is obvious from the last line in (7.13), the Zinn-Justin equation (7.11) can then be written as

$$\boxed{(\Gamma,\Gamma) = 0 .} \tag{7.15}$$

7.1.4 Invariance of the measure under the BRST transformation

Let us now come back to the question of whether the functional integral measure is invariant under the BRST transformation or not. Clearly, even if the measure is invariant under gauge transformations, this does not immediately imply invariance under BRST transformations. The reason is that the latter are non-linear transformations of the fields $A_\mu, \psi, \overline{\psi}, \omega, \omega^*, h$ while the former are linear transformations of A_μ, ψ and $\overline{\psi}$ only. A further technical complication appears since the BRST transformation mixes commuting and anticommuting field variables. Nevertheless, we will now show that the relevant Jacobian equals unity provided the measure for the matter fields alone is gauge invariant.

[32]A bosonic functional is a sum of terms each of which contains an *even* number of anticommuting fields. Hence it is commuting. Similarly, a fermionic functional is made from terms containing an *odd* number of anticommuting fields and hence is anticommuting.

We want to compute the Jacobian for a change of integration variables of the form $\chi^n \to {\chi'}^n = \chi^n + \theta\Delta^n \equiv \chi^n + \theta s\chi^n$ where χ^n stands for the various commuting and anticommuting fields and θ is a single anticommuting parameter satisfying $\theta^2 = 0$. Although the Δ^n depend non-linearly on the χ^r, the fact that $\theta^2 = 0$ will simplify the Jacobian enormously. We let

$$J^n_{\ r}(x,y) = \frac{\delta_L {\chi'}^n(x)}{\delta\chi^r(y)} = \delta^n_r\, \delta^{(4)}(x-y) + \frac{\delta_L\big(\theta\Delta^n(x)\big)}{\delta\chi^r(y)} \ \equiv\ \delta^n_r\, \delta^{(4)}(x-y) + R^n_{\ r}(x,y)\ . \tag{7.16}$$

Note that R is linear in θ, i.e. $R \equiv \theta\widehat{R}$, and hence $R^2 = 0$, so that $\log(1+R) = R$. We can then use the standard relation between the determinant and the trace of the logarithm to get

$$\mathrm{Det}\, J = \exp\big(\,\mathrm{Tr}\, \log(\mathbf{1}+R)\big) = \exp\big(\,\mathrm{Tr}\, R\big) = 1 + \,\mathrm{Tr}\, R\ , \tag{7.17}$$

where Tr includes a sum over the different field types of functional traces for every field. Now, for anticommuting χ^r (i.e. the ghost, antighost or matter fermions), one has $\frac{\delta_L\theta\Delta^n}{\delta\chi^r} = -\theta\frac{\delta_L\Delta^n}{\delta\chi^r}$. It follows that one has

$$\mathrm{Tr}\, R = \theta\ \mathrm{Str}\, \widehat{R}\ , \tag{7.18}$$

where Str is a functional "supertrace" which is just an ordinary functional trace but with minus signs inserted for the anticommuting fields. With obvious notations we have

$$\mathrm{Str}\widehat{R} = \,\mathrm{Tr}\, \widehat{R}_A - \,\mathrm{Tr}\, \widehat{R}_\psi - \,\mathrm{Tr}\, \widehat{R}_{\overline{\psi}} + \,\mathrm{Tr}\, \widehat{R}_h - \,\mathrm{Tr}\, \widehat{R}_\omega - \,\mathrm{Tr}\, \widehat{R}_{\omega^*}\ . \tag{7.19}$$

Recall that $A'_\mu = A_\mu + \theta s A_\mu = A_\mu + \theta(\partial_\mu\omega - i[A_\mu,\omega])$ so that $\Delta^{A_\mu} = (\partial_\mu\omega - i[A_\mu,\omega])$. Similarly, $\Delta^\omega = i\omega\omega$ and $\Delta^{\omega^*} = -h$, as well as $\Delta^\psi = i\omega\psi$ and $\Delta^h = 0$. Obviously then, $\mathrm{Tr}\, \widehat{R}_{\omega^*} = \,\mathrm{Tr}\, \widehat{R}_h = 0$. Also $(\widehat{R}_\omega)_{\alpha\beta}(x,y) = C_{\alpha\beta\gamma}\omega^\gamma(x)\delta^{(4)}(x-y)$ so that its trace vanishes (after an appropriate gauge invariant regularization). Similarly,

$$(\widehat{R}_A)^\mu_{\alpha\beta\nu}(x,y) \ =\ \frac{\delta\Delta^\mu_\alpha(x)}{\delta A^\nu_\beta(y)} \ =\ \delta^\mu_\nu\, C_{\alpha\beta\gamma}\,\omega^\gamma\,\delta^{(4)}(x-y)\ . \tag{7.20}$$

This is again antisymmetric and its trace vanishes (after an appropriate gauge invariant regularization). Actually, this is the same operator as the one we encountered when we discussed the invariance of the gauge field measure under gauge transformations (except that now $\epsilon \to \omega$). Obviously, $\mathrm{Tr}\, \widehat{R}_A = 0$ is equivalent to the statement that the gauge field measure is gauge invariant. Finally, one has to study $\mathrm{Tr}\, R_\psi$ and $\mathrm{Tr}\, R_{\overline{\psi}}$. Once more, the vanishing of the trace is equivalent to the gauge invariance of the matter measure. Although formally this trace vanishes, just as for the other fields, here we may encounter a problem if it is not possible to regularize the trace in a gauge invariant way. This is precisely the case for chiral fermions and one has a so-called anomaly as we will extensively discuss in the next section.

We conclude that in the absence of anomalies (no chiral matter fermions), the full functional integral measure is indeed invariant under BRST transformations. Moreover, the BRST invariance of the measure is equivalent to the gauge invariance of the matter measure alone.

7.2 Renormalization of gauge theories theories

7.2.1 The general structure and strategy

As already emphasized, non-abelian gauge theories are renormalizable by power counting. As discussed in sect. 4, it follows from power counting that there only are finitely many Green-functions (or equivalently only finitely many 1PI n-point vertex functions $\Gamma^{(n)}$) that are divergent with their superficial degree of divergence being some fixed finite number (independent of the order of perturbation theory). Relying on the BPHZ theorem, we then know that all divergences can be removed by finitely many local counterterms whose coefficients can be determined order by order in perturbation theory. However, what this does not tell us is whether the divergent parts of the Green functions or of the $\Gamma^{(n)}$ – and hence the corresponding counterterms – share the same symmetries as the original action. If one needed to add non gauge invariant (or actually non BRST invariant) counterterms at some order N of perturbation theory to cancel some non-invariant divergence, these non-invariant counterterms would almost certainly lead to non-invariant contributions at some higher order $N' > N$ of perturbation theory. Thus, when one asks whether a (non-abelian) gauge theory is renormalizable the question is whether all counterterms share the same symmetries as the original (gauge-fixed) action, namely the BRST symmetry.

The proof of the renormalizability of non-abelian gauge theories thus amounts to showing that the divergent parts Γ_∞ of the quantum effective action Γ (which generates the 1PI vertex functions $\Gamma^{(n)}$) due to the loops still has the BRST symmetry, or some deformation thereof obtained after changing the coupling constant and field normalizations. Instead of actually computing the loop diagrams, we will exploit the algebraic structure coded in the Zinn-Justin equation to show that this is indeed the case.

As usual, we rewrite the original bare action (including now the extra couplings $\sim \int \Delta^n K_n$) in terms of renormalized fields and coupling constants and masses:

$$S_B[\chi_B, K_B] = S[\chi, K] = S_R^{(0)}[\chi, K] + S_{\text{c.t.}}[\chi, K] \ , \tag{7.21}$$

where $S_R^{(0)}$ has the same form as the original (bare) action but with all masses and couplings equal to their renormalized values, and $S_{\text{c.t.}}$ are the counterterms, c.f. the general discussion and examples studied in sections 2 and 3. There are now two ways to rephrase the question of renormalizability of the non-abelian gauge theories. If one insists that S_B is BRST invariant, as well as the "tree-level" renormalized action $S_R^{(0)}$, then the counterterms will also be BRST invariant. The non-trivial question then is whether these counterterms are enough to cancel all the divergent parts that arise in any N-loop diagram. Equivalently, we may just begin with the usual BRST invariant $S_R^{(0)}$ and compute the divergent parts Γ_∞ of Γ and adjust the counterterms $S_{\text{c.t.}}$ order by order in perturbation theory to cancel these divergent parts. The question then is whether these counterterms are BRST invariant, i.e. whether the bare action can be BRST invariant.

Actually, there is is slight subtlety here: in the non-abelian case, the gauge and BRST transformations explicitly depend on the gauge coupling constant (which we have hidden in the generators t_α, respectively in the structure constants). Thus, when one states that some quantity is gauge or

BRST invariant, one has to specify what is the coupling constant. Now the bare action S_B is BRST invariant using the bare coupling constant, while the renormalized action $S_R^{(0)}$ should be BRST invariant using the renormalized coupling constant. The precise formulation of renormalizability then is whether we can find counterterms that cancel the divergent part Γ_∞ and which equal the difference of the g_B-BRST invariant bare action S_B and the g_R-BRST invariant action $S_R^{(0)}$. Equivalently, starting from a g_R-BRST invariant $S_R^{(0)}$, we must show that $S_R^{(0)} - \Gamma_\infty$ is g_B-BRST invariant. In the sequel, we will call a conterterm or Γ_∞ simply "BRST-invariant" if it fulfills this requirement.

In perturbation theory, Γ has an expansion

$$\Gamma[\chi, K] = \sum_{N=0}^{\infty} \Gamma_N[\chi, K] \quad , \quad \Gamma_0 = S_R^{(0)} \quad , \quad \Gamma_N = \Gamma_{N-\text{loop}} + \sum_{M=1}^{N} \Gamma^{\text{c.t.},M}_{(N-M)-\text{loop}} \ , \tag{7.22}$$

where $\Gamma^{\text{c.t.},M}_{(N-M)-\text{loop}}$ is an $(N-M)$-loop contribution involving "vertices" from lower-order counterterms of total order M, resulting in a contribution at the same order in perturbation theory as $\Gamma_{N-\text{loop}}$. (In QED e.g., the order α^2 contributions to the vacuum polarisation correspond to $\Gamma_2^{\mu\nu}$ and are given by the genuine two-loop diagrams ($\Gamma^{\mu\nu}_{2-\text{loop}}$), one-loop diagrams with one-loop (order α) counterterms inserted ($\Gamma^{\mu\nu\ \text{c.t.},1}_{1-\text{loop}}$), as well as a new order-α^2 counterterm ($\Gamma^{\mu\nu\ \text{c.t.},2}_{0-\text{loop}}$).

