

Plan

miércoles, 8 de octubre de 2025

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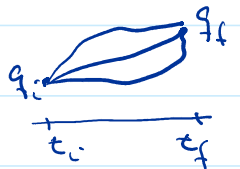
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Einstein-Hilbert action + GHY boundary term

We want to derive the Einstein eqs from a Lagrangian, or from an action, by extremizing it for variations of the metric, $g_{\mu\nu} \rightarrow g_{\mu\nu}^{(x)} + \delta g_{\mu\nu}^{(x)}$ that remain fixed at the boundary $\delta g_{\mu\nu}^{(x)}|_{\partial M} = 0$

In Classical Mech $q(t) \rightarrow q(t) + \delta q(t)$

with $q(t_i)$ and $q(t_f)$ fixed



Dirichlet bc's.

These allow to have conserved energy and a well-defined variational problem

$$I = \int dt \frac{1}{2} \dot{q}(t)^2 \rightarrow \frac{\delta I}{\delta q(t)} = 0 \Rightarrow \ddot{q} = 0$$

$$\begin{aligned} q &\rightarrow q + \delta q \\ \dot{q} &\rightarrow \dot{q} + \delta \dot{q} \end{aligned}$$

$$I + \delta I = \int dt \frac{1}{2} (\dot{q} + \delta \dot{q})^2 = \int dt \left(\frac{1}{2} \dot{q}^2 + \dot{q} \delta \dot{q} \right)$$

$$\delta I = \int dt \dot{q} \delta \dot{q} = \int dt \left(-\frac{d}{dt} \dot{q} \right) \delta q + \int dt \frac{d}{dt} (\dot{q} \delta q)$$

$$= - \int dt \ddot{q} \delta q + \dot{q} \delta q \Big|_{t_i}^{t_f}$$

$$\underbrace{\quad}_{=0} \text{ since } \delta q(t_i, t_f) = 0$$

$$\Rightarrow \frac{\delta I}{\delta q} = 0 \Rightarrow \ddot{q} = 0$$

For gravity There's a simple candidate:

$$I_g^{(0)} = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g} R \quad \text{Einstein-Hilbert}$$

This action is very natural! It is manifestly diff-invariant, since the integrand is a scalar. It also contains two derivatives of the metric, so we expect that it'll give 2nd-order equations for the metric, as are Einstein's equations. So in this respect, it is almost unique

Almost, but not quite. One can add other terms ("Lovelock Terms") to the action and still find e.o.m.'s that are of 2nd order

$$\begin{aligned} \text{Vary } g^{ij} &\rightarrow g^{ij} + \delta g^{ij} & \delta \sqrt{-g} &= -\frac{1}{2} g_{ij} \delta g^{ij} \sqrt{-g} \\ & & \delta(R_{ij} g^{ij}) &= R_{ij} \delta g^{ij} + g^{ij} \delta R_{ij} \\ \delta I_g^{(0)} &= \frac{1}{16\pi G} \int_M \sqrt{-g} (R_{ij} - \frac{1}{2} g_{ij} R) \delta g^{ij} - \frac{1}{8\pi G} \int_{\partial M} \sqrt{h} \delta K \end{aligned}$$

$K = \overset{\text{mean}}{\text{extrinsic curvature of boundary } \partial M}$

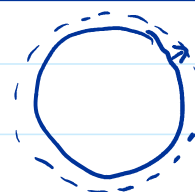
$h_{ij} = \text{metric induced at boundary}$

$n_i = \text{unit normal}$

$h_{ij} = g_{ij} \mp n_i n_j$

$$\begin{aligned} M &: 4d \\ \partial M &: 3d \end{aligned}$$

$$K = \frac{\Delta_a a^{(3)}}{a^{(3)}}$$



relative variation

$$K = \frac{1}{\sqrt{h}} n^i \partial_i \sqrt{h} = \frac{1}{\sqrt{h}} \partial_n \sqrt{h}$$

relative variation
of boundary area
along normal

δK does not vanish if we fix $\delta g^{ij}|_{\partial M} = 0$

δK involves variations of derivatives of $g_{ij}|_{\partial M}$

Define

$$I_g = I_g^{(0)} + \frac{1}{8\pi G} \int_{\partial M} \sqrt{h} K$$

↳ York-Gibbons-Hawking Term
(GHY)

so that

$$\frac{\delta I_g}{\delta g^{ij}} = \frac{1}{16\pi G} (R_{ij} - \frac{1}{2} g_{ij} R)$$

with matter $I_m(\phi, g)$

$$T_{ij} = -\frac{2}{\sqrt{-g}} \frac{\delta I_m}{\delta g^{ij}}$$

$$\frac{\delta (I_g + I_m)}{\delta g^{ij}} = \frac{1}{16\pi G} (R_{ij} - \frac{1}{2} g_{ij} R - 8\pi G T_{ij}) = 0$$

⇒ Einstein eqs

- Why is GHY needed? Because $\sqrt{-g} R$ has second derivatives of the metric: $R_{\text{iemann}} \sim \partial \Gamma + \Gamma \Gamma$ and $\Gamma \sim \partial g$
so $\int \sqrt{-g} R \sim \int g^{-1} \partial^2 g$ instead of $(\partial g)^2$

Consider as a Toy model

$$I = -\frac{1}{2} \int dt \dot{q} \ddot{q}$$

$$q \rightarrow q + \delta q \quad \dot{q} \rightarrow \dot{q} + \delta \dot{q} \quad \ddot{q} \rightarrow \ddot{q} + \delta \ddot{q}$$

$$\delta I = - \int_I dt \dot{q} \delta \ddot{q} + \frac{1}{2} \int_I dt \frac{d}{dt} (-\dot{q} \delta \dot{q} + \ddot{q} \delta q)$$

$$= - \int_I dt \dot{q} \delta \ddot{q} + \frac{1}{2} (-\dot{q} \delta \dot{q} + \ddot{q} \delta q) \Big|_{\partial I}$$

$$\text{and if we impose } \delta q \Big|_{\partial I} = 0$$

Then we don't cancel the term $\delta \dot{q} \Big|_{\partial I}$

We fix it by adding an extra bdy term:

$$\tilde{I} = I + \frac{1}{2} \dot{q} \ddot{q} \Big|_{\partial I} = -\frac{1}{2} \int dt \dot{q} \ddot{q} + \frac{1}{2} \int dt \frac{d}{dt} (\dot{q} \ddot{q}) = \frac{1}{2} \int dt \dot{q}^2 \quad : \text{ so we recover the usual action, w/ only first derivatives}$$

$$\delta \tilde{I} = - \int_I \ddot{q} \delta \dot{q} + \dot{q} \delta \ddot{q} \Big|_{\partial I}$$

$$\hookrightarrow \delta \dot{q} \Big|_{\partial I} = 0$$

$$\frac{\delta \tilde{I}}{\delta \dot{q}} = -\ddot{q} \quad : \text{ we obtain the correct eqn w/ Dirichlet b.c.s}$$

With a negative cosmological constant

in $D = d+1$

$$I = \frac{1}{16\pi G} \int_M d^{d+1}x \sqrt{g} \left(R + \frac{d(d-1)}{L^2} \right) + \frac{1}{8\pi G} \int_{\partial M} d^d x \sqrt{h} K$$

$$\frac{\delta I}{\delta g^{ij}} = 0 \Rightarrow R_{ij} - \frac{1}{2} g_{ij} \left(R + \frac{d(d-1)}{L^2} \right) = 0$$

$$\Rightarrow R_{ij} = -\frac{d}{L^2} g_{ij}$$

$$R = -\frac{d(d+1)}{L^2}$$

$$ds^2 = - \left(\frac{r^2}{L^2} + 1 \right) dt^2 + \frac{dr^2}{\frac{r^2}{L^2} + 1} + r^2 d\Omega_{d-1} \quad : \text{ AdS}_{d+1}$$

For a vacuum solution g_{cl} , $R_{ij}(g_{cl})=0$ $R(g_d)=0$

$$I_g[g_d] = \frac{1}{8\pi G} \int_{\partial M} \sqrt{h} K$$

In asymp flat or AdS This is infinite because of infinite volume. It requires regularization of infinity and subtraction of divergences: renormalization of classical action. In AdS This is holographic renormalization.

Causal diagrams (Penrose)

What's the structure of infinity in an asymptotically flat / AdS / dS space?

Is it the same if we go to infinity along Timelike Trajectories, spacelike, null?

Must assign points at ∞ while preserving causal relationships.

For a spacetime M with metric $ds^2 = g_{ij}(x) dx^i dx^j$ introduce an auxiliary spacetime $\tilde{M}$ with

$$d\tilde{s}^2 = \Omega^{-2}(x) ds^2 \quad \tilde{g}_{ij} = \Omega^{-2} g_{ij}$$

such that $\Omega^2(x) \rightarrow \infty$ as $|x| \rightarrow \infty$

and $\tilde{g}_{ij} \rightarrow \text{finite}$ (more precisely $\Omega \sim |x|$ for non-singular ∞)

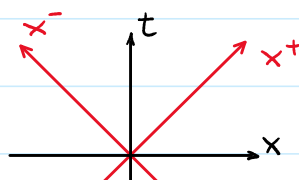
$\tilde{M}$ has the same causal structure as M

$$ds^2 \leq 0 \Leftrightarrow d\tilde{s}^2 \leq 0$$

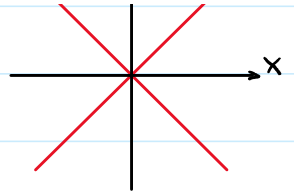
but $\tilde{M}$ is compact: include points at boundary

$\partial\tilde{M}$: conformal boundary of M

Minkowski₁₊₁: $ds^2 = -dt^2 + dx^2$ $x^\pm = t \pm x$
 $= -dx^+ dx^-$



$$= -dx^+ dx^-$$



Bring infTy To finite distance
Through a conformal Transformation

$$x^+ = \tan u$$

$$x^\pm \rightarrow \pm \infty$$

$$u, v \rightarrow -\pi/2, \pi/2$$

$$x^- = \tan v$$

finite range

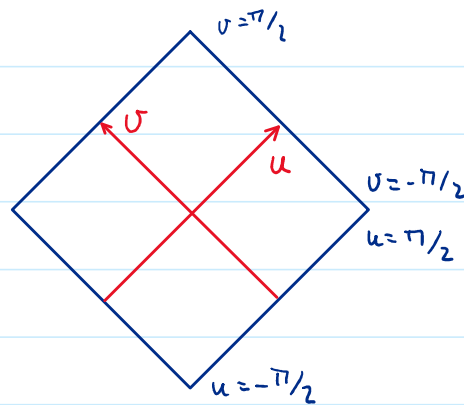
$$-\pi/2 < u, v < \pi/2$$

$$ds^2 = -dx^+ dx^- = -\frac{1}{\cos^2 u \cos^2 v} du dv$$

Define

$$ds^2 = \Omega^{-2} ds^2 \quad \text{w/} \quad \Omega = \frac{1}{\cos u \cos v}$$

$$= -du dv$$



Infinites:

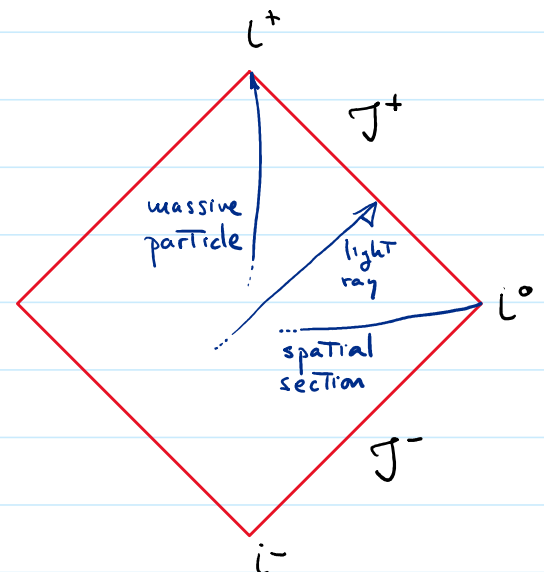
$i^+ = \{t \rightarrow \infty, \text{fixed } x\}$: future Timelike ∞

$i^- = \{t \rightarrow -\infty, \text{fixed } x\}$: past Timelike ∞

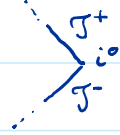
$i^0 = \{x \rightarrow \infty, \text{fixed } t\}$: spacelike ∞

$\mathcal{J}^+ = \{x^+ \rightarrow \infty, \text{fixed } x^-\}$: future null ∞

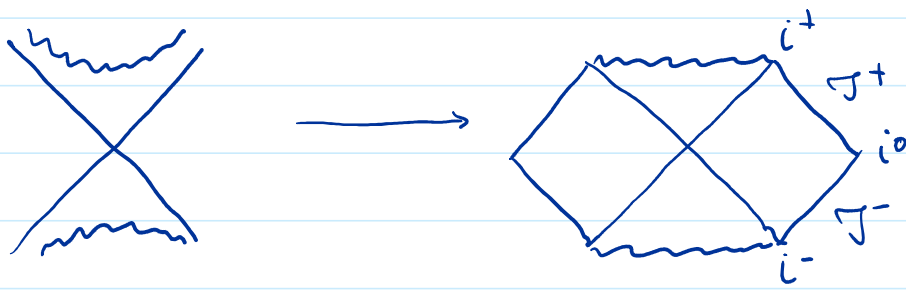
$\mathcal{J}^- = \{x^- \rightarrow -\infty, \text{fixed } x^+\}$: past null ∞



Asymptotically flat spaces have this same structure near infinity:



Schwarzschild:

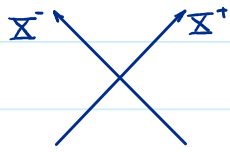


Null coordinates

$$\underline{X}^{\pm} = T \pm X = \pm \left(\frac{r}{2M} - 1 \right)^{1/2} e^{\frac{r \pm t}{4M}}$$

$$x^{\pm} = t \pm r$$

$$= \pm \left(\frac{r}{2M} - 1 \right)^{1/2} e^{\pm x^{\pm}/4M}$$



$$\underline{X}^+ = \frac{1}{4} \left(\frac{r}{2M} - 1 \right)^{1/2} e^{\pm x^{\pm}/4M}$$



$$\underline{X}^+ = \left(1 - \frac{r}{2M} \right)^{1/2} e^{\mp x^{\pm}/4M}$$



$$\underline{X}^+ = - \left(1 - \frac{r}{2M} \right)^{1/2} e^{\mp x^{\pm}/4M}$$



$$ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-d\underline{X}^+ d\underline{X}^- \right) + r^2 d\Omega$$

$$-\underline{X}^+ \underline{X}^- = \left(\frac{r}{2M} - 1 \right) e^{r/2M}$$

$$-\frac{\underline{X}^+}{\underline{X}^-} = e^{t/2M}$$

$$\left[\begin{array}{l} r = \text{const} \Rightarrow \underline{X}^+ \underline{X}^- = \text{const} \\ r = 0 \Leftrightarrow \underline{X}^+ \underline{X}^- = 1 \\ r = 2M \Rightarrow \underline{X}^+ \underline{X}^- = 0 \end{array} \right. \left. \begin{array}{l} \underline{X}^+ = 0 \\ \underline{X}^- = 0 \end{array} \right]$$

$$\underline{X}^+ = \tan u$$

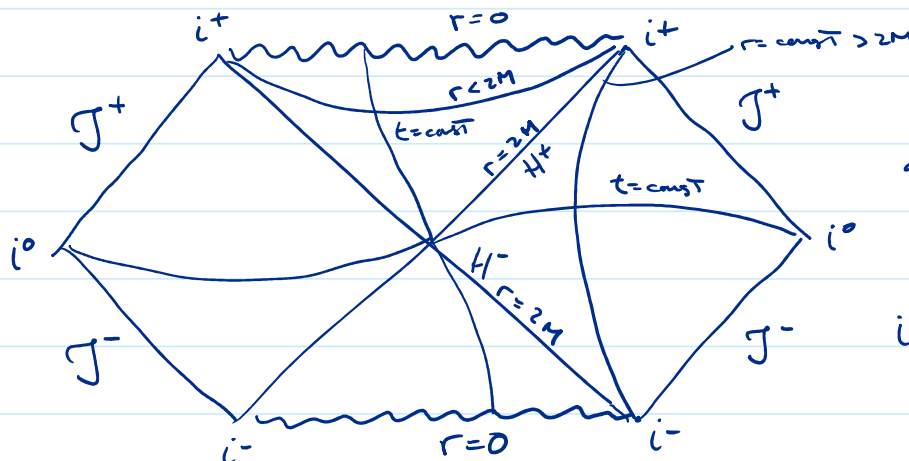
$$-\pi/2 \leq u, v \leq \pi/2$$

$$\underline{X}^- = \tan v$$

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} \frac{1}{\cos^2 u \cos^2 v} (-du dv) + r^2 d\Omega$$

$$d\tilde{s} = \cos^2 u \cos^2 v r e^{r/2M} ds^2 = -32M^3 du dv + () r^2 d\Omega$$

$$r = 0 \Leftrightarrow \tan u \tan v = 1$$



asymptotically flat

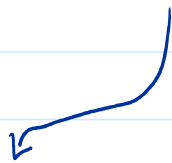
i^+, i^- : distinct from $r=0$

Anti de Sitter

$$ds^2 = -\left(\frac{r^2}{L^2} + 1\right) dt^2 + \frac{dr^2}{\frac{r^2}{L^2} + 1}$$

$$= -\cosh^2 \frac{\rho}{L} dt^2 + d\rho^2$$

$$= \frac{1}{\cos^2 \theta} (-dt^2 + L^2 d\theta^2)$$

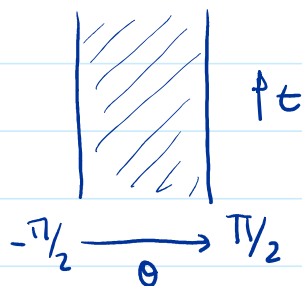


$$\sinh \frac{\rho}{L} = \frac{r}{L} = \tan \theta$$

$$\cosh \frac{\rho}{L} = \frac{1}{\cos \theta}$$

$$-\pi/2 \leq \theta \leq \pi/2$$

Spatial interval of length L

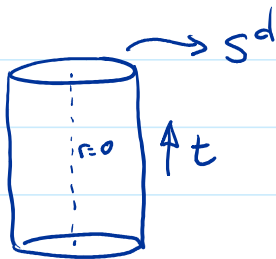


AdS_{d+1}

$$ds^2 = -\left(\frac{r^2}{L^2} + 1\right) dt^2 + \frac{dr^2}{\frac{r^2}{L^2} + 1} + r^2 d\Omega_{d-1}$$

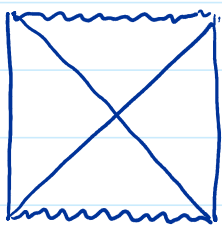
$$= -\cosh^2 \frac{\rho}{L} + d\rho^2 + \sinh^2 \frac{\rho}{L} d\Omega_{d-1}$$

$$= \frac{1}{\cos^2 \theta} \left(-dt^2 + L^2 \underbrace{(d\theta^2 + \sin^2 \theta d\Omega_{d-1})}_{S^d} \right)$$

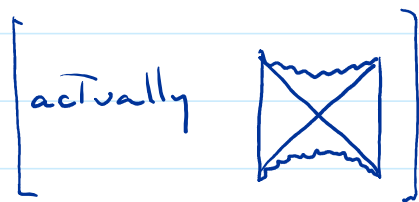


$$0 \leq \theta < \pi/2$$

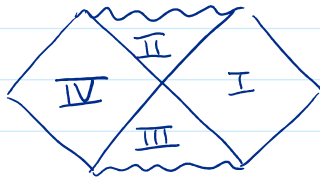
Anti-de Sitter black hole



$$ds^2 = -\left(\frac{r^2}{L^2} + 1 - \frac{r_0}{r}\right) dt^2 + \frac{dr^2}{\frac{r^2}{L^2} + 1 - \frac{r_0}{r}} + r^2 d\Omega_2$$



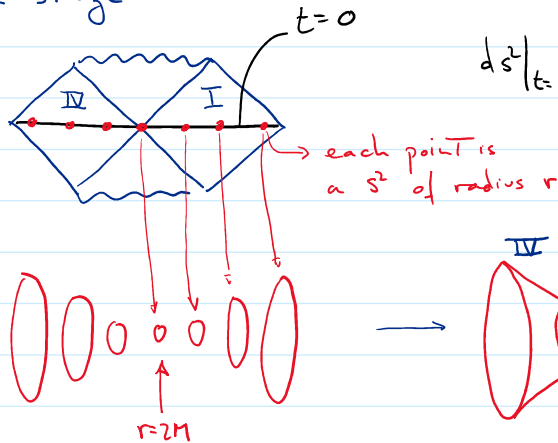
"Eternal black hole"



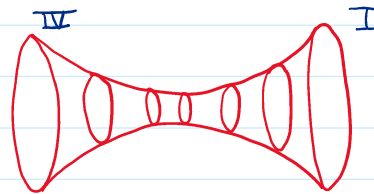
III : white hole : can't enter. Light goes out
I and IV : asymptotically flat, causally disconnected regions.