One can again introduce a loop-counting parameter λ and formally replace $S_R^{(0)}$ by $\frac{1}{\lambda} S_R^{(0)}$. Then an N-loop term will have a coefficient λ^{N-1}, and $\Gamma_{N-\text{loop}}$ will be multiplied by λ^{N-1}. We can similarly assign a factor λ^M to every counterterm that arises from an M-loop diagram, in addition to an overall $\frac{1}{\lambda}$ in front of $S_{\text{c.t.}}$: $\frac{1}{\lambda} S_{\text{c.t.}} = \frac{1}{\lambda} \sum_{M=1}^{\infty} \lambda^M S^M_{\text{c.t.}}$. Then $\Gamma^{\text{c.t.},M}_{(N-M)-\text{loop}}$ will be accompanied by a factor $\lambda^{N-M-1}\lambda^M = \lambda^{N-1}$. As a result, we see that $\Gamma = \sum_{N=0}^{\infty} \lambda^{N-1} \Gamma_N$. Inserting this into the Zinn-Justin equation $(\Gamma, \Gamma) = 0$ and collecting the coefficients of the λ^{N-2}-terms (recall that the antibracket is symmetric) yields

$$\sum_{N'=0}^{N} (\Gamma_{N'}, \Gamma_{N-N'}) = 0 \quad , \qquad N \geq 0 \ . \tag{7.23}$$

We want to show that Γ can be made finite by choosing "BRST-invariant" local counterterms order by order in perturbation theory. We will show this by induction in N. First, for $N = 0$ one has $\Gamma_0 = S_R^{(0)}$. Now, $S_R^{(0)}$ is expressed in terms of the renormalized (finite) parameters (couplings and masses) and the renormalized fields. Obviously, no counterterms are needed at this order. Suppose then that all $\Gamma_{N'}$ with $N' \leq N-1$ are finite, i.e. we assume that all divergences in $\Gamma_{N'-\text{loop}}$ have been cancelled by the contributions $\sum_{M=1}^{N'} \Gamma^{\text{c.t.},M}_{(N'-M)-\text{loop}}$, induced by appropriately chosen ("BRST-invariant") counterterms $S^M_{\text{c.t.}}$ with $M \leq N'$. This shows that in the sum (7.23) the only terms that can involve divergences are the terms $N' = 0$ and $N' = N$. Thus, isolating the possibly divergent parts, eq. (7.23) yields

$$(S_R^{(0)}, \Gamma_{N,\infty}) = 0 \ . \tag{7.24}$$

In agreement with our above remark, we can interpret this constraint in two different ways.

- First, we may consider that Γ_N contains the N-loop contributions $\Gamma_{N-\text{loop}}$, the contributions from lower-order counterterms in loops $\sum_{M=1}^{N-1} \Gamma^{\text{c.t.},M}_{(N-M)-\text{loop}}$ (the corresponding counterterms are

"BRST-invariant" by the induction hypothesis) and an order-N counterterm $\Gamma_{\rm tree}^{{\rm c.t.},N}$. The latter is taken "BRST invariant" and we must then show that it can indeed cancel the divergent part from the other contributions. This will be the case if the latter are "BRST invariant". Since the counterterm is already supposed to be "BRST invariant", it is equivalent to show that the divergent part $\Gamma_{N,\infty}$ of the full Γ_N is "BRST invariant". (Of course, if this is the case, the divergent parts just cancel and $\Gamma_{N,\infty} = 0$.)

- Second, we may consider that we have not yet included an order-N counterterm $\Gamma_{\rm tree}^{{\rm c.t.},N}$ in the computation of Γ_N which now only contains $\Gamma_{N-{\rm loop}}$ and the contributions from lower-order counterterms in loops $\sum_{M=1}^{N-1} \Gamma_{(N-M)-{\rm loop}}^{{\rm c.t.},M}$. Then, to be able to cancel the divergent part $\Gamma_{N,\infty}$ of Γ_N by an appropriate "BRST invariant" counterterm, we must show that $\Gamma_{N,\infty}$ is "BRST invariant".

We see that, whatever interpretation we adopt, it is necessary and sufficient to show that $\Gamma_{N,\infty}$ is "BRST invariant". This will be done by :

- exploiting the content of (7.24) which we derived from the Zinn-Justin equation,
- using the various linear symmetries[33] of the tree-level action $S_R^{(0)}$ which must be also be symmetries of the effective action Γ at every order, i.e. of Γ_N, and in particular also of its diverging part,
- using the fact that $\Gamma_{N,\infty}$ is a local functional of the fields of dimension less or equal to 4 (here we rely of course on BPHZ to exclude any trouble with overlapping divergences).

Let us recall that all computations are done in the presence of the extra couplings $\sim \Delta^n K_n$, i.e. with $S_R^{(0)}[\chi^n, K_n] = \int(\mathcal{L}_{{\rm new},R}[\chi^n] + \Delta^n K_n)$. Here $\Delta^n = s\chi^n$ with the BRST-transformation s being the one involving the same renormalized coupling as appears in $S_R^{(0)}$. Let us also insist, that eq. (7.24) will only tell us something about the diverging part of Γ_N, not its finite part.

7.2.2 Constraining the divergent part of Γ

Let us now constrain the divergent part $\Gamma_{N,\infty}$ of Γ_N using dimensional arguments, the linear symmetries and equation (7.24). We begin by showing that $\Gamma_{N,\infty}$ can depend at most linearly on the various K_n.

Dimensional arguments

The gauge field A_μ has dimension 1, just as scalar matter fields. The fermionic matter fields, ψ and $\overline{\psi}$ have dimension $\frac{3}{2}$. The dimensions of the ghost and antighost fields can be read from their kinetic term (or their propagator). This depends on the choice of gauge-fixing function f_α. If $f_\alpha = \partial_\mu A^\mu + \ldots$, the ghost Lagrangian is $\sim \omega^* \partial_\mu D^\mu \omega + \ldots$ and $\dim \omega^* + \dim \omega = 2$. Since in any ghost-number zero

[33] These are Lorentz transformations, global gauge transformations, ghost phase transformations related to ghost number conservation and possibly antighost translations for certain choices of the gauge-fixing function like e.g. $f_\alpha = \partial_\mu A^\mu_\alpha$.

functional one always has as many ghosts as antighosts it does not matter how we distribute the 2 between the ghost and antighost and we can choose $\dim\omega = \dim\omega^* = 1$. As one sees from $s\,A_\mu = D_\mu\omega$, $s\psi = i\omega\psi$, etc, s changes dimensions by $\dim\omega = 1$. It follows that for any field χ^n one has $\dim\Delta^n = \dim\chi^n + 1$. Finally, $\Delta^n K_n$ must have dimension 4 so that $\dim K_n = 4 - \dim\Delta^n$. To summarize:

$$\begin{aligned} \dim A_\mu \;=\; \dim\omega \;=\; \dim\omega^* \;=\; 1 \quad &, \quad \dim\psi = \dim\overline{\psi} = \frac{3}{2}\,, \\ \dim\Delta^A = \dim\Delta^\omega = \dim\Delta^{\omega^*} = 2 \quad &, \quad \dim\Delta^\psi = \dim\Delta^{\overline{\psi}} = \frac{5}{2}\,, \\ \dim K_A = \dim K_\omega = \dim K_{\omega^*} = 2 \quad &, \quad \dim K_\psi = \dim K_{\overline{\psi}} = \frac{3}{2}\,. \end{aligned} \tag{7.25}$$

Since the divergent part of Γ must be a local functional of all the fields of at most dimension 4, we see that it can be at most quadratic in the K_n.

<u>Ghost number conservation</u>

Since $\mathcal{L}_{\rm new}$ is invariant under global ghost/antighost phase rotations $\omega \to e^{i\alpha}\omega$, $\omega^* \to e^{-i\alpha}\omega^*$ it follows that the ghost number is conserved (ω has ghost number $+1$ and ω^* ghost number -1). This will remain true in the presence of the extra couplings $\sim \Delta^n K_n$ if we assign ghost numbers to the K_n which are the opposite of the ghost numbers of Δ^n. Since $\Delta^n = s\,\chi^n$ and s increases the ghost numbers by one unit, one has $n_{\rm gh}\Delta^n = n_{\rm gh}\chi^n + 1$. Thus

$$\begin{aligned} n_{\rm gh}(A_\mu) = n_{\rm gh}(\psi) = n_{\rm gh}(\overline{\psi}) = 0 \quad &, \quad n_{\rm gh}(\omega) = 1 \;\;,\;\; n_{\rm gh}(\omega^*) = -1\,, \\ n_{\rm gh}(\Delta^A) = n_{\rm gh}(\Delta^\psi) = n_{\rm gh}(\Delta^{\overline{\psi}}) = 1 \quad &, \quad n_{\rm gh}(\Delta^\omega) = 2 \;\;,\;\; n_{\rm gh}(\Delta^{\omega^*}) = 0\,, \\ n_{\rm gh}(K_A) = n_{\rm gh}(K_\psi) = n_{\rm gh}(K_{\overline{\psi}}) = -1 \quad &, \quad n_{\rm gh}(K_\omega) = -2 \;\;,\;\; n_{\rm gh}(K_{\omega^*}) = 0\,,\;. \end{aligned} \tag{7.26}$$

Hence, for all K_n have negative or zero ghost number.

Now, $\Gamma_{N,\infty}$ must have ghost number zero and we have seen above that it can be at most quadratic in the K_n. Let's see which quadratic terms could appear. Terms involving two K_A, K_ω or K_{ω^*} have dimension 4 and hence cannot involve any other field. Such terms have negative ghost numbers and are excluded except for $K_{\omega^*}K_{\omega^*}$. The latter term, however, is excluded since $\Delta^{\omega^*} \equiv s\omega^* = -h$ is a *linear* transformation so that $\frac{\delta_R\Gamma}{\delta K_{\omega^*}} = \langle\Delta^{\omega^*}\rangle = \Delta^{\omega^*} \equiv -h$ tells us that $\Gamma[\chi^n, K_n]$ must be linear in K_{ω^*} and cannot have a term $\sim K_{\omega^*}K_{\omega^*}$. Actually, this argument tells us even a bit more: the K_{ω^*} dependence of $\Gamma[\chi, K]$ must be precisely a term $\int {\rm d}^4x\,(-h)K_{\omega^*}$. This is just the (finite) term already present in the tree-level action $S_R^{(0)}[\chi, K]$ and, in particular, the diverging part Γ_∞ cannot contain any term linear in K_{ω^*}. Similarly, $\Delta^h = sh = 0$ and we conclude that Γ_∞ cannot contain a term linear in K_h either.