Connected by a non-traversable wormhole : Einstein-Rosen bridge

E-R bridge



$$ds^2|_{t=0} = \frac{r dr^2}{r-2M} + r^2 d\Omega$$

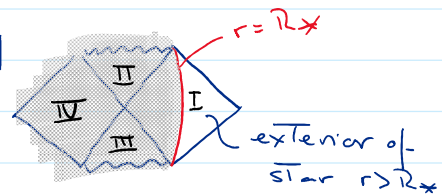


Non-Traversable wormhole
(closes faster than light speed)

• Penrose diagrams for stars: static and collapsing

Static star: surface at constant $r=R_*$

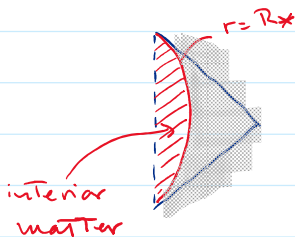
Exterior of star $r \geq R_*$: Schwarzschild



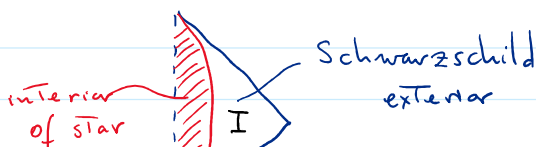
contains I but not II, III, IV

Interior of star $r \leq R_*$: matter solution

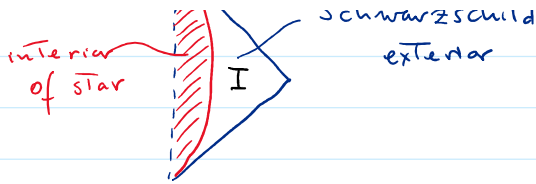
must be qualitatively like Minkowski causally



Gluing The Two diagrams at $r=R_*$



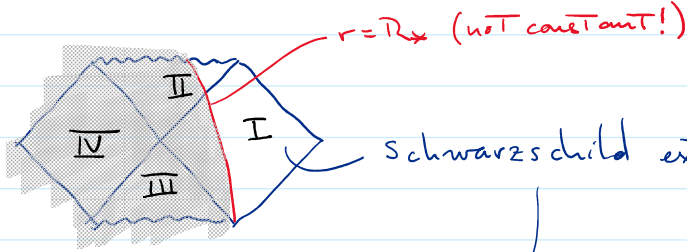
: causally the same as Minkowski



: causally the same as Minkowski

Collapsing star: R_* shrinks to zero

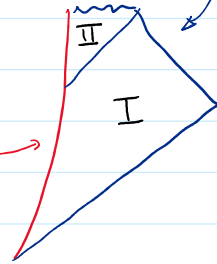
Exterior:



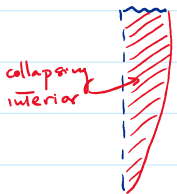
Schwarzschild exterior: contains I and part of II but not III nor IV:

- no white hole
- no disconnected second asymptotic region

in a conformal diagram, this line can be distorted as long as it remains timelike

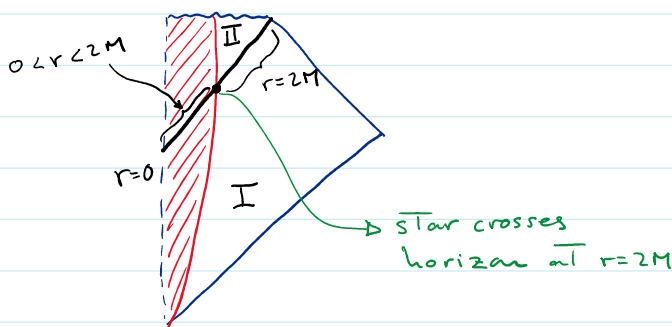


Interior: depends on matter theory. Typically it will be like a matter cosmology (anisotropic) collapsing in a big crunch



Details may vary depending on matter model

Glue them to obtain the Penrose diagram of a collapsed black hole



Eddington-Finkelstein coordinates

jueves, 11 de marzo de 2021 10:56

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2$$

Let's shine light:

ingoing radial light rays:

θ, ϕ constant

$$ds^2 = 0$$

$$\Rightarrow dt^2 = \frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2}$$

$$dt = - \frac{dr}{\left|1 - \frac{2M}{r}\right|}$$

$$\Rightarrow t = -r_* + \text{constant}$$

where $dr_* = \frac{dr}{\left|1 - \frac{2M}{r}\right|}$

$$r_* = r + 2M \log|r - 2M| \quad (\text{"Tortoise coord"})$$

Note: $r_* \rightarrow r \rightarrow \infty$ as $r \rightarrow \infty$
 $r_* \rightarrow -\infty$ as $r \rightarrow 2M$

Ingoing light rays are

$$v \equiv t + r_* = \text{constant}$$

As $r \rightarrow 2M$ along a light ray, $t \rightarrow +\infty$

BUT t is a coordinate whose meaning is clear near ∞ , less so at smaller r .

We may choose v as coordinate, adapted to ingoing light rays, instead of t .
 i.e. change $(t, r) \rightarrow (v, r)$

$$dt = dv - dr_* = dv - \frac{dr}{1 - \frac{2M}{r}}$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2 dv dr + r^2 d\Omega_2$$

$$(g_{ij}) = \begin{pmatrix} -\left(1 - \frac{2M}{r}\right) & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & r^2 \Omega_2 \end{pmatrix}$$

: non-singular
 at $r=2M$

$$g_{vv} = 1 - \frac{2M}{r} : \text{finite}$$

$$\det g_{ij} = -2 r^2 \sin^2 \theta$$

finite

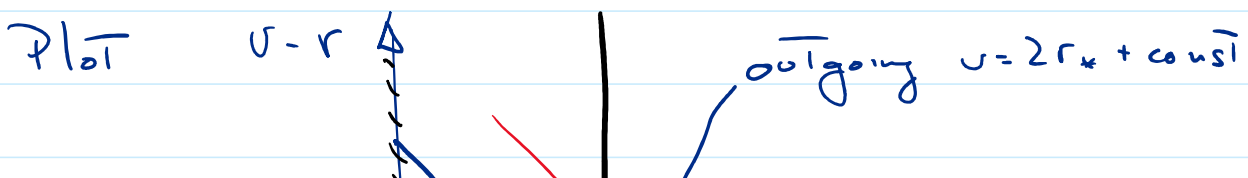
Ingoing light rays encounter $r=2M$ as a smooth place.

Analyze all radial light rays:

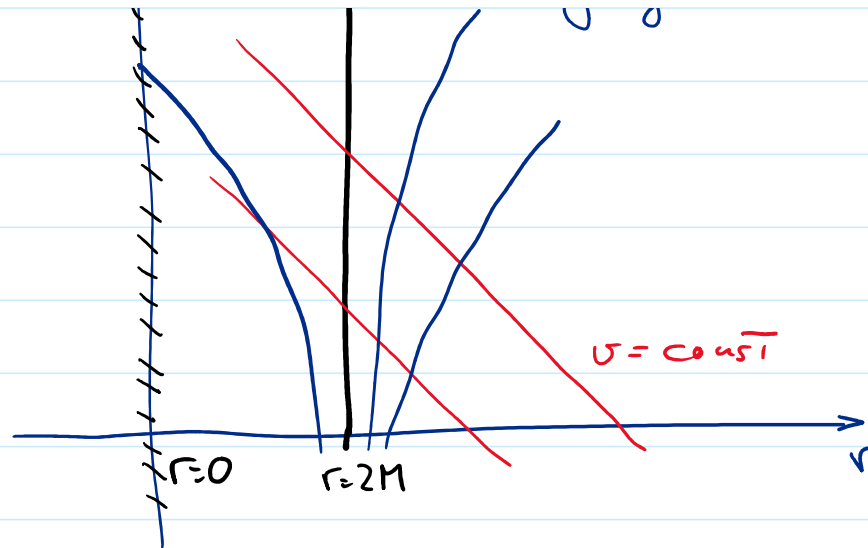
$$ds^2 = 0 \Rightarrow \left(1 - \frac{2M}{r}\right) dv^2 = 2 dv dr$$

$$\Rightarrow \begin{aligned} v &= \text{const} & : \text{ingoing} \\ r &= 2M & : \text{constant radius} \end{aligned}$$

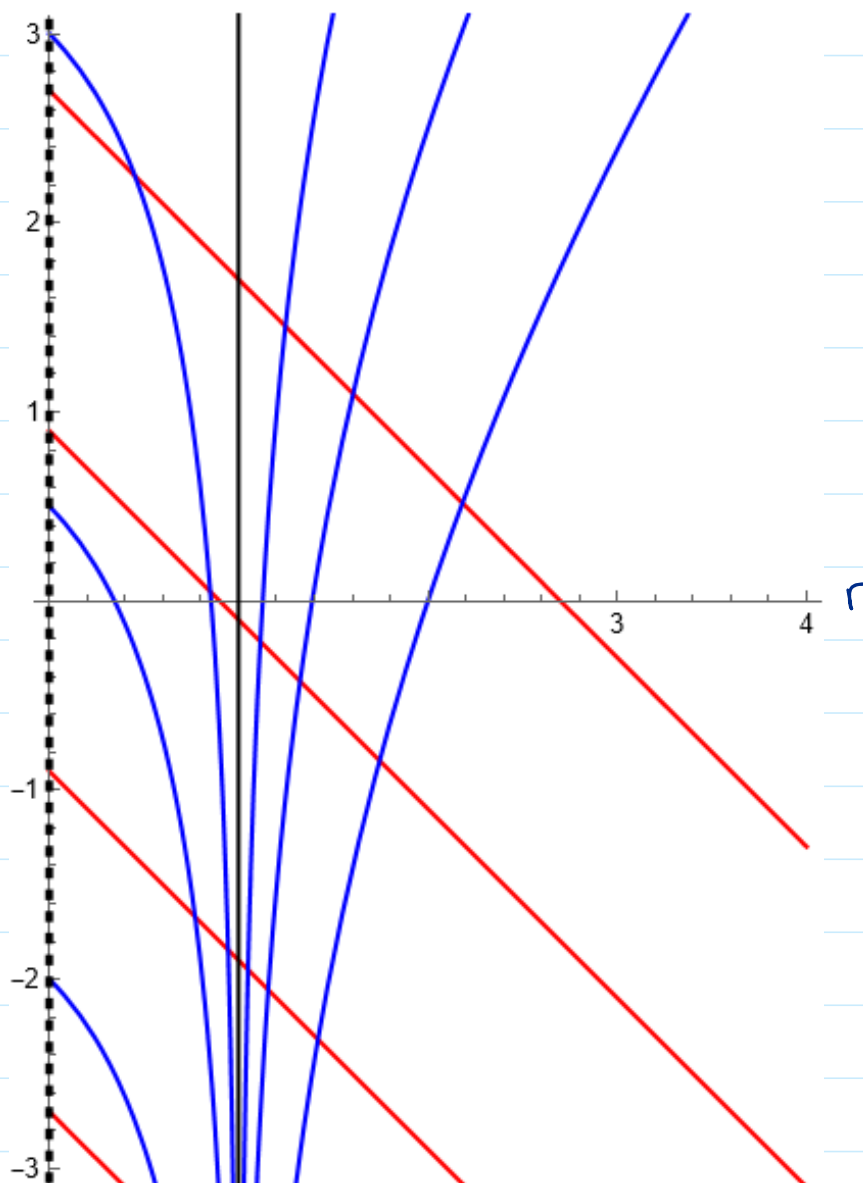
$$\left(1 - \frac{2M}{r}\right) dv = 2 dr \Rightarrow v = 2r_* + \text{const} : \text{outgoing}$$