Next, since the ghost is a Lorentz scalar, all Δ^n transform under the same representation of the Lorentz group as the fields χ^n and the K_n must transform in such a way that $\Delta^n K_n$ is a Lorentz scalar. It follows that K_ω and K_{ω^*} are Lorentz scalars, K_{A^μ} is a four-vector and K_ψ transforms as the spinor $\overline{\psi}$, while $K_{\overline{\psi}}$ transforms as ψ. Thus a term quadratic in the K_n and involving at least one K_ψ or $K_{\overline{\psi}}$ must necessarily involve both of them, i.e. be $\sim K_\psi K_{\overline{\psi}}$. This expression has ghost number -2 and dimension 3. There is no dimension 1 field of ghost number $+2$ available to make a dimension 4, ghost number 0 term. Thus we conclude that $\Gamma_{N,\infty}$ is at most linear in any of the K_n:

$$\Gamma_{N,\infty}[\chi, K] = \Gamma_{N,\infty}[\chi] + \int \mathrm{d}^4x\, \widetilde{\Delta}^n_N(x) K_n(x) \ , \tag{7.27}$$

with $\Gamma_{N,\infty}[\chi] \equiv \Gamma_{N,\infty}[\chi, 0] = \int \mathrm{d}^4x\, \gamma_{N,\infty}(x)$. Similarly, we had for the (tree-level) action

$$S^{(0)}_R[\chi, K] = S^{(0)}_R[\chi] + \int \mathrm{d}^4x\, \Delta^n(x) K_n(x) \ . \tag{7.28}$$

Just as Δ^n is a local expression in the fields χ^n, the same must be true for the $\widetilde{\Delta}^n_N$. Note also that the $\widetilde{\Delta}^n_N$ must have the same ghost numbers and dimensions as the Δ^n. Finally, we have seen above that $\Gamma_{N,\infty}$ cannot contain any terms linear in K_{ω^*} or K_h and we conclude that

$$\widetilde{\Delta}^{\omega^*}_N = \widetilde{\Delta}^h_N = 0 \ . \tag{7.29}$$

Exploiting $(S^{(0)}_R, \Gamma_{N,\infty}) = 0$

In order to get further information on the form of $\gamma_{N,\infty}(x)$ and $\widetilde{\Delta}^n_N(x)$, we insert (7.27) and (7.28) into $(S^{(0)}_R[\chi, K], \Gamma_{N,\infty}[\chi, K]) = 0$:

$$\begin{aligned} 0 \ &= \ \int \mathrm{d}^4x \left(\frac{\delta_R S^{(0)}_R[\chi,K]}{\delta K_n(x)} \frac{\delta_L \Gamma_{N,\infty}[\chi,K]}{\delta \chi^n(x)} + \frac{\delta_R \Gamma_{N,\infty}[\chi,K]}{\delta K_n(x)} \frac{\delta_L S^{(0)}_R[\chi,K]}{\delta \chi^n(x)} \right) \\ &= \ \int \mathrm{d}^4x \left\{ \Delta^n(x) \left(\frac{\delta_L \Gamma_{N,\infty}[\chi]}{\delta \chi^n(x)} + \int \mathrm{d}^4y\, \frac{\delta_L \widetilde{\Delta}^m_N(y)}{\delta \chi^n(x)} K_m(y) \right) \right. \\ &\qquad \left. + \widetilde{\Delta}^n_N(x) \left(\frac{\delta_L S^{(0)}_R[\chi]}{\delta \chi^n(x)} + \int \mathrm{d}^4y\, \frac{\delta_L \Delta^m(y)}{\delta \chi^n(x)} K_m(y) \right) \right\} \ . \end{aligned} \tag{7.30}$$

The terms without K_m and the coefficients of $K_m(y)$ must vanish separately, giving two equations:

$$\int \mathrm{d}^4x \left(\Delta^n(x) \frac{\delta_L \Gamma_{N,\infty}[\chi]}{\delta \chi^n(x)} + \widetilde{\Delta}^n_N(x) \frac{\delta_L S^{(0)}_R[\chi]}{\delta \chi^n(x)} \right) = 0 \ , \tag{7.31}$$

and

$$\int \mathrm{d}^4x \left(\Delta^n(x) \frac{\delta_L \widetilde{\Delta}^m_N(y)}{\delta \chi^n(x)} + \widetilde{\Delta}^n_N(x) \frac{\delta_L \Delta^m(y)}{\delta \chi^n(x)} \right) = 0 \ . \tag{7.32}$$

The second equation (7.32) is a set of functional first-order partial linear differential equations for the $\widetilde{\Delta}^n_N$. Note that these equations do constrain the functional form of the $\widetilde{\Delta}^n_N$ but not their overall normalization. Inserting the solutions $\widetilde{\Delta}^n_N$ into the first equation then should determine the functional form of $\Gamma_{N,\infty}[\chi]$. Note that these two equations (7.31) and (7.32) do not explicitly involve the K_n anymore. Indeed, all we really want to determine in the end is the form of $\Gamma_{N,\infty}[\chi] = \Gamma_{N,\infty}[\chi, 0]$ although, once we know $\widetilde{\Delta}^n_N(x)$, we also know $\Gamma_{N,\infty}[\chi, K] = \Gamma_{N,\infty}[\chi, 0] + \int \widetilde{\Delta}^n_N(x) K_n(x)$.

To solve these equations in practice, it is useful to give them a more physical interpretation. Define

$$\Gamma_N^{(\epsilon)}[\chi] = S_R^{(0)}[\chi] + \epsilon\,\Gamma_{N,\infty}[\chi] \qquad \text{and} \qquad \Delta_N^{(\epsilon)n}(x) = \Delta^n(x) + \epsilon\,\widetilde{\Delta}_N^n(x)\ . \tag{7.33}$$

Consider the "deformed BRST transformation"

$$s^{(\epsilon)}\chi^n(x) = \Delta_N^{(\epsilon)n}(x)\ . \tag{7.34}$$

Obviously, for $\epsilon = 0$ this reduces to the ordinary BRST transformation. It is then not difficult to see that, *up to first order in* ϵ, $\Gamma_N^{(\epsilon)}[\chi]$ is invariant under this transformation and that this transformation is nilpotent:

$$s^{(\epsilon)}\Gamma_N^{(\epsilon)}[\chi] = 0 + \mathcal{O}(\epsilon^2) \quad \text{by (7.31)} \qquad \text{and} \qquad s^{(\epsilon)}s^{(\epsilon)}\chi^n = 0 + \mathcal{O}(\epsilon^2) \quad \text{by (7.32)}\ . \tag{7.35}$$

Deformed BRST symmetry

First note (again) that, since $\widetilde{\Delta}_N^n$ couples to K_n in $\Gamma_{N,\infty}[\chi, K]$, in the same way as Δ^n couples to K_n in $S_R^{(0)}$, both $\widetilde{\Delta}_N^n$ and Δ^n must have the same ghost numbers, dimensions and Lorentz transformation properties.

The strategy now is to write the most general nilpotent transformation of the fields that increases the ghost numbers of the fields by one unit, increases their dimensions by $\dim\omega$ and that is a deformation of the usual BRST transformation in the sense that it depends on some small parameter ϵ and reduces to the ordinary BRST transformation in the $\epsilon \to 0$ limit. (Of course, nilpotency is only required up to terms of order ϵ^2.) These requirements will turn out to be stringent enough to show that any such deformed BRST transformation must be identical to the ordinary BRST transformation, up to changes in the normalization of the fields and a change of the gauge coupling constant.

It is easy to see that the deformed BRST symmetry must act as

$$\begin{aligned}
\delta_\theta\psi &= i\theta\,\omega^\alpha T_\alpha\psi\ ,\\
\delta_\theta A_{\alpha\mu} &= \theta\left(B_{\alpha\beta}\,\partial_\mu\omega_\beta + D_{\alpha\beta\gamma}\,A_{\beta\mu}\omega_\gamma\right)\ ,\\
\delta_\theta\omega_\alpha &= -\frac{1}{2}\theta\,E_{\alpha\beta\gamma}\,\omega_\beta\omega_\gamma\ .
\end{aligned} \tag{7.36}$$

(Having the correct ghost number would also allow e.g. a term of the form $\delta\psi \sim i\omega_\alpha\omega_\beta\omega^*_\gamma F_{\alpha\beta\gamma}^{\ \ \ \ \delta}T_\delta\psi$, but this is excluded by the argument about the dimensions.) Recall from (7.29) that $\widetilde{\Delta}_N^{\omega^*} = \widetilde{\Delta}_N^h = 0$ and, hence, the deformed BRST transformation of ω^* and h just equals the undeformed one:

$$\delta\omega^*_\alpha = -h_\alpha \quad , \qquad \delta h_\alpha = 0\ . \tag{7.37}$$

Let us now exploit the nilpotency of these transformations (7.36) and (7.37). First, $\delta_{\theta_1}\delta_{\theta_2}\omega = 0$ yields $E_{\alpha\beta\gamma}E_{\beta\delta\epsilon}\,\omega_\delta\omega_\epsilon\omega_\gamma = 0$ so that $E_{\alpha\beta\gamma}E_{\beta\delta\epsilon}$ must vanish after antisymmetrizing in δ, ϵ, γ. i.e. the $E_{\alpha\beta\gamma}$ satisfy the Jacobi identity and must be the structure constants of some Lie algebra. As $\epsilon \to 0$, these are just the $C_{\alpha\beta\gamma}$ and we conclude that this Lie algebra is the same as the original Lie algebra $\mathcal{G}$ and the $E_{\alpha\beta\gamma}$ differ from the $C_{\alpha\beta\gamma}$ just by the normalization:

$$E_{\alpha\beta\gamma} = \mathcal{Z}\, C_{\alpha\beta\gamma}\ . \tag{7.38}$$

(Note that if the Lie algebra is semi-simple one may have a different $\mathcal{Z}_i$ for every simple factor.) Next, looking at $\delta_{\theta_1}\delta_{\theta_2}A_\mu$ gives $D_{\alpha\beta\gamma}D_{\beta\delta\epsilon} - D_{\alpha\beta\epsilon}D_{\beta\delta\gamma} = E_{\beta\epsilon\gamma}D_{\alpha\delta\beta} = \mathcal{Z}\, C_{\beta\epsilon\gamma}D_{\alpha\delta\beta}$ and $B_{\alpha\beta}E_{\beta\gamma\delta} = D_{\alpha\beta\delta}B_{\beta\gamma}$. The first equation implies that the matrices $\widehat{D}_\gamma$ with $(\widehat{D}_\gamma)_{\alpha\beta} = \frac{1}{\mathcal{Z}}D_{\gamma\alpha\beta}$ are the generators of the adjoint representation of the Lie algebra $\mathcal{G}$, so that

$$D_{\alpha\beta\gamma} = \mathcal{Z} C_{\alpha\beta\gamma}\ . \tag{7.39}$$

Hence, $D_{\alpha\beta\gamma} = E_{\alpha\beta\gamma}$ and the second condition involving $B_{\alpha\beta}$ just states that the matrix B commutes with all matrices $\widehat{D}$ and thus is proportional to the unit matrix:

$$B_{\alpha\beta} = \mathcal{Z}\, \mathcal{N}\, \delta_{\alpha\beta}\ . \tag{7.40}$$

(Again, for a semi-simple $\mathcal{G}$ we can have a different $\mathcal{N}_i$ for every simple factor.) Finally, looking at $\delta_{\theta_1}\delta_{\theta_2}\psi$ implies $[T_\alpha, T_\beta] = iE_{\alpha\beta\gamma}T_\gamma = i\mathcal{Z}C_{\alpha\beta\gamma}T_\gamma$ and we identify

$$T_\alpha = \mathcal{Z}\, t_\alpha\ . \tag{7.41}$$

We conclude that, apart from the new constants $\mathcal{Z}$ and $\mathcal{N}$, the deformed BRST symmetry $\widetilde{s}$ must act exactly as the ordinary BRST symmetry s we started with. Let us summarize:

$$\begin{aligned}
\widetilde{s}\,\psi &= i\mathcal{Z}\,\omega\,\psi\ ,\\
\widetilde{s}\,A_\mu &= \mathcal{Z}\left(\mathcal{N}\partial_\mu\omega - i[A_\mu,\omega]\right)\ ,\\
\widetilde{s}\,\omega &= i\mathcal{Z}\,\omega\,\omega\\
\widetilde{s}\,\omega^* &= -h\\
\widetilde{s}\,h &= 0\ .
\end{aligned} \tag{7.42}$$

We can absorb these new constants by redefining the normalizations of the ghost fields and redefining the coupling constant (hidden in the $C_{\alpha\beta\gamma}$) as follows:

$$\widetilde{\omega}^\alpha = \mathcal{Z}\,\mathcal{N}\,\omega^\alpha \quad , \quad \widetilde{A}^\alpha_\mu = A^\alpha_\mu \quad , \quad \widetilde{C}_{\alpha\beta\gamma} = \frac{1}{\mathcal{N}}\,C_{\alpha\beta\gamma}\ . \tag{7.43}$$

Then the generators are redefined accordingly by $\widetilde{t}_\alpha = \frac{1}{\mathcal{N}}\,t_\alpha$ and thus $\widetilde{\omega} = \widetilde{\omega}^\alpha\widetilde{t}_\alpha = \mathcal{Z}\omega^\alpha t_\alpha = \mathcal{Z}\omega$ and $\widetilde{A}_\mu = \frac{1}{\mathcal{N}}A_\mu$. Furthermore, one can redefine the normalization of ψ by some factor and the one of ω^* and h by another (common) factor. These latter factors are not yet determined by (7.42). In terms of the redefined fields and couplings, the deformed BRST transformations (7.42) take exactly the form of the undeformed ones. In particular, we have

$$\widetilde{s}^2 = 0 \tag{7.44}$$

and any gauge invariant functional $F[\widetilde{A}_\mu, \psi]$ (with structure constants $\widetilde{C}_{\alpha\beta\gamma}$) automatically also is $\widetilde{s}$ invariant.

Let us now construct $\Gamma_N^{(\epsilon)}[\chi] = \int \mathcal{L}_N^{(\epsilon)}[\chi]$ with a Lagrangian $\mathcal{L}_N^{(\epsilon)}$ that is invariant under the deformed BRST transformation $\widetilde{s}$. As we have just seen, the latter is the same as the original BRST transformation – up to changes of normalizations. Hence, our task amounts to constructing the most general BRST invariant Lagrangian in the redefined fields and couplings. Strictly speaking, we should only require invariance to first order in the deformation parameter ϵ, but because of the rather rigid algebraic structure, this actually results in invariance to all orders in ϵ. Of course, as explained before, we also require that it is of dimension less or equal to four and invariant under all linear symmetries of S_R, namely Lorentz symmetry, global gauge symmetry, ghost phase rotations (implying total ghost number zero), and (in most gauges) antighost translations, in which case ω^* must appear as $\partial_\mu\omega^*$. Obviously, once we have constrained the form of $\mathcal{L}_N^{(\epsilon)} = \mathcal{L}_{\rm new} + \epsilon\,\mathcal{L}_{N,\infty}$ we also have constrained the form of the diverging part $\mathcal{L}_{N,\infty}$ of $\mathcal{L}_N$.

It now follows from the above cohomology theorem applied to $\widetilde{s}$ that the most general $\widetilde{s}$ invariant local function is of the form $\mathcal{L}_N^{(\epsilon)} = \mathcal{L}'_N[A,\psi] + \widetilde{s}\widetilde{\Psi}$ with $\mathcal{L}'_N[A,\psi]$ invariant under the gauge symmetry with the redefined coupling and $\widetilde{\Psi}$ of ghost number -1. The additional requirements cited above imply that $\mathcal{L}'_N[A,\psi]$ is of dimension less or equal four, while $\widetilde{\Psi}$ must be of dimension less or equal 3 and contain ω^* only as $\partial_\mu\omega^*$. Taking also into account Lorentz and global gauge invariance, we arrive at

$$\begin{aligned}\mathcal{L}_N^{(\epsilon)} &= -\frac{1}{4}\mathcal{Z}_A\,\widetilde{F}_\alpha^{\mu\nu}\widetilde{F}^{\alpha\mu\nu} + \widetilde{\mathcal{L}}'_N[\psi,\widetilde{D}_\mu\psi] + \widetilde{s}\left(-\frac{\xi'}{2}\omega_\alpha^* h_\alpha + \frac{\mathcal{Z}_\omega}{\mathcal{Z}\mathcal{N}}\partial_\mu\omega_\alpha^* A_\alpha^\mu\right)\\ &= -\frac{1}{4}\mathcal{Z}_A\,\widetilde{F}_\alpha^{\mu\nu}\widetilde{F}^{\alpha\mu\nu} + \widetilde{\mathcal{L}}'_N[\psi,\widetilde{D}_\mu\psi] + \frac{\xi'}{2}\,h_\alpha h_\alpha + \frac{\mathcal{Z}_\omega}{\mathcal{Z}\mathcal{N}}\,h_\alpha\partial_\mu A_\alpha^\mu - \mathcal{Z}_\omega\,\partial_\mu\omega_\alpha^*\left(\widetilde{D}^\mu\omega\right)_\alpha\,, \end{aligned} \tag{7.45}$$

where

$$\widetilde{F}_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha + \widetilde{C}_{\alpha\beta\gamma}A_\mu^\beta A_\nu^\gamma \quad , \quad \widetilde{D}_\mu\psi = \partial_\mu\psi - iA_\mu^\alpha\widetilde{t}_\alpha^{\mathcal{R}}\psi \quad , \quad (\widetilde{D}_\mu\omega)_\alpha = \partial_\mu\omega^\alpha + \widetilde{C}_{\alpha\beta\gamma}A_\mu^\beta\omega^\gamma\,. \tag{7.46}$$

This is of the same form as the initial $\mathcal{L}_{\rm new}$ except for the appearence of the "renormalization constants" $\mathcal{Z}_A$, $\mathcal{Z}_\omega$, a changed coupling via the $\frac{1}{\mathcal{N}}$ in $\widetilde{C}_{\alpha\beta\gamma}$, as well as further "renormalization constants" hidden in $\widetilde{\mathcal{L}}'_N[\psi,\widetilde{D}_\mu\psi]$, and the replacement $\xi\to\xi'$. Let us insist that, on the redefined fields, the deformed BRST symmetry is just a BRST symmetry with a redefined coupling constant $\widetilde{g} = g/\mathcal{N}$, and $\mathcal{L}_N^{(\epsilon)}$ is invariant under this $\widetilde{g}$-BRST symmetry for all ϵ.

7.2.3 Conclusion and remarks

It is enough to recall that $\mathcal{L}_N^{(\epsilon)} = \mathcal{L}_{\rm new} + \epsilon\mathcal{L}_{N,\infty}$ to see that the diverging part $\mathcal{L}_{N,\infty}$ of $\mathcal{L}_N$ is "BRST-invariant" in the sense explained above, i.e. it is the difference of a $\widetilde{g}$-BRST and a g-BRST invariant local term. Since ϵ was arbitrary, the same is true for $-\mathcal{L}_{N,\infty}$ which is the required counterterm. One can now redo the argument order by order in the loop-expansion to see that the sum of all counterterms up to a given order N is "BRST-invariant" in this sense. Hence we conclude that to any order in perturbation theory the renormalized action is g-BRST invariant precisely if the bare action is g_B-BRST invariant. Said differently, by appropriately choosing the field renormalization constants and coupling constant renormalization we get a finite quantum effective action Γ_N at every order N of perturbation theory.