$v = r/t_1 (r_2 - r)$
 $\sim t$ for $r \gg 2M$

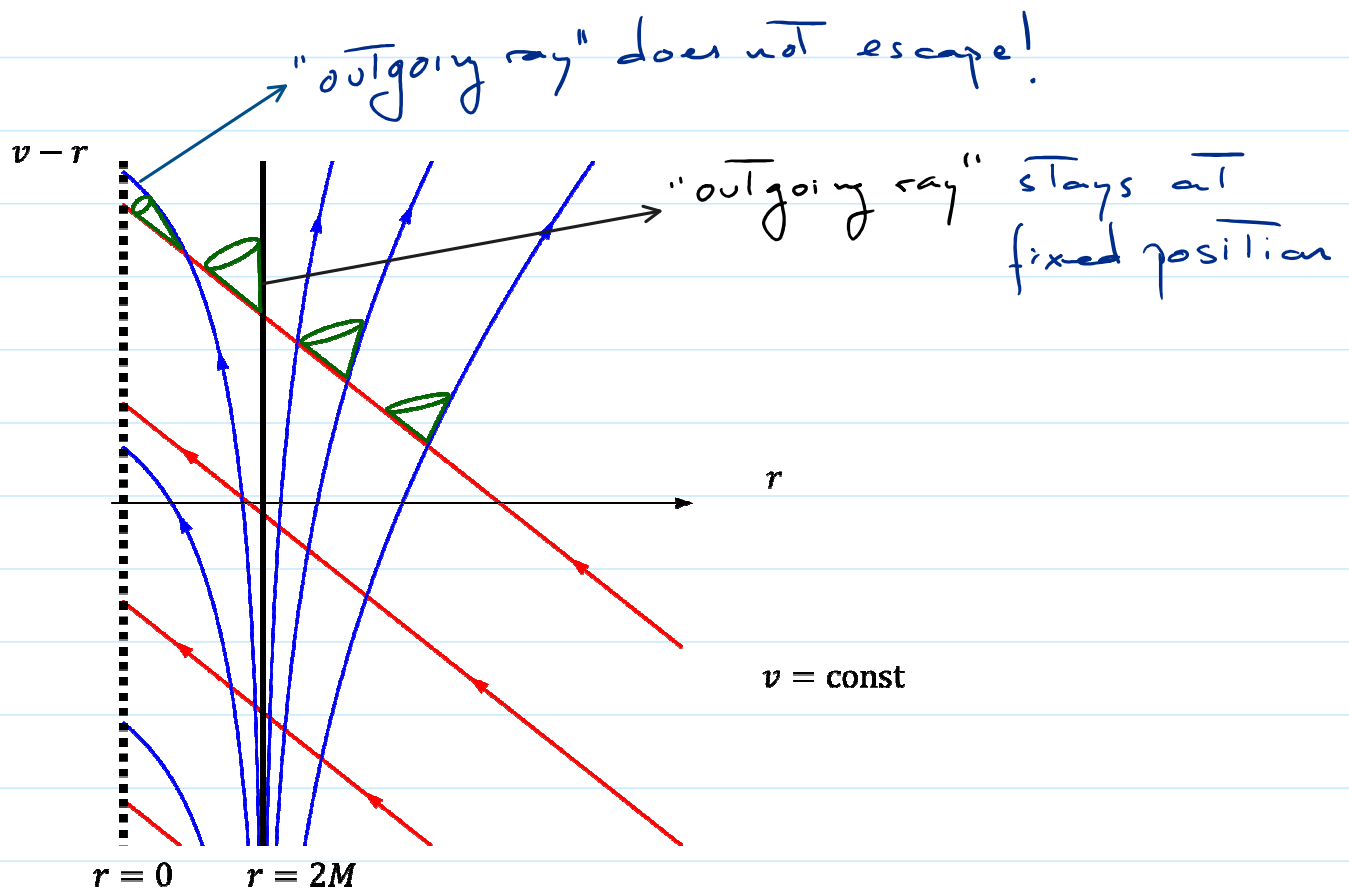


$v-r$



Light cones Tip over

"Tangential ray" does not escape!



Since we (massive objects) move inside light cones,
Then inside $r < 2M$ we cannot escape to $r > 2M$

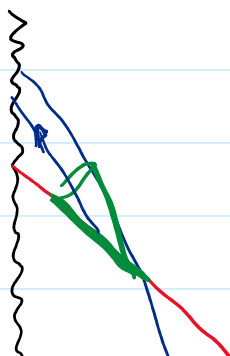
- $r=2M$: "frozen light rays" : HORIZON

one per point in the sphere:

"horizon is generated by null rays"

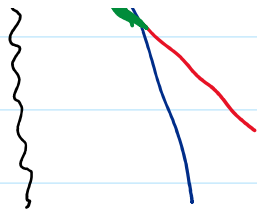
Null hypersurface $S^2 \times \mathbb{R}$

- $r=0$ is a singularity ($R_{ijkl} R^{ijkl} \rightarrow \infty$ at $r=0$)



No light rays emanate from the singularity: it cannot be seen!

It is experienced as an event in the future.



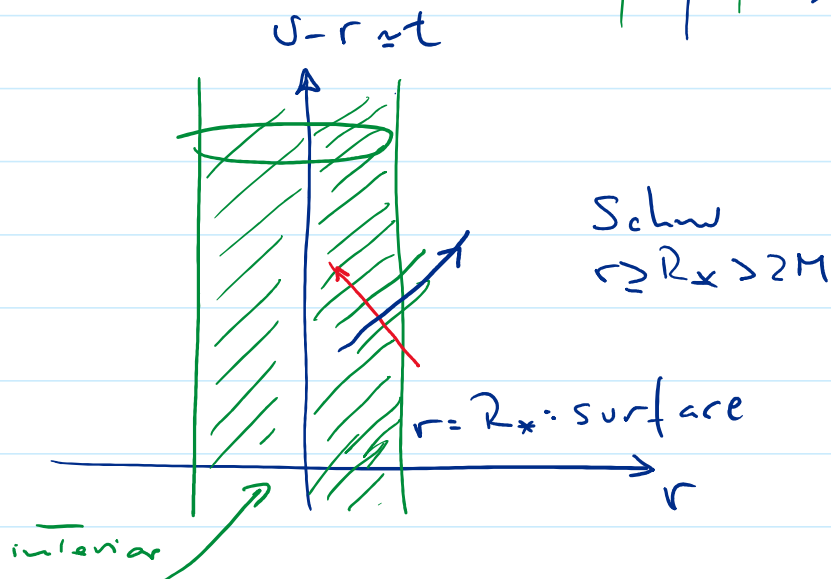
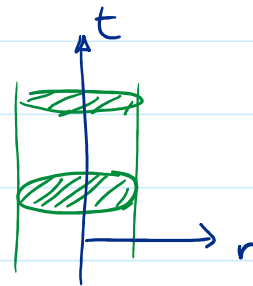
in the future.

Singularity is not a point in space, but a moment in time

All particle Trajectories in $r < 2M$ end at $r=0$ in finite proper time.

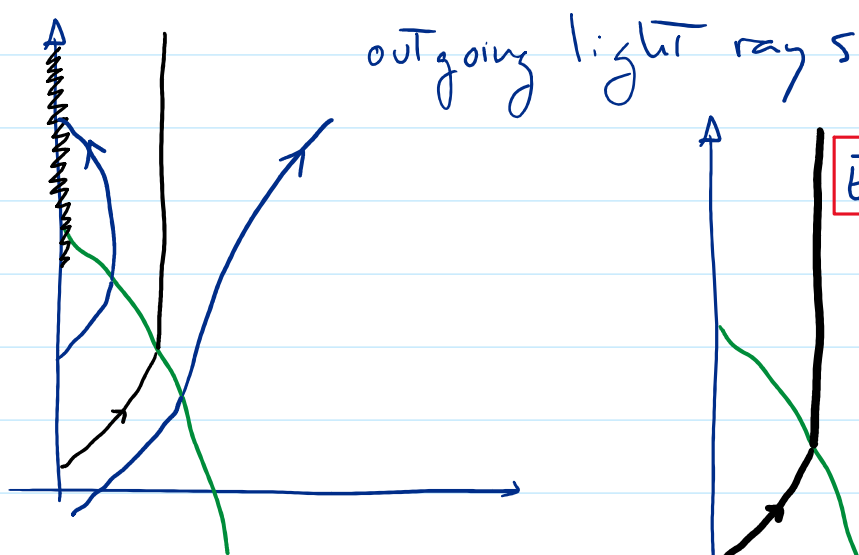
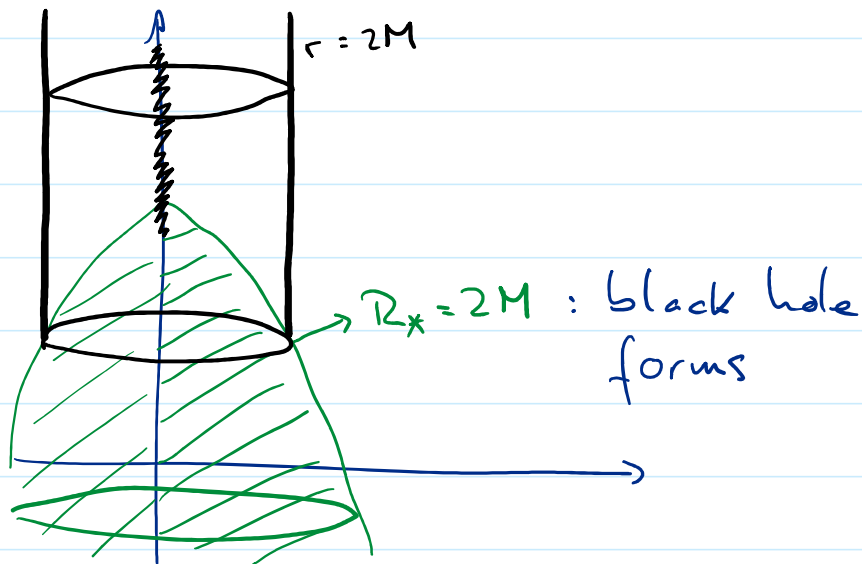
• Collapsing star