Let us look more explicitly at the renormalization of the gauge field and of the gauge coupling constant g. Recall that the latter was included in the Lie algebra generators as $t_\alpha = g\,\widehat{t}_\alpha$ and accordingly in the structure constants as $C_{\alpha\beta\gamma} = g\,\widehat{C}_{\alpha\beta\gamma}$ with $\widehat{t}_\alpha$ and $\widehat{C}_{\alpha\beta\gamma}$ the more conventionally normalized, coupling-independent generators and structure constants. Thus $F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + g\,\widehat{C}_{\beta\gamma}{}^\alpha A^\beta_\mu A^\gamma_\nu$, and similarly for $F^\alpha_{B\mu\nu}$ with A_B and g_B. If we let

$$A^\alpha_{B\mu} = \sqrt{Z_A}\; A^\alpha_\mu \quad , \quad g_B = \sqrt{Z_g}\; g \ , \tag{7.47}$$

the relation between the bare and renormalized F^2 term is

$$\begin{aligned} -\frac{1}{4} F^\alpha_{B\mu\nu} F_B^{\alpha\mu\nu} = -\frac{1}{4} F^\alpha_{\mu\nu} F^{\alpha\mu\nu} & -\frac{1}{4}(Z_A - 1)\ (\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu)(\partial^\mu A^{\alpha\nu} - \partial^\nu A^{\alpha\mu}) \\ & -\frac{1}{2} g(\sqrt{Z_g} Z_A^{3/2} - 1)\ (\partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu) \widehat{C}_{\beta\gamma}{}^\alpha A^{\beta\mu} A^{\gamma\nu} \\ & -\frac{1}{4} g^2 (Z_g Z_A^2 - 1)\ \widehat{C}_{\beta\gamma}{}^\alpha A^\beta_\mu A^\gamma_\nu\, \widehat{C}_{\delta\epsilon}{}^\alpha A^{\delta\mu} A^{\epsilon\nu} \ . \end{aligned} \tag{7.48}$$

To determine only Z_A, at a given order, it is enough to compute the two-gauge boson function, i.e. the vacuum-polarization. However, to determine Z_g and hence the renormalization of the coupling constant, one has to determine two of the three counterterms on the rhs of (7.48) and hence compute e.g. the vacuum-polarization and the 3-gauge boson vertex function. This is related to the fact that the bare and renormalized actions have their BRST invariance defined with different couplings. In an abelian gauge theory like QED, there is no coupling dependence of the BRST symmetry and the Ward identity implies $(Z_g Z_A)_{\rm abelian} = 1$ so that we could read the coupling constant renormalization from the vacuum-polarization. Below, we will discuss a special type of gauge fixing, the so-called background gauge, where an analogous relation holds also in the non-abelian theory and, hence, it is enough to compute a single Green's function to obtain the coupling constant renormalization and the β-function.

We have used the anti-ghost translation invariance of the so-called ξ-gauges $f_\alpha = \partial_\mu A^\mu_\alpha$, and this is why we get back this form. However, if one uses a gauge-fixing with $f_\alpha = \partial_\mu A^\mu_\alpha + a_{\alpha\beta\gamma} A^\mu_\beta A_{\beta\mu}$ with some global G-tensor $a_{\alpha\beta\gamma}$, there is no ω^*-translation invariance to enforce the absence of BRST invariant terms with two ghosts and two anti-ghosts, like e.g. $\widetilde{s}\left(b_{\alpha\beta\gamma}\omega^*_\alpha\omega^*_\beta\omega_\gamma\right) = -b_{\alpha\beta\gamma}\big(2h_\alpha\omega^*_\beta\omega_\gamma + \frac{1}{2}\mathcal{Z}C_{\gamma\delta\epsilon}\omega^*_\alpha\omega^*_\beta\omega_\delta\omega_\epsilon\big)$. Hence, such terms must be allowed in the action (and counterterms) from the beginning. However, such terms cannot be obtained from the Faddeev-Popov procedure (which always gives a ghost term linear in the ghost and linear in the anti-ghost). To get these terms necessitates a more general Lagrangian $\mathcal{L}_{\rm new} = \mathcal{L}[A,\psi] + s\Psi[A,\psi,\omega,\omega^*,h]$, as already discussed above.

We can conclude that, as long as we start with a general Lagrangian of this form, with gauge invariant $\mathcal{L}$ of dimension less or equal 4, there is a counterterm available to cancel any divergence at any loop order, and the theory is renormalizable in the usual sense.

Heavy use was made of the Slavnov-Taylor identity $\int {\rm d}^4x\, \langle s\,\chi^n\rangle \frac{\delta\Gamma}{\delta\chi^n} = 0$ from which the Zinn-Justin equation was obtained. The derivation of the Slavnov-Taylor identity assumed that

$\mathcal{D}\chi_n e^{i\int \mathcal{L}_{\rm new}[\chi^n]}$ is BRST invariant and in particular that $\mathcal{D}\psi\mathcal{D}A_\mu e^{i\int \mathcal{L}[A,\psi]}$ is gauge invariant. Now $\int \mathcal{L}[A,\psi]$ is gauge invariant by construction but it may happen that $\mathcal{D}\psi$ is not gauge invariant. This typically shows up at one loop and then $(S_R,\Gamma_1)\neq 0$. Equivalently, certain one-loop diagrams are then seen to violate the corresponding Ward identities of covariant current conservation. In this case the theory is said to be anomalous. Anomalies are the subject of the next chapter. Fortunately, as we will see, anomalies can only appear for certain gauge groups and only in the presence of chiral fermions. Moreover, in potentially anomalous theories, by carefully arranging the content of the chiral matter fermions, the anomalies can be made to cancel. This is the case, in particular, for the standard model based on the gauge group $SU(3)\times SU(2)\times U(1)$ with the content of chiral matter fermions observed in nature.

7.3 Background field gauge

Suppose we can find a gauge-fixing procedure such that the one-loop quantum effective action[34] $\Gamma_{1-\rm loop}[a,\psi_0,\omega_0,\omega_0^*]$ must be gauge invariant under the *same* transformation as was the tree-level renormalized action $S_R^{(0)}$, i.e. $\delta a_\mu^\alpha = \partial_\mu\epsilon^\alpha + g\,\widehat{C}_{\beta\gamma}{}^\alpha a_\mu^\beta\epsilon^\gamma$ and $\delta\psi_0 = ig\,\epsilon^\alpha\,\widehat{t}_\alpha^{\mathcal{R}}\,\psi_0$, as well as some appropriate transformations for ω_0 and ω_0^*. Then necessarily we have

$$S_R^{(0)}[A,\ldots]+S_{\rm c.t.}^{1-\rm loop}[A,\ldots]+\ldots = -\frac{1}{4}F_{\mu\nu}^\alpha F^{\alpha\mu\nu}-\frac{1}{4}(Z_A-1)F_{\mu\nu}^\alpha F^{\alpha\mu\nu}+\ldots\ = -\frac{1}{4}Z_A F_{\mu\nu}^\alpha F^{\alpha\mu\nu}+\ldots. \tag{7.49}$$

On the other hand, this must be the bare action. Comparing with (7.48), we see that then $Z_g Z_A=1$ and, hence, $g_B=(Z_A)^{-1/2}g$. Since the one-loop β-function is given by the coefficient of $\frac{1}{\epsilon}$ in the expression of g_B in terms of g, we then find that[35]

$$\beta_{1-\rm loop} = -\frac{1}{2}\,g\times\left(\text{coeff of }\frac{1}{\epsilon}\text{ in }Z_A\right)\ . \tag{7.50}$$

This will allow us to extract the one-loop β-function from the computation of the single coefficient of the F^2-term in $\Gamma_{1-\rm loop}$.

With this motivation in mind, let us now introduce the background field gauge. Recall equations (1.89) and (1.91) which compute $\Gamma[a]$ from $S[a+A]$ by integrating over the quantum gauge field A_μ (with the restriction to 1PI diagrams - which is irrelevant at one loop). We want to find a gauge-fixing for A_μ such that $\Gamma[a]$ still is invariant under gauge transformations of the "background" field a_μ.

We introduce two types of gauge transformations: the "background field gauge transformations" δ_B and the "quantum field gauge transformations" δ_g. The background field gauge transformation is defined to act on the background *and* quantum fields as follows:

$$\begin{aligned}\delta_B a_\mu &= \partial_\mu\epsilon - i[a_\mu,\epsilon]\equiv D_\mu^B\epsilon &\Leftrightarrow\quad& \delta_B a_\mu^\alpha = \partial_\mu\epsilon^\alpha + C^\alpha{}_{\beta\gamma}a_\mu^\beta\epsilon^\gamma\ ,\\ \delta_B A_\mu &= -i[A_\mu,\epsilon] &\Leftrightarrow\quad& \delta_B A_\mu^\alpha = C^\alpha{}_{\beta\gamma}A_\mu^\beta\epsilon^\gamma\ .\end{aligned} \tag{7.51}$$

[34] Although we do not want to compute scattering amplitudes or Green's functions for external ghosts, nothing prevents us from computing Γ also with ghost "background fields" ω_0 and ω_0^*.

[35] To lowest order in perturbation theory $g_B=(Z_A)^{-1/2}g=(1+(Z_A-1))^{-1/2}g\simeq(1-\frac{1}{2}(Z_A-1))g$.

Thus under this background gauge transformation a_μ transforms as a gauge field and A_μ as an adjoint matter field. Obviously, on the sum of both fields δ_B acts as an ordinary gauge transformation:

$$\delta_B(a_\mu + A_\mu) = \partial_\mu \epsilon - i[a_\mu + A_\mu, \epsilon] \ . \tag{7.52}$$

Similarly, the action on the matter quantum and background fields is defined as

$$\delta_B \psi_0 = i\epsilon^\alpha t_\alpha^{\mathcal{R}} \psi_0 \quad , \quad \delta_B \psi = i\epsilon^\alpha t_\alpha^{\mathcal{R}} \psi \quad \Rightarrow \quad \delta_B(\psi_0 + \psi) = i\epsilon^\alpha t_\alpha^{\mathcal{R}} (\psi_0 + \psi) \ , \tag{7.53}$$

and for the ghost fields

$$\delta_B \omega_0^\alpha = C^\alpha_{\ \beta\gamma} \omega_0^\beta \epsilon^\gamma \quad , \quad \delta_B \omega^\alpha = C^\alpha_{\ \beta\gamma} \omega^\beta \epsilon^\gamma \quad \Rightarrow \quad \delta_B(\omega_0^\alpha + \omega^\alpha) = C^\alpha_{\ \beta\gamma} (\omega_0^\beta + \omega^\beta) \epsilon^\gamma \ , \tag{7.54}$$

and idem for the antighost fields. On the other hand, the quantum field gauge transformations δ_g should not act on the background fields and will be defined such that they do act as standard gauge transformations on the sums $a_\mu + A_\mu$ and $\psi_0 + \psi$. Thus

$$\delta_g a_\mu = \delta_g \psi_0 = \delta_g \omega_0 = \delta_g \omega_0^* = 0 \ , \tag{7.55}$$

and

$$\begin{aligned} \delta_g A_\mu = \partial_\mu \epsilon - i[a_\mu + A_\mu, \epsilon] \quad &\Rightarrow \quad \delta_g(a_\mu + A_\mu) = \partial_\mu \epsilon - i[a_\mu + A_\mu, \epsilon] \\ \delta_g \psi = i\epsilon^\alpha t_\alpha^{\mathcal{R}} (\psi_0 + \psi) \quad &\Rightarrow \quad \delta_g(\psi_0 + \psi) = i\epsilon^\alpha t_\alpha^{\mathcal{R}} (\psi_0 + \psi) \ . \end{aligned} \tag{7.56}$$

In order to compute the one-loop quantum effective action as the functional integral (1.91) one has to choose a gauge-fixing function f_α in the action S. The choice for the background field gauge is

$$f_\alpha = (D_\mu^B A^\mu)_\alpha \equiv \partial_\mu A_\alpha^\mu + C_{\alpha\beta\gamma} a_\mu^\beta A^{\gamma\mu} \ . \tag{7.57}$$

Note that this is a generalization of the standard gauge fixing function to a non-vanishing background field a_μ. It is an appropriate choice for the present purpose, since D_μ^B is covariant under δ_B and thus $D_\mu^B A^\mu$ transforms as A^μ under δ_B :

$$\delta_B f_\alpha = C_{\alpha\beta\gamma} f^\beta \epsilon^\gamma \quad \Rightarrow \quad \delta_B(f_\alpha f_\alpha) = 0 \ , \tag{7.58}$$

and we see that the gauge-fixing term is invariant under the background gauge transformations δ_B. On the other hand, if f_α is to fix the gauge in the functional integral, it better be not invariant under the quantum field gauge transformations δ_g. Indeed, we have

$$\delta_g f_\alpha = \delta_g (D_\mu^B A^\mu)_\alpha = (D_\mu^B \delta_g A^\mu)_\alpha = (D_\mu^B D_B^\mu \epsilon)_\alpha - i(D_\mu^B [A^\mu, \epsilon])_\alpha \ , \tag{7.59}$$

from which we read the corresponding ghost Lagrangian

$$\mathcal{L}_{\rm gh}[a_\mu, A_\mu, \omega, \omega^*] = -\big(D_\mu^B \omega^*\big)_\alpha \big(D_B^\mu \omega - [A^\mu, \omega]\big)_\alpha \ . \tag{7.60}$$

Since ω, ω^* and A_μ transform as adjoint matter fields under δ_B, and since D_μ^B is covariant with respect to δ_B, we conclude that this ghost Lagrangian is invariant under the background field gauge transformations, $\delta_B \mathcal{L}_{\rm gh} = 0$. Since also ω_0 and ω_0^* transform as adjoint matter fields under δ_B, the same remains true if one replaces ω by $\omega_0 + \omega$ and ω^* by $\omega_0^* + \omega^*$:

$$\delta_B \mathcal{L}_{\rm gh}[a_\mu, A_\mu, \omega_0 + \omega, \omega_0^* + \omega^*] = 0 \ , \tag{7.61}$$

so that finally

$$\delta_B \mathcal{L}_{\rm mod}[a_\mu, A_\mu, \psi_0 + \psi, \omega_0 + \omega, \omega_0^* + \omega^*] = 0 \quad , \quad \mathcal{L}_{\rm mod} = \mathcal{L} - \frac{1}{2\xi} f_\alpha f_\alpha + \mathcal{L}_{\rm gh} \ . \tag{7.62}$$

Note that $\mathcal{L}[a + A, \psi_0 + \psi] = -\frac{1}{4} F_{\mu\nu}[a + A] F^{\mu\nu}[a + A] + \mathcal{L}_{\rm matter}[a + A, \psi_0 + \psi]$ with $F_{\mu\nu}[a + A] = F_{\mu\nu}[a] + D^B_\mu A_\nu - D^B_\nu A_\mu - i[A_\mu, A_\nu]$.

Our goal was to determine $\Gamma[a_\mu, \ldots]$ as $\int_{\rm 1PI} \mathcal{D}A \ldots \exp\left(i \int \mathcal{L}_{\rm mod}(a + A, \ldots)\right)$. We see that due to the gauge fixing and ghost terms, $\mathcal{L}_{\rm new}$ does not only depend on the sum $a + A$ but on a and A separately. It is then not obvious any more that this does generate *all* 1PI diagrams. However, if we restrict ourselves to the one-loop quantum effective action, things are simpler and the separate dependence on a and A of the gauge-fixing and ghost terms does not cause any trouble. Since $\mathcal{L}_{\rm mod}$ is invariant under the background gauge transformations δ_B, as is also the measure $\mathcal{D}A \mathcal{D}\psi \mathcal{D}\omega \mathcal{D}\omega^*$ (excluding anomalies), it is then obvious that $\Gamma_{1-{\rm loop}}[a, \psi_0, \ldots]$ must also be invariant under these transformations:

$$\delta_B \Gamma_{1-{\rm loop}}[a, \psi_0, \ldots] = 0 \ . \tag{7.63}$$

But the background gauge transformations act on the background fields just as ordinary gauge transformations (in particular with the same normalisation of $g\, a^\alpha_\mu \widehat{t}_\alpha$ as explained at the beginning of this subsection). Then eq. (7.49) must hold, and hence $Z_g Z_A = 1$ so that we can use (7.50) to compute the one-loop β-function.

7.4 One-loop β-functions for Yang-Mills and supersymmetric Yang-Mills theories

To extract the one-loop β-function we need to compute the coefficient of $\frac{1}{\epsilon}$ in Z_A in the background field gauge. $(Z_A - 1)$ is the coefficient of $-\frac{1}{4} F^2$ in the counterterm. Hence we need the coefficient of $\frac{1}{\epsilon}$ in front of the $-\frac{1}{4} F^2$ piece of the counterterm. To extract this term, we may simply consider a constant background gauge field a_μ and all other background fields vanishing. In particular then, $F^\alpha_{\mu\nu}(a) = C^\alpha{}_{\beta\gamma} a^\beta_\mu a^\gamma_\nu$.

7.4.1 β-function for Yang-Mills theory

As explained above, to compute the one-loop contribution to $\Gamma[a]$ we have to identify the part in $\int \mathcal{L}_{\rm mod}[a, A, \psi, \omega, \omega^*]$ that is quadratic in the quantum fields $A, \psi, \omega, \omega^*$. In particular, $F_{\mu\nu}[a + A]\, F^{\mu\nu}[a + A]\big|_{A^2-{\rm part}} = (D^B_\mu A_\nu - D^B_\nu A_\mu)(D^\mu_B A^\nu - D^\nu_B A^\mu) - 2 F_{\mu\nu}[a]\, [A^\mu, A^\nu]$. The expansion of $\mathcal{L}_{\rm mod}$ then yields

$$\int \mathcal{L}_{\rm mod}[a, A, \psi, \omega, \omega^*]\big|_{\rm quadratic\ part} = \int {\rm d}^4x {\rm d}^4y \Big(-\frac{1}{2} A^{\alpha\mu}(x) \mathcal{D}^A_{x\alpha\mu, y\beta\nu}[a]\, A^{\beta\nu}(y)$$
$$- \bar\psi_k(x) \mathcal{D}^\psi_{xk, yl}[a]\, \psi_l(y) - \omega^{*\alpha}(x) \mathcal{D}^\omega_{x\alpha, y\beta}[a]\, \omega^\beta(y) \Big) \ , \tag{7.64}$$

where

$$\mathcal{D}^{\omega}_{x\alpha,y\beta}[a] = \Big(-\delta_{\alpha\gamma}\frac{\partial}{\partial x^\mu} - C_{\alpha\delta\gamma}\, a^\delta_\mu(x)\Big)\Big(-\delta_{\gamma\beta}\frac{\partial}{\partial y_\mu} + C_{\gamma\epsilon\beta}\, a^\mu_\epsilon(y)\Big)\ \delta^{(4)}(x-y)\ , \tag{7.65}$$

or, using a matrix notation with $a^{\rm adj}_\mu = a^\delta_\mu t^{\rm adj}_\delta$ and the generators of the adjoint representation given in (6.6),

$$\mathcal{D}^{\omega}_{x,y}[a] = \Big(-\frac{\partial}{\partial x^\mu} + i\, a^{\rm adj}_\mu(x)\Big)\Big(-\frac{\partial}{\partial y_\mu} - i\, a^\mu_{\rm adj}(y)\Big)\ \delta^{(4)}(x-y)\ . \tag{7.66}$$

Similarly, one has

$$\mathcal{D}^{\psi}_{x,y}[a] = \Big(-\gamma^\mu\frac{\partial}{\partial y_\mu} - i\gamma^\mu a^{\mathcal{R}}_\mu(y) + m\Big)\ \delta^{(4)}(x-y)\ , \tag{7.67}$$

and

$$\begin{aligned}\mathcal{D}^{A}_{x\mu,y\nu}[a] = \Big\{ &\eta_{\mu\nu}\big(-\frac{\partial}{\partial x^\rho} + i\, a^{\rm adj}_\rho(x)\big)\big(-\frac{\partial}{\partial y_\rho} - i\, a^\rho_{\rm adj}(y)\big) - \big(-\frac{\partial}{\partial x^\nu} + i\, a^{\rm adj}_\nu(x)\big)\big(-\frac{\partial}{\partial y^\mu} - i\, a^{\rm adj}_\mu(y)\big)\\ &-iF^{\rm adj}_{\mu\nu}[a]\ + \frac{1}{\xi}\big(-\frac{\partial}{\partial x^\mu} + i\, a^{\rm adj}_\mu(x)\big)\big(-\frac{\partial}{\partial y^\nu} - i\, a^{\rm adj}_\nu(y)\big)\Big\}\ \delta^{(4)}(x-y)\ . \end{aligned}\tag{7.68}$$

Then

$$i\Gamma_{1-{\rm loop}}[a] = -\frac{1}{2}{\rm Tr}\log\mathcal{D}^A[a] + {\rm Tr}\log\mathcal{D}^\psi[a] + {\rm Tr}\log\mathcal{D}^\omega[a]\ . \tag{7.69}$$

These traces are evaluated as usual. We will sketch the computation for ${\rm Tr}\log\mathcal{D}^\omega[a]$, the others being similar.