First, non-collapsing star



$r=0$ is simply origin of coords

Now collapse: R_* decreases with Time



Outgoing E-F coordinates:

$$u = t - r_*$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2 du dr + r^2 d\Omega^2$$

Radial light rays: $\left(1 - \frac{2M}{r}\right) du^2 = 2 du dr$

$$u = \text{const}$$

$$r = 2M$$

: outgoing

: horizon

$$u = -2r_* + \text{const} : \text{ingoing}$$

Now we can only move out of The horizon,
not inside it. This is a white hole

Lightning review:

$R_{\mu\nu}=0$: neutral black hole : Schwarzschild

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2$$

Horizon at $r=2M$: regular

$$dr_* = \frac{dr}{f} \quad r_* = \int \frac{dr}{f} \quad f = 1 - \frac{2M}{r}$$

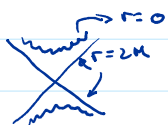
$$v = t + r_* \quad (\sigma, r, \theta, \phi)$$

$$ds^2 = -f dv^2 + 2 dr dv + r^2 d\Omega^2 : \text{regular at } r=2M$$

$f=0$

$g_{\mu\nu}$ finite, $\det g_{\mu\nu} \neq 0$ finite

Surface gravity : $\kappa = \frac{1}{2} f' \Big|_{r=2M} = \frac{1}{4M}$

Kruskal extension  $r=0$ is spacelike
not a point but a moment in time

Penrose : 

Surface gravity in E-F coords

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{g(r)} + d\Sigma$$

$$dr_* = \frac{dr}{\sqrt{fg}} \quad v = t + r_* \quad dv = dt + \frac{dr}{\sqrt{fg}}$$

$$dt = dv - \frac{dr}{\sqrt{fg}}$$

$$-f dt^2 + \frac{dr^2}{g} = -f \left(dv - \frac{dr}{\sqrt{fg}} \right)^2 + \frac{dr^2}{g} = -f dv^2 + 2\sqrt{\frac{f}{g}} dv dr$$

$$\sqrt{\frac{f}{g}} dr = dp \quad \kappa = \frac{1}{2} \frac{df}{dp} = \frac{1}{2} \sqrt{\frac{g}{f}} \frac{df}{dr} = \frac{1}{2} \sqrt{\frac{g}{f}} f'$$

$$dr = \sqrt{\frac{g}{f}} dp \quad \left. \begin{array}{l} \text{1.2. } r \text{ is } \rightarrow d.r \text{ and } d\Sigma \end{array} \right\}$$

$$\left. \begin{aligned} dr &= \sqrt{\frac{g}{f}} dp \\ dt &= dr - \frac{1}{f} dp \end{aligned} \right\} \Rightarrow ds^2 = -f dr^2 + 2 dr dp + d\bar{z}$$

$$\kappa = \frac{1}{2} \frac{df}{dr} \Big|_{r_0} = \frac{1}{2} \sqrt{\frac{g}{f}} \frac{df}{dr} \Big|_{r_0}$$

$$f(r_0) = 0$$

Charged black holes

$$R_{ij} - \frac{1}{2} g_{ij} R = 8\pi G T_{ij}^{EM}$$

$$\nabla_i F^{ij} = 0$$

$$T_{ij}^{EM} = F_{ik} F_j{}^k - \frac{1}{4} F^2$$

$$I = I_G - \frac{1}{4} \int \sqrt{-g} F^2$$

$$F^2 = F_{ij} F^{ij}$$

What can we expect in the metric?

The electromagnetic field energy will gravitate

$$\text{Say } A_t \sim \frac{Q}{r}$$

$$F_{tr} \sim \frac{Q}{r^2} \quad T_{ij} \sim \frac{Q^2}{r^4}$$

$$\text{since } \partial^2 g \sim T$$

$$\text{we expect from EM a term} \sim \frac{Q^2}{r^2}$$

in the metric

Indeed, solving the Einstein-Maxwell eqs gives

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q^2}{r^2}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$A_t = -\frac{Q}{r} \quad \text{Reissner-Nordström solution}$$

$$\text{Mass} = M \quad \text{from } -g_{tt} = -1 + \frac{2M}{r} + \dots$$

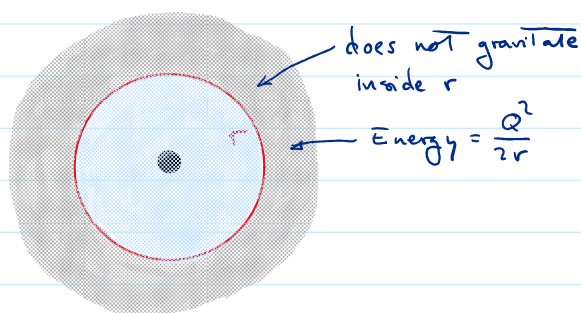
$$\text{Charge} = Q \quad \text{from } A_t = -\frac{Q}{r} \text{ and Gauss' law}$$

$$\left[\text{Charge} = \frac{1}{4\pi} \int_{S^2} d\vec{s} \cdot \vec{E} \quad \text{with} \quad F^{tr} = \frac{Q}{r^2} = E^r \right]$$

$$\left[\begin{aligned} \text{Charge} &= \frac{1}{4\pi} \int_{\Sigma^2} d\vec{s} \cdot \vec{E} \quad \text{with} \quad F^{tr} = \frac{Q}{r^2} = E^r \\ &= \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta r^2 \frac{Q}{r^2} = Q \end{aligned} \right]$$

M is the Total mass (energy) in the entire space, out to infinity. Some of it is due to the energy of the electromagnetic field, which spreads out through all of the spacetime.

At finite r , we can consider the amount of mass/energy that is creating a gravitational attraction on a unit mass placed at that radius



effective mass inside r :

$$M_{\text{eff}}(r) = M - \frac{Q^2}{2r}$$

$$-g_{tt}^{-1} = \frac{2M_{\text{eff}}(r)}{r}$$

"repulsion": There's less energy of the EM field in a sphere of radius r than there's in the entire space, so $M_{\text{eff}}(r) < M$

This can be formalized in terms of the "Komar mass"

It's a purely gravitational effect of the energy of the electric field created by a charge, and therefore it affects all particles, even neutral ones.

It is independent of the electric attraction or repulsion that a charged particle would experience (observe that it is $\propto Q^2$ and therefore the same for $Q \gtrless 0$).

Special loci: horizons & singularities

Singularity: $r=0$ is a curvature singularity: $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\rho\sigma}R^{\mu\rho\sigma} \xrightarrow{r \rightarrow 0} \infty$
 [It is a timelike singularity whenever $Q \neq 0$, as we will see.]

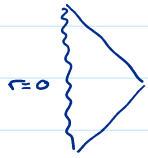
Horizons: look for zeroes of $\overset{\text{only if static!}}{V} = -g_{tt} : V=0 \Leftrightarrow r^2 - 2Mr + Q^2 = 0$
 $r = r_{\pm} = M \pm \sqrt{M^2 - Q^2}$

Always assume $M > 0$ (negative mass is unphysical)
 w.l.o.g. $Q \geq 0$ (in the metric only Q^2 appears)

We'll discuss separately The Three cases $M < Q$, $M > Q$, $M = Q$

- $M < Q$: no horizon at real r .

$r=0$: naked singularity (Timelike)



Notice That for all known charged particles

we have $m \ll q$

$$q \sim 10^{-1}$$

so They're not black holes

$$m < 10^{-17}$$

(Similarly for spin, since $j \sim 1$)

- $M > Q$: There are metric singularities at $r = r_{\pm}$
Eddington-Finkelstein coordinates

$$dr_* = \frac{dr}{V} \quad r_* = \int \frac{dr}{V}$$

$$v = t + r_*$$

$$(v, r, \theta, \phi)$$

$$ds^2 = -V dv^2 + 2 dr dv + r^2 d\Omega^2 : \text{regular at } r = r_{\pm}$$

r_+ : outer horizon

r_- : inner horizon

$$\kappa_{\pm} = \pm \frac{\sqrt{M^2 - Q^2}}{r_{\pm}^2}$$

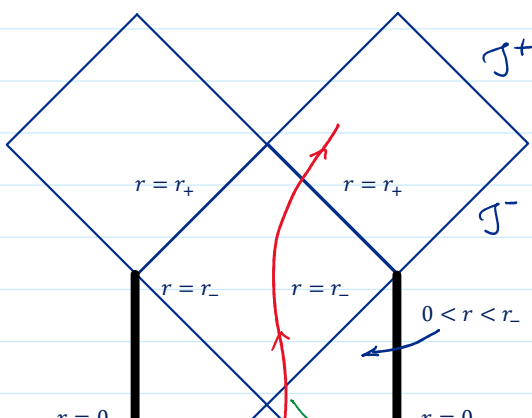
$\kappa_- < 0$ indicates repulsion: one has to accelerate to get to it