Since we will only consider constant fields a_μ it it most convenient to Fourier transform. One has in general $\mathcal{D}_{x,y} \equiv \langle x|\,\mathcal{D}\,|y\rangle$ and hence

$$\langle p|\,\mathcal{D}\,|q\rangle = \int {\rm d}^4x{\rm d}^4y\ \langle p|\,x\rangle\ \langle x|\,\mathcal{D}\,|y\rangle\ \langle y|\,q\rangle = \frac{1}{(2\pi)^4}\int {\rm d}^4x{\rm d}^4y\ e^{-ipx}\ \langle x|\,\mathcal{D}\,|y\rangle\ e^{iqy}\ . \tag{7.70}$$

With a $\mathcal{D}_{x,y}$ of the form $\mathcal{D}_{x,y} = f(-\frac{\partial}{\partial x^\mu})\, g(-\frac{\partial}{\partial y_\mu})\ \delta^{(4)}(x-y)$ this yields

$$\begin{aligned}\langle p|\,\mathcal{D}\,|q\rangle &= \frac{1}{(2\pi)^4}\int {\rm d}^4x{\rm d}^4y\ e^{-ipx}\left[f(-\frac{\partial}{\partial x^\mu})\, g(-\frac{\partial}{\partial y_\mu})\ \delta^{(4)}(x-y)\right] e^{iqy} = \frac{1}{(2\pi)^4}\int {\rm d}^4x\, e^{i(q-p)x} f(-ip)\, g(iq)\\ &= f(-ip)\, g(ip)\, \delta^{(4)}(p-q) \equiv M(p)\, \delta^{(4)}(p-q)\ . \end{aligned}\tag{7.71}$$

Hence $\langle p|\,\mathcal{D}^2\,|q\rangle = \int {\rm d}^4k\ \langle p|\,\mathcal{D}\,|k\rangle\ \langle k|\,\mathcal{D}^2\,|q\rangle = M^2(p)\,\delta^{(4)}(p-q)$, and similarly for any power n and any function of $\mathcal{D}$, and thus

$${\rm Tr}\ \log\mathcal{D} = \int {\rm d}^4p\ \langle p|\ {\rm tr}\ \log\mathcal{D}\,|p\rangle = \int {\rm d}^4p\, \delta^{(4)}(p-p)\, {\rm tr}\ \log M(p)\ . \tag{7.72}$$

Of course, $\delta^{(4)}(p-p)$ arises because we work with constant fields and it has to be interpreted as $\int \frac{{\rm d}^4x}{(2\pi)^4}\, e^{ix(p-p)} = \frac{1}{(2\pi)^4}\int {\rm d}^4x$. Our $M(p)$ are all of the form $M(p) = M_0(p) + M_1(p) + M_2(p)$ with $iM_0(p)$ being the inverse propagator for the given field and M_1 and M_2 are linear and bilinear in the background fields a_μ. One has

$${\rm tr}\ \log M = \ {\rm tr}\ \log M_0 + \ {\rm tr}\ \log\big(1 + M_0^{-1}(M_1+M_2)\big) = \ {\rm tr}\ \log M_0 + \ {\rm tr}\sum_{n=1}^\infty \frac{(-)^{n-1}}{n}\big(M_0^{-1}(M_1+M_2)\big)^n\ . \tag{7.73}$$

We need to pick out the quartic term in a_μ:

$${\rm tr}\ \log M\big|_{a^4} = -\frac{1}{2}\,{\rm tr}\,\big(M_0^{-1}M_2\big)^2 + \ {\rm tr}\,\big(M_0^{-1}M_1\big)^2 M_0^{-1}M_2 - \frac{1}{4}\,{\rm tr}\,\big(M_0^{-1}M_1\big)^4\ . \tag{7.74}$$

Since $-iM_0^{-1}$ is just the propagator this expansion generates the relevant one-loop Feynman diagrams with 4 external (background) gauge fields a_μ attached. There are vertices with one and vertices with two gauge fields, corresponding to M_1 and M_2. Obviously, one could have evaluated these diagrams directly using the appropriate Feynman rules. The present alternative computation has the advantage of giving all combinatorial factors in a straightforward way.

Obviously, the $\mathcal{D}^\omega[a]$ corresponds to the ghost loop and the above expression gives

$$\begin{aligned}
\mathrm{Tr}\,\log \mathcal{D}^\omega[a] &= \int \mathrm{d}^4x \int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \mathrm{tr}\,\log M^\omega(p)\ ,\\
M^\omega(p) &= \big(-ip_\mu + i\,a_\mu^{\rm adj}\big)\big(ip^\mu - i\,a^\mu_{\rm adj}\big) = p_\mu p^\mu - 2p^\mu a_\mu^{\rm adj} + a_\mu^{\rm adj}a^\mu_{\rm adj}\ .
\end{aligned} \tag{7.75}$$

We read off $M_0^\omega(p) = p_\mu p^\mu$, $M_1^\omega(p) = -2p^\mu a_\mu^{\rm adj}$ and $M_2^\omega(p) = a_\mu^{\rm adj}a^\mu_{\rm adj}$ and (7.74) then gives

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \mathrm{tr}\,\log M^\omega\big|_{a^4} = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\Big(-\frac{1}{2}\frac{\mathrm{tr}\,a_\lambda a^\lambda\, a_\sigma a^\sigma}{(p^2-i\epsilon)^2} + 4\frac{\mathrm{tr}\,p_\mu a^\mu\, p_\nu a^\nu\, a_\lambda a^\lambda}{(p^2-i\epsilon)^3} - 4\frac{\mathrm{tr}\,(p_\mu a^\mu)^4}{(p^2-i\epsilon)^4}\Big)\ , \tag{7.76}$$

where all the a_μ are in the adjoint representation although we did not write it explicitly. Since all a_μ are constant, corresponding to vanishing external momenta, the loop integrals are particularly simple. Using dimensional regularization ($d = 4-\epsilon$) and introducing an IR regulator μ, we have[36]

$$\begin{aligned}
I \equiv \int \frac{\mathrm{d}^dp}{(2\pi)^4}\,\frac{1}{(p^2+\mu^2-i\epsilon)^2} &= \frac{i}{16\pi^2}(\pi\mu^2)^{-\epsilon/2}\left(\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\right)\\
I_{\mu\nu} \equiv \int \frac{\mathrm{d}^dp}{(2\pi)^4}\,\frac{p_\mu p_\nu}{(p^2+\mu^2-i\epsilon)^3} &= \frac{1}{4}\eta_{\mu\nu}\, I\\
I_{\mu\nu\rho\sigma} \equiv \int \frac{\mathrm{d}^dp}{(2\pi)^4}\,\frac{p_\mu p_\nu p_\rho p_\sigma}{(p^2+\mu^2-i\epsilon)^4} &= \frac{1}{24}\big(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\big)\ I\ .
\end{aligned} \tag{7.77}$$

Thus

$$\begin{aligned}
\int \frac{\mathrm{d}^4p}{(2\pi)^4}\ \mathrm{tr}\,\log M^\omega\big|_{a^4} &= I\,\Big(-\frac{1}{2}\,\mathrm{tr}\,a_\lambda a^\lambda\, a_\sigma a^\sigma + \eta_{\mu\nu}\,\mathrm{tr}\,a^\mu\, a^\nu\, a_\lambda a^\lambda - \frac{1}{6}\big(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}\big)\,\mathrm{tr}\,a^\mu a^\nu a^\rho a^\sigma\Big)\\
&= \frac{1}{6}\,I\,\big(\mathrm{tr}\,a_\lambda a^\lambda\, a_\sigma a^\sigma - \ \mathrm{tr}\,a_\lambda a_\sigma\, a^\lambda a^\sigma\big) = -\frac{1}{6}\,I\,\big(\mathrm{tr}\,a_\lambda a_\sigma\, a^\lambda a^\sigma - \ \mathrm{tr}\,a_\sigma a_\lambda a^\lambda\, a^\sigma\big)\ .
\end{aligned} \tag{7.78}$$

Recall that for constant a_μ in the adjoint representation one has $\mathrm{tr}_{\,\rm adj}F_{\mu\nu}F^{\mu\nu} = \ \mathrm{tr}_{\,\rm adj}(-i)[a_\mu,a_\nu](-i)[a^\mu,a^\nu] = -2\,\mathrm{tr}_{\,\rm adj}\big(a_\mu a_\nu a^\mu a^\nu - a_\mu a_\nu a^\nu a^\mu\big)$, so that finally

$$\mathrm{Tr}\log\mathcal{D}^\omega[a] = \int \mathrm{d}^4x\,\frac{1}{12}\,I\ \ \mathrm{tr}_{\,\rm adj}F_{\mu\nu}F^{\mu\nu}\ . \tag{7.79}$$

The evaluation of $\mathrm{Tr}\log\mathcal{D}^A[a]$ is very similar, except that the tensorial structure is a bit more complicated. In the final result, the $\frac{1}{12}$ is replaced by a $-\frac{5}{3}$. The matter determinant $\mathrm{Tr}\log\mathcal{D}^\psi[a]$ depends on the mass of the fermions. However, the structure of the divergence is mass independent, i.e. the term $\sim\frac{1}{\epsilon}$ does not depend on the mass, and if our only purpose is to compute the β-function we can just as well neglect the fermion mass.

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[36]Note that when computing $I_{\mu\nu}$ one replaces $p_\mu p_\nu$ by $\frac{1}{d}\eta_{\mu\nu}p^2$. The remaining scalar integral yields $\frac{d}{4}I$ so that $I_{\mu\nu} = \frac{1}{4}\eta_{\mu\nu}I$ and *not* $\frac{1}{d}\eta_{\mu\nu}I$ as one might have naively expected. Similarly for $I_{\mu\nu\rho\sigma}$.

In the limit where the fermion masses can be neglected, the three traces are:

$$\begin{aligned}
\mathrm{Tr}\log\mathcal{D}^A|_{a^4-\text{piece}} &= I\ \int \mathrm{d}^4x\, \left(-\frac{5}{3}\,\mathrm{tr}_{\,\mathrm{adj}}F_{\mu\nu}F^{\mu\nu}\right)\\
\mathrm{Tr}\log\mathcal{D}^\omega|_{a^4-\text{piece}} &= I\ \int \mathrm{d}^4x\, \left(\frac{1}{12}\,\mathrm{tr}_{\,\mathrm{adj}}F_{\mu\nu}F^{\mu\nu}\right)\\
\mathrm{Tr}\log\mathcal{D}^\psi|_{a^4-\text{piece}} &= I\ \int \mathrm{d}^4x\, \left(-\frac{1}{3}\,\mathrm{tr}_{\,\mathcal{R}}F_{\mu\nu}F^{\mu\nu}\right)\ ,
\end{aligned} \tag{7.80}$$

where (μ is an IR regulator) $I=\int \frac{\mathrm{d}^dp}{(2\pi)^4}\frac{1}{(p^2+\mu^2-i\epsilon)^2} = \frac{i}{8\pi^2}\left(\frac{1}{\epsilon}+\text{finite}\right)$. If we let

$$\mathrm{tr}_{\,\mathrm{adj}}\, t_\alpha t_\beta = g^2\, C_1\,\delta_{\alpha\beta} \quad , \quad \mathrm{tr}_{\,\mathcal{R}}\, t_\alpha t_\beta = g^2\, C_2\,\delta_{\alpha\beta}\ , \tag{7.81}$$

we get

$$\Gamma^{\mathrm{div}}_{1-\mathrm{loop}}[a] = -\frac{g^2}{2\pi^2}\frac{1}{\epsilon}\left(\frac{5}{6}C_1+\frac{1}{12}C_1-\frac{1}{3}C_2\right)\int \mathrm{d}^4x\left(-\frac{1}{4}F^\alpha_{\mu\nu}F^{\alpha\mu\nu}\right)\ , \tag{7.82}$$

and, hence

$$Z_A|^{\mathrm{div}}_{1-\mathrm{loop}} = +\frac{g^2}{2\pi^2}\frac{1}{\epsilon}\left(\frac{11}{12}C_1-\frac{1}{3}C_2\right)\int \mathrm{d}^4x\left(-\frac{1}{4}F^\alpha_{\mu\nu}F^{\alpha\mu\nu}\right)\ . \tag{7.83}$$