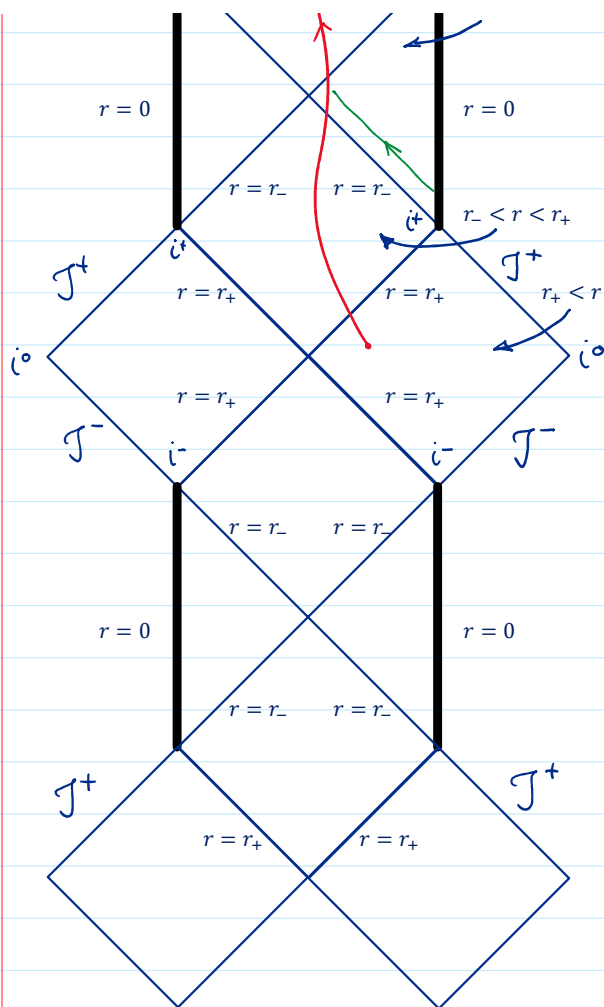
$r > r_+$: AF region

$r_- < r < r_+$: r Timelike, t spacelike

$0 < r < r_-$: r spacelike, t Timelike

$r=0$: Timelike singularity





starts in one AT region and travels to a different one, emerging through a white hole

Past r_- , the singularity at $r=0$ is visible

We expect the inner horizon, to be unstable to perturbations, and possibly then the geometry cannot be extended past that singularity.

← singularity. It may have portions that are null or spacelike
"Strong Cosmic Censorship Conjecture"

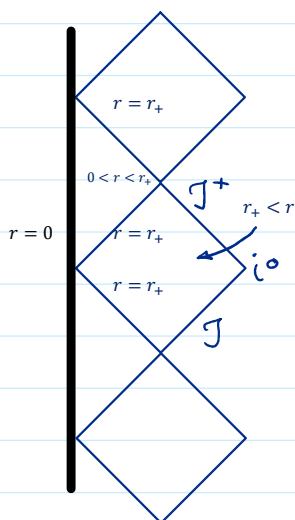
There's good evidence that SCC holds, at least when the backreaction

There's good evidence That SCC holds, at least when The backreaction of quantum fields is Taken into account.

• $M = Q$: extremal RN black hole

$$1 - \frac{2M}{r} + \frac{M^2}{r^2} = \left(1 - \frac{M}{r}\right)^2 \quad r_+ = r_- = M = Q : \text{double root}$$

$\kappa_+ = 0$: degenerate horizon



$r = r_+$: event horizon
Cauchy horizon

$r = 0$: Timelike singularity

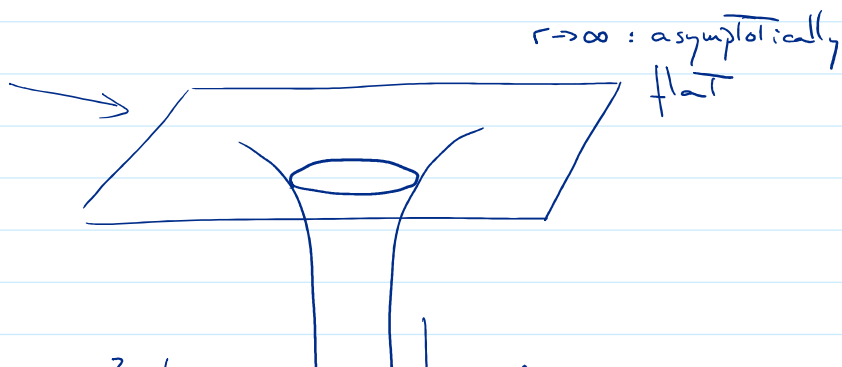
Nowadays we know That
The classical extremal black
hole is a mathematical
artifact of a wrong order
of limits: (i) $t \rightarrow 0$ (ii) $T \rightarrow 0$

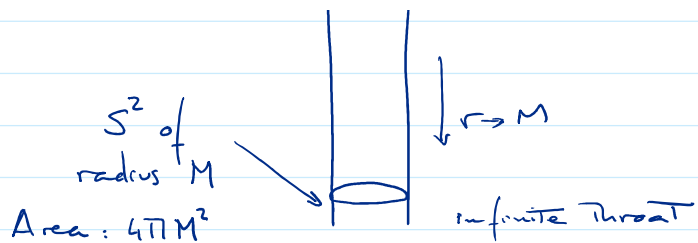
If $t \neq 0$, quantum fluctuations
of The geometry become
uncontrollably large as $T \rightarrow 0$

Proper radial distance To $r=M$ is infinite

$$\int_{r>M}^M \frac{dr}{1 - \frac{M}{r}} = \int_{r>M}^M \frac{r dr}{r-M} : \text{diverges at } r=M$$

$$ds^2|_t = \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} + r^2 d\Omega$$





This infinite Throat is a classical artifact.
 With quantum effects, The length of The Throat has very large variance.

Near-horizon geometry near extremality

Rewrite the metric by shifting the radial coordinate:

First reparametrize $(M, Q) \rightarrow (r_0, q)$

$$M = q + r_0/2 \quad r_+ = M + \sqrt{M^2 - Q^2} = r_0 + q$$

$$Q^2 = q(q + r_0) \quad M^2 - Q^2 = r_0^2/4$$

Charge $\propto q$ $r_0 = \text{near-extremality parameter}$ $r_0 = 0$: extremal

Now shift the radial coordinate

$$r = \bar{r} + q$$

$$\begin{aligned} 1 - \frac{2M}{r} + \frac{Q^2}{r^2} &= \frac{(r-r_+)(r-r_-)}{r^2} = \frac{(r-q)(r-q-r_0)}{r^2} \\ &= \frac{\bar{r}(\bar{r}-r_0)}{(\bar{r}+q)^2} = \frac{f(\bar{r})}{\left(1 + \frac{q}{\bar{r}}\right)^2} \\ f(\bar{r}) &= 1 - \frac{r_0}{\bar{r}} \end{aligned}$$

In the following, drop the bar $\bar{r} \rightarrow r$

$$ds^2 = -\frac{f(r)}{\left(1 + \frac{q}{r}\right)^2} dt^2 + \left(1 + \frac{q}{r}\right)^2 \left(\frac{dr^2}{f(r)} + r^2 d\Omega_2 \right)$$

This is not the usual area-radius gauge!

Horizon at $r = r_0$. Area = $4\pi(r_0 + q)^2$

Extremal limit: $r_0 = 0$, horizon at $r=0$, finite area

In this case Taking $r \ll q$, near-horizon limit:

$$ds^2 \approx -\frac{r^2}{q^2} dt^2 + q^2 \frac{dr^2}{r^2} + q^2 d\Omega_2^2 \quad r = \frac{q^2}{z}$$

$$= q^2 \left(-\frac{dt^2 + \frac{dz^2}{z^2}}{z^2} + d\Omega_2^2 \right)$$

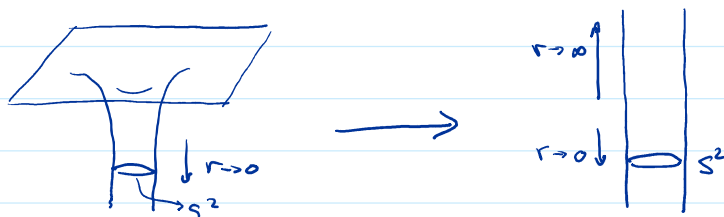
Poincaré-AdS₃ $\times$ S^2

spatial distance $\propto \int \frac{dr}{r} = \log r \xrightarrow{r \rightarrow 0} -\infty$

Horizon at $r=0 \quad z=\infty$

Boundary (mouth of Throat)

at $r \rightarrow \infty \quad z \rightarrow 0$



Now consider near-extremal, near-horizon

Take $r, r_0 \ll q$

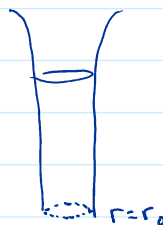
$$ds^2 \approx -\frac{r(r-r_0)}{q^2} dt^2 + q^2 \frac{dr^2}{r(r-r_0)} + q^2 d\Omega_2^2$$

change $r = \frac{r_0}{2} (\tilde{r} + 1)$ $t = \frac{2q^2}{r_0} \tilde{t}$. Then

$$ds^2 \approx q^2 \left(-(\tilde{r}^2 - 1) d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 - 1} + d\Omega_2^2 \right)$$

Rindler-AdS₃ $\times$ S^2
aka Thermal AdS₃

long Throat
capped by horizon



or $r = r_0 \cosh^2 p/2$ $t = \frac{2q^2}{r_0} \tilde{t}$ $p = 2 \operatorname{arctanh} \sqrt{\frac{r-r_0}{r}}$

$$ds^2 \approx q^2 \left(-\sinh^2 p d\tilde{t}^2 + dp^2 + d\Omega_2^2 \right)$$

Directly in the extremal case:

Directly in the extremal case:

$$\bar{r} = r - Q$$

$$ds^2 = -\left(1 - \frac{Q}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{Q}{r}\right)^2} + r^2 d\Omega = -\frac{1}{H^2} dt^2 + H^2 (d\bar{r}^2 + \bar{r}^2 d\Omega)$$

$$A_t = -\frac{Q}{r}$$

$$A_t = -\frac{1}{H} + \text{const}$$

$$H = 1 + \frac{Q}{\bar{r}}$$

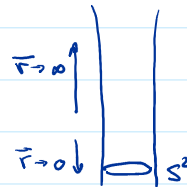
Horizon at $\bar{r} = 0$: not a point, but a S^2 w/ radius Q
(This is independent of the coordinates)

$$\text{Near } \bar{r} = 0 \quad H \approx \frac{Q}{\bar{r}}$$

$$ds^2 \approx -\frac{\bar{r}^2}{Q^2} dt^2 + \frac{Q^2}{\bar{r}^2} d\bar{r}^2 + Q^2 d\Omega_2 \quad : \quad \text{AdS}_2 \times S^2 \quad \text{Throat}$$

radius Q

$$\text{AdS}_2 \text{ in "Poincaré coordinates"} \quad \frac{\bar{r}}{Q} = \frac{Q}{z}$$



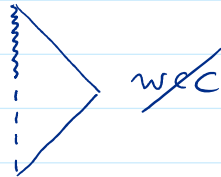
$$ds^2(\text{AdS}_2) = Q^2 \frac{-dt^2 + dz^2}{z^2}$$

horizon at $z = \infty$

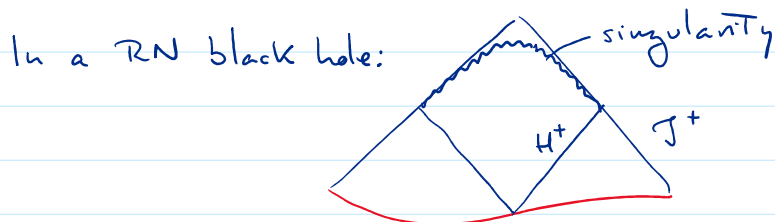
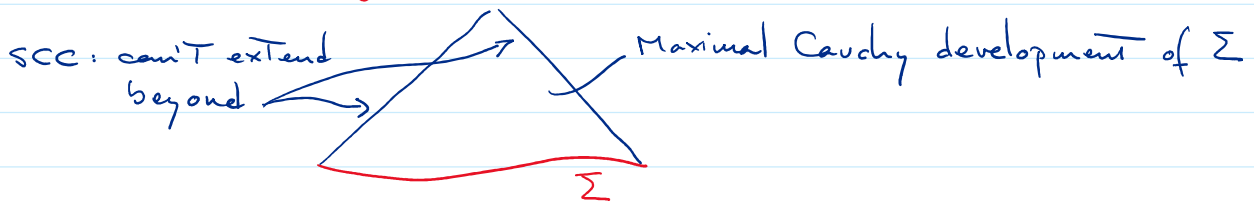
boundary at $z = 0$

Cosmic Censorship Conjectures (CCC)

Weak CCC: Starting from smooth initial data, the evolution following the classical Einstein equations, with reasonably physical matter, never forms a naked singularity.
Mathematically, $\mathcal{I}^+$ is complete

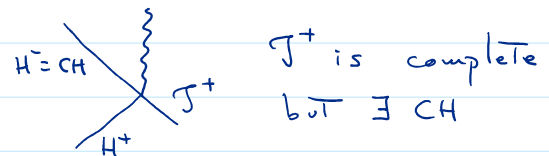


Strong CCC: maximal Cauchy evolution of smooth initial data following the Einstein eqs w/ reasonable matter is generically inextendible



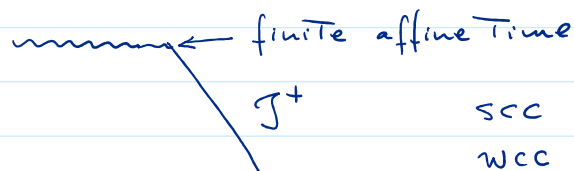
WCC and SCC are logically independent: weak CC $\nleftrightarrow$ strong CC

WCC $\nleftrightarrow$ SCC: inside a black hole



$\mathcal{I}^+$ is complete
but $\exists$ CH

SCC $\nleftrightarrow$ WCC: "sudden end of Time"



SCC not violated

WCC violated since $\mathcal{I}^+$ is incomplete

Most cases of WCC violations would be SCC violations.

SCC violations are most interesting when they're not WCC violations. Physically, then, SCC is a rather different than the (original) WCC.

. In the mathematical definitions $\begin{cases} \text{WCC: } \mathcal{I}^+ \text{ complete} \\ \text{SCC: } D^+(\Sigma) \text{ is inextendible} \end{cases}$

There's no mention of singularities, only of limits to what may be predicted from the classical evolution of a gravitating system.

The formulations avoid any characterization of the nature of the singularities.

This is

- Mathematically convenient
- Physically insufficient
 - We're interested in knowing whether we can access Planck-scale quantum gravity regimes
- When talking about singularities (that can be approached in a physical process) I'll always have in mind a **physical cutoff, most often at the Planck scale**

Then it'll be interesting to know how the Planck regime is approached — not simply that something, somewhere, some time, becomes ∞ — but what, where, when, how

Physical version

WCC: Starting from low energy densities, can we ^{from afar} access Planck-energy densities in a quick, classical ^{scale} time?

SCC $\rightarrow$ WCC: "Does time come to an end inside a black hole?"

May or not add a refinement about genericity. In my view, it is useful and important to consider the physics of non-generic phenomena.

• What is the status of the validity of these conjectures?

Weak CC: can be violated but only in very mild ways, so that

Weak CC : can be violated but only in very mild ways, so that evolution is very accurately predictable

Strong CC : classical fluctuations can create mild singularities at CH

but quantum fluctuations create strong singularities at CH

Strong CC is valid, for all we know

Rotating black holes (Kerr 1963)

Collapse typically involves angular momentum, so the geometry in the exterior of a collapsing object won't be Schwarzschild.

If the collapse doesn't produce a black hole, there are many possible exterior (vacuum) solutions. But if a bh forms, then the only possible final state, when all transients have died away, is the Kerr solution.

This is a consequence of the bh uniqueness theorems.

The first one was proven by Werner Israel for static black holes:

Israel: The only static solution of the $R_{\mu\nu} = 0$ that is regular on and outside an event horizon is the Schwarzschild bh

Do not confuse it w/ Birkhoff's Theorem: it's a very different result

Birkhoff: assume: sphericity $\Rightarrow$ staticity
Schwarzschild soln
no reference to regularity nor to horizons

Israel: assume: -staticity
-horizon $\Rightarrow$ sphericity
-regularity Schwarzschild bh

(none of them assumes asymptotic flatness)

Israel's original proof assumes non-degenerate horizon, $\kappa \neq 0$. This can be relaxed.

Carter-Robinson-Hawking: The only stationary, AF solution of $R_{\mu\nu} = 0$ that is regular on and outside a (non-degenerate) EH is the Kerr black hole w/ Mass M and angular momentum J .

In the presence of charge (Einstein-Maxwell Theory) the theorems extend to uniqueness of

-Reissner-Nordström (static) by Israel

-Kerr-Newman (stationary) by Bunting + Mazur

Moreover, these solutions are dynamically stable:

Moreover, these solutions are dynamically stable:

perturbative (linear) stability : $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$

solve for $\delta g_{\mu\nu} \Rightarrow$ perturbation decays in time

Proving non-linear stability is a much more difficult problem.

Moreover, the mathematical proof of the uniqueness theorem assumes analyticity, or a very high degree of differentiability, which is not very physical.

BUT in numerical simulations, the Kerr solution always appears as the unique, stable final state of collapse to a black hole, so even if the uniqueness and stability of the Kerr black holes may not have a rigorous mathematical proof, they are strongly believed to be true.

They have a striking consequence:

The Kerr solution gives the exact description of all astrophysical black holes in the universe

Kerr solution

$Q=1$

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - 2a \sin^2 \theta \frac{2Mr}{\Sigma} dt d\phi + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta d\phi^2 + \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2 \right)$$

$$\Delta = r^2 - 2Mr + a^2$$

M, a : parameters

$$\Sigma = r^2 + a^2 \cos^2 \theta$$

$a=0$: Schwarzschild

$$g_{\phi\phi} = \left(r^2 + a^2 + \frac{2Mr a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta$$

$$\frac{\Delta}{r^2} = 1 - \frac{2M}{r} + \frac{a^2}{r^2}$$

$\hookrightarrow$ centrifugal potential barrier

$\partial/\partial t$: Killing vector : stationary but not static (not h's orthogonal)
because of $g_{t\phi} \neq 0$

$\partial/\partial \phi$: " : axisymmetric

Mass: as $r \rightarrow \infty$ $-g_{tt} \approx 1 - \frac{2M}{r} + O(1/r^2)$ Mass = M

Angular momentum: $g_{t\phi} \approx -\frac{2J}{r} \sin^2 \theta + O(1/r^2)$ Angular momentum = J

Kerr $g_{t\phi} \approx -\frac{2Ma}{r} \sin^2 \theta + O(1/r^2)$ $J = Ma$

$$a = J/M$$

Discrete symmetries

- $a \rightarrow -a$ $\phi \rightarrow -\phi$: reverse sense of rotation
- $a \rightarrow -a$ $t \rightarrow -t$: reverse Time
- $t \rightarrow -t$ $\phi \rightarrow -\phi$

Will Take $a \geq 0$ w.l.o.g.

Metric has singularities at $\Sigma=0$ and $\Delta=0$ ($\theta=0, \pi$: symmetry axis)

$$\det g_{\mu\nu} = -\Sigma^2 \sin^2 \theta$$

$\Sigma=0 = r^2 + a^2 \cos^2 \theta$: curvature singularity (Riemann) $^2 \rightarrow \infty$
($r=0, \theta=\pi/2$)

$\Delta=0$: we'll see that there will be horizons

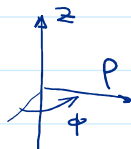
$r = r_{\pm} = M \pm \sqrt{M^2 - a^2}$ i.f. $r_{\pm} \in \mathbb{R}$ Then there are Killing horizons
 $M \geq a$

• Singularity $\Sigma = 0$ ($r=0, \theta=\pi/2$) : ring singularity

$$M=0 \quad a \neq 0$$

$$ds^2 = -dt^2 + \Sigma \left(\frac{dr^2}{r^2+a^2} + d\theta^2 \right) + (r^2+a^2) \sin^2\theta d\phi^2 \quad \text{flat space in ellipsoidal coordinates}$$

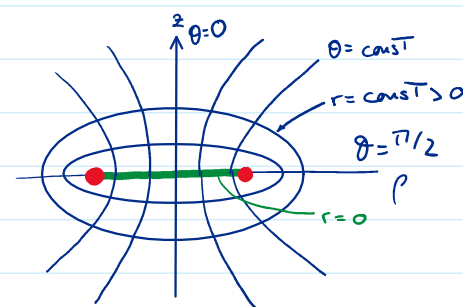
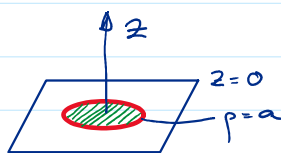
Change coords: $z = r \cos\theta$
 $(r, \theta) \rightarrow (z, \rho)$ $\rho = \sqrt{r^2+a^2} \sin\theta$



$$ds^2 = -dt^2 + dz^2 + d\rho^2 + \rho^2 d\phi^2$$

$$\Sigma = 0 \Rightarrow r=0 \quad \Leftrightarrow \quad z=0$$

$$\theta = \pi/2 \quad \rho = a$$



$$r=0$$

$$0 \leq \theta < \pi/2$$

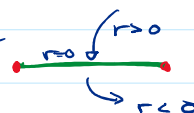


$$r=0, \theta=\pi/2$$

In The full Kerr solution $\Sigma=0$ is a ring singularity

In flat space $r \geq 0$ (and $r < 0$ is isometric)