Finally we can read off the one-loop β-function as

$$\boxed{\beta_{1-\mathrm{loop}} = -\frac{g^3}{4\pi^2}\left(\frac{11}{12}C_1-\frac{1}{3}C_2\right)\ ,} \tag{7.84}$$

which is negative as long as $C_2<\frac{11}{4}C_1$, i.e. as long as there are not "too many" matter fields. Of course, this is a famous result. For example, if the gauge group is $SU(N)$ and the matter representation $\mathcal{R}$ is the defining "vector" representation N, one has $C_1=N$ and $C_2=\frac{1}{2}$. For n_f flavours of quarks in the N-representation one has $C_2=\frac{n_f}{2}$. Thus

$$\beta_{1-\mathrm{loop}}^{SU(N)-QCD \text{ with } n_f \text{ flavours of quarks}} = -\frac{g^3}{4\pi^2}\left(\frac{11}{12}N-\frac{n_f}{6}\right)\ , \tag{7.85}$$

Standard QCD with gauge group $SU(3)$ could accommodate as many as $n_f=16$ flavours of quarks and still remain asymptotically free. Recall from our earlier discussion that the relevant β-function at a given scale μ is the one that corresponds to the number of flavours of quarks having masses (well) below μ. Any quarks having masses (well) above μ do not contribute at this scale. Let us write $\beta_{n_f}(g)=-\beta_0^{n_f}\frac{g^3}{16\pi^2}$ and also use $\alpha(\mu)=\frac{g^2(\mu)}{4\pi}$. As shown before, the running of $g(\mu)$ is such that

$$\Lambda_{n_f} = \mu\,\exp\left(-\frac{8\pi^2}{\beta_0^{n_f}g^2(\mu)}\right) = \mu\,\exp\left(-\frac{2\pi}{\beta_0^{n_f}\alpha(\mu)}\right) \tag{7.86}$$

does not depend on μ. It is the RG-invariant mass scale characterizing the strength of the interaction. More precisely, it is independent of μ as long as μ does not come close to any of the quark masses and the effective number of flavours does not change. Since the running of the coupling below m_{n_f+1}

is well approximated by β_{n_f} and above m_{n_f+1} by β_{n_f+1}, matching the coupling at $\mu = m_{n_f+1}$ leads to the RG scale matching condition

$$\left(\frac{\Lambda_{n_f}}{m_{n_f+1}}\right)^{\beta_0^{n_f}} = \left(\frac{\Lambda_{n_f+1}}{m_{n_f+1}}\right)^{\beta_0^{n_f+1}} . \tag{7.87}$$

It is useful to isolate the contributions of the various fields to the one-loop β-function. There are 3 basic contributions: the gauge and ghost fields, spin-$\frac{1}{2}$ matter fields and scalar matter fields. We already have computed the first two types of contribution, while the contribution of a complex scalar matter field can be obtained from the one of the ghost fields. Indeed, the ghosts are (anticommuting) scalars in the adjoint representation and with the obvious changes we get ($C_{\rm adj} \equiv C_1$ and $C_{\mathcal{R}} \equiv C_2$)

$$\begin{aligned}
\beta_{1-{\rm loop}}^{\text{gauge and ghost}} &= -\frac{g^3}{4\pi^2}\frac{11}{12}\, C_{\rm adj} \\
\beta_{1-{\rm loop}}^{\text{Dirac fermion in } \mathcal{R}} &= -\frac{g^3}{4\pi^2}\left(-\frac{1}{3}\, C_{\mathcal{R}}\right) \\
\beta_{1-{\rm loop}}^{\text{complex scalar in } \mathcal{R}} &= -\frac{g^3}{4\pi^2}\left(-\frac{1}{12}\, C_{\mathcal{R}}\right) .
\end{aligned} \tag{7.88}$$

Note that we considered standard Dirac fermions. If one considers instead chiral (Weyl) fermions or Majorana fermions, one has to divide the corresponding matter contribution by a factor of 2.

7.4.2 β-functions in supersymmetric gauge theories

In supersymmetric theories the fields are part of supersymmetry multiplets and thus the field content is subject to important constraints. We will now show that this leads to simpler expressions for the one-loop β-functions.

$\underline{\mathcal{N} = 1}$:

In gauge theories with the minimal amount of supersymmetry, so called unextended supersymmetry, often referred to as $\mathcal{N} = 1$ supersymmetry, the gauge fields have a corresponding partner, called the gaugino field, which is a spin-$\frac{1}{2}$ Majorana fermion $\lambda^\alpha(x)$, also in the adjoint representation and also massless. More precisely, the boson-fermion correspondence holds for *physical* states, so that the Majorana fermion field is the partner of the gauge and ghost fields. Together they form the gauge or vector multiplet. The matter fields are organized in so-called chiral multiplets, each multiplet consisting of a Majorana fermion $\psi(x)$ and a complex scalar $\phi(x)$, both fields in the same representation $\mathcal{R}$ of the gauge group. Since a Dirac field is equivalent to two Majorana fields, the β-function of a Majorana fermion is half that of a Dirac fermion. Hence, $\beta_{1-{\rm loop}}^{\rm gaugino} = \frac{g^3}{4\pi^2}\frac{1}{6}\, C_{\rm adj}$, so that

$$\beta_{1-{\rm loop}}^{\text{vector multiplet}} = \beta_{1-{\rm loop}}^{\text{gauge and ghost and gaugino}} = -\frac{g^3}{4\pi^2}\frac{3}{4}\, C_{\rm adj} . \tag{7.89}$$

Similarly, for a chiral multiplet

$$\beta_{1-{\rm loop}}^{\text{chiral multiplet}} = \frac{1}{2}\beta_{1-{\rm loop}}^{\text{Dirac fermion in } \mathcal{R}} + \beta_{1-{\rm loop}}^{\text{complex scalar in } \mathcal{R}} = -\frac{g^3}{4\pi^2}\left(-\frac{1}{4}\, C_{\mathcal{R}}\right) . \tag{7.90}$$

Obviously then, in an $\mathcal{N}=1$ supersymmetric gauge theory with $n_{C,i}$ chiral multiplets in the representation $\mathcal{R}_i$:

$$\beta_{1-\text{loop}}^{\mathcal{N}=1 \text{ with } n_{C,i} \text{ in } \mathcal{R}_i} = -\frac{g^3}{4\pi^2}\left(\frac{3}{4}\,C_{\text{adj}} - \frac{1}{4}\sum_i n_{C,i} C_{\mathcal{R}_i}\right) \ . \tag{7.91}$$

Note that such a theory with 3 chiral multiplets in the adjoint representation has a vanishing one-loop β-function. Actually, with appropriate masses and couplings, this theory actually has an extended supersymmetry.

$\underline{\mathcal{N}=2}$:

In $\mathcal{N}=2$ extended supersymmetric gauge theory the $\mathcal{N}=1$ multiplets are grouped into larger $\mathcal{N}=2$ multiplets. Thus the $\mathcal{N}=2$ vector or gauge multiplet consists of the $\mathcal{N}=1$ vector multiplet together with a massless $\mathcal{N}=1$ chiral multiplet in the adjoint representation. Hence

$$\beta_{1-\text{loop}}^{\mathcal{N}=2 \text{ vector multiplet}} - \frac{g^3}{4\pi^2}\,\frac{1}{2}\,C_{\text{adj}} \ . \tag{7.92}$$

$\mathcal{N}=2$ matter multiplets are so-called hypermultiplets which consist of two $\mathcal{N}=1$ chiral multiplets:

$$\beta_{1-\text{loop}}^{\mathcal{N}=2 \text{ hypermultiplet}} = -\frac{g^3}{4\pi^2}\left(-\frac{1}{2}\,C_{\mathcal{R}}\right) \ , \tag{7.93}$$

so that an $\mathcal{N}=2$ theory with $n_{H,i}$ hypermultiplets in representations $\mathcal{R}_i$ has its one-loop β-function given by

$$\beta_{1-\text{loop}}^{\mathcal{N}=2 \text{ with } n_{H,i} \text{ in } \mathcal{R}_i} = -\frac{g^3}{4\pi^2}\left(\frac{1}{2}\,C_{\text{adj}} - \frac{1}{2}\sum_i n_{C,i} C_{\mathcal{R}_i}\right) \ . \tag{7.94}$$

Again, we see that the special theory with a single adjoint hypermultiplet has a vanishing one-loop β-function.

$\underline{\mathcal{N}=4}$:

The maximally extended (global) supersymmetry is $\mathcal{N}=4$ extended supersymmetric gauge theory. It has a single $\mathcal{N}=4$ multiplet consisting of an $\mathcal{N}=2$ vector and a massless $\mathcal{N}=2$ adjoint hypermultiplet. As just noted, this theory has a vanishing one-loop β-function:

$$\beta_{1-\text{loop}}^{\mathcal{N}=4} = 0 \ . \tag{7.95}$$

PART IV:

ANOMALIES

For detailed notes on anomalies, see my

"Lectures on Anomalies", e-Print: arXiv:0802.0634 [hep-th].

8 Anomalies : basics I

8.1 Transformation of the fermion measure: abelian anomaly

8.2 Anomalies and non-invariance of the effective action

8.3 Anomalous Slavnov-Taylor-Ward identities

8.4 Anomaly from the triangle Feynman diagram: AVV

9 Anomalies : basics II

9.1 Triangle diagram with chiral fermions only

9.2 Locality and finiteness of the anomaly

9.3 Cancellation of anomalies, example of the standard model

10 Anomalies : formal developments

10.1 Differential forms and characteristic classes in arbitrary even dimensions

10.2 Wess-Zumino consistency conditions and descent equation

11 Anomalies in arbitrary dimensions

11.1 Relation between anomalies and index theorems

11.2 Gravitational and mixed gauge-gravitational anomalies

11.3 Anomaly cancellation in ten-dimensional type IIB supergravity and in type I $SO(32)$ or $E_8 \times E_8$ heterotic superstring