In Kerr $r \neq -r$: $r > 0$ and $r < 0$ are not isometric and both are different and present in the solution



(see Hawking+Ellis for more details)

Small enough r : CTCs

$$\text{Orbits of } \frac{\partial}{\partial \phi} \text{ at constant } t \text{ and } \theta = \pi/2$$

$$r=0$$

$$\left| \frac{\partial}{\partial \phi} \right|^2 = g_{\phi\phi} \approx + \frac{2Ma^2}{r}$$

when $r < 0$ $\frac{\partial}{\partial \phi}$ is Timelike
 $\theta = \pi/2$

$\partial/\partial\phi$: circular orbits $\phi \sim \phi + 2\pi$: in $\downarrow$ These are closed Timelike curves

$$a > M, \quad M > a, \quad M = a$$

• $a > M$ $\Delta > 0$ $r_+ \notin \mathbb{R}$ $\nexists$ horizon and The singularity is naked
 CTCs are naked

$$J > M^2$$

$$(G=1, c=1)$$

Unphysical

$$J > M^2$$
$$(G=1, c=1)$$

Unphysical

bits are naked

- $M > a$: $r = r_{\pm} \in \mathbb{R}$ are null hypersurfaces, Killing horizons
 $J < M^2$ EF coords: $dr = dt + (r^2 + a^2) \frac{dr}{\Delta}$ $d\tilde{\phi} = d\phi + a \frac{dr}{\Delta}$

$$ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dr^2 + 2drdr - \frac{4Mar \sin^2 \theta}{\Sigma} d\tilde{\phi} dr - 2a \sin^2 \theta d\tilde{\phi} dr + \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma} \sin^2 \theta d\tilde{\phi}^2 + \Sigma d\theta^2$$

This is perfectly regular where $\Delta = 0$, at $r = r_{\pm}$

The vectors that become null at, resp., $r = r_{\pm}$ are

$$\xi_{\pm} = \frac{\partial}{\partial t} + \frac{a}{r_{\pm}^2 + a^2} \frac{\partial}{\partial \phi} \quad \xi_{\pm} \text{ is null at } r = r_{\pm}$$

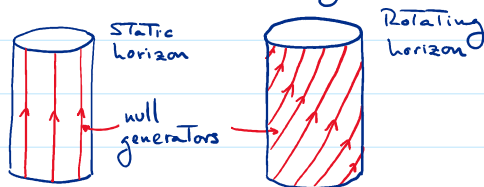
↪ linear combination w/ constant coeffs of Two Killings → Killing

$$|\xi_{\pm}|^2 = g_{tt} + \frac{2a}{r_{\pm}^2 + a^2} g_{t\phi} + \left(\frac{a}{r_{\pm}^2 + a^2}\right)^2 = 0 \quad \text{at } r = r_{\pm}$$

$$\chi_{\pm} = \frac{\pm \sqrt{M^2 - a^2}}{r_{\pm}^2 + a^2}$$

Notice that the Killing generator of the horizon is not $\partial/\partial t$.

This allows us to specify in what sense the black hole is rotating. The null geodesic generators are rotating relative to static asymptotic observers who follow trajectories with four-velocity $u = \frac{\partial}{\partial t}$



The null generators of the horizon are null orbits of the Killing vector

$$\xi_{\pm} = \xi_{\pm}^{\mu} \partial_{\mu}$$

On these orbits we have

$$\xi_{\pm}^{\mu} \partial_{\mu} \left(\phi - \frac{a}{r_{\pm}^2 + a^2} t \right) = 0$$

so the null generators satisfy $\phi = \frac{a}{r_{\pm}^2 + a^2} t + \text{const}$

$$\Rightarrow \frac{d\phi}{dt} = \frac{a}{r_{\pm}^2 + a^2} = \Omega_{\pm} \text{ is the velocity of rotation of the horizon(s).}$$

This is the velocity relative to observers that are static relative to $\partial/\partial t$ i.e. static observers at infinity

CP diagram

Cannot draw a diagram that captures all since we don't have full spherical symmetry but only axial symmetry. The geometry depends not only on r but also on θ and we'd need three dimensions (t, r, θ) to capture all the causal structure (and even then it wouldn't be quite clear).

Draw sections: along the symmetry axis $\theta = 0, \pi$

(doesn't intersect the singularity $r=0$ $\theta = \pi/2$)

$$ds^2|_{\text{axis}} = - \frac{1 - \frac{2M}{r} + \frac{a^2}{r^2}}{1 + \frac{a^2}{r^2}} dt^2 + \frac{1 + \frac{a^2}{r^2}}{1 - \frac{2M}{r} + \frac{a^2}{r^2}} dr^2 \quad : \text{non-singular at } r=0 \text{ but otherwise similar to RN}$$

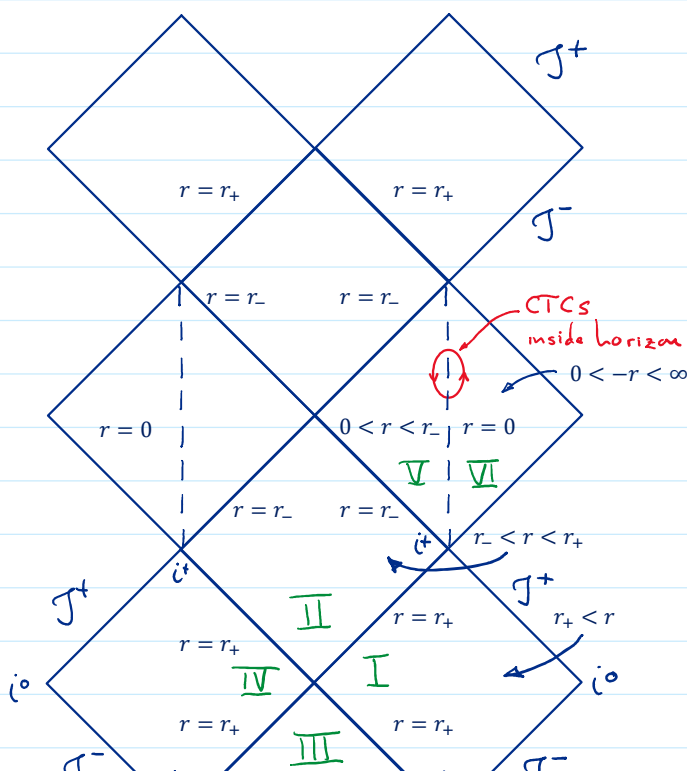
Near $r=0$ this is $ds^2|_{\text{axis}} \approx -dt^2 + dr^2$ which is non-singular (center of ring)

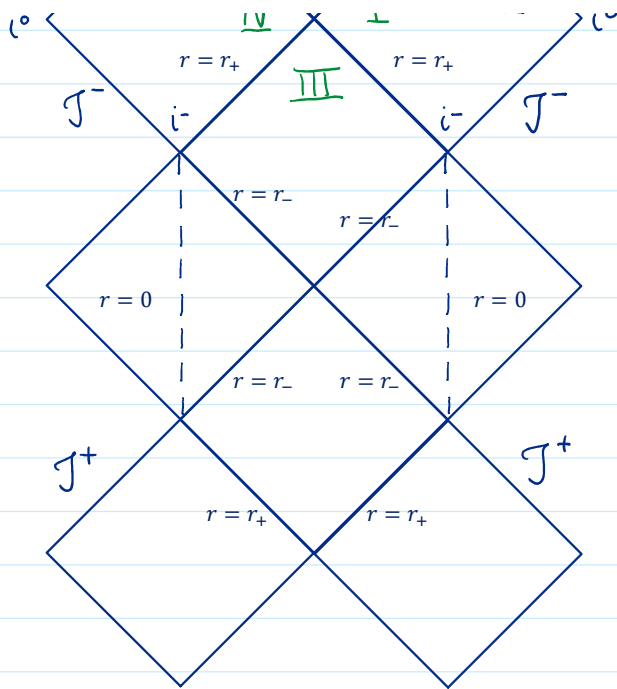
The term M/r allows to extend into $r < 0$ in a non-isometric way.

When $r \rightarrow -\infty$ the geometry is asymptotically flat and actually it looks like it had negative mass but not singular at $r=0$

(In the full geometry, the ring singularity would be naked in the region of $r < 0$)

Away from $|r| \lesssim a$ the geometry is much like Reissner-Nordström





$r = r_+$: event horizon

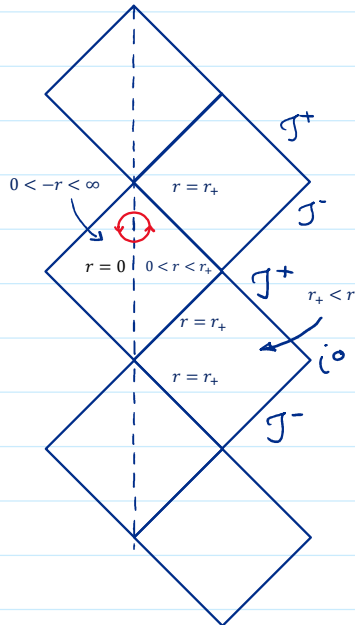
$r = r_-$: Cauchy horizon : unstable to fluctuations, will become singular
(strongest singularity comes from QFT divergences)
Regions w/ CTCs V and VI will be removed

In realistic bls from collapse only I and part of region II will be present

Can also draw a diagram along a radial line in the equatorial plane.
This is like Reissner-Nordström.

Extremal Kerr solution: $M=a$. Degenerate horizon $\kappa=0$, $\Omega_H = \frac{1}{2a}$

Axial section $\theta=0, \pi$



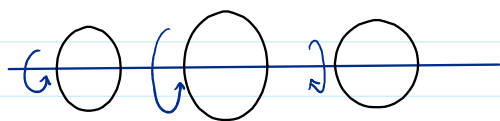
Near the horizon the solution develops an infinite throat, but the geometry does not become $AdS_2 \times S^2$ but instead a warped product
(Bardeen+Horowitz hep-th/9905097)

This is not a supersymmetric solution.

∃ extremal multibh solutions that are non-regular on and outside the event horizons. The reason is that there's no exact cancellation of forces analogous to Newton $\leftrightarrow$ Coulomb.

There is a spin-spin repulsion for aligned spins, but the force doesn't fall-off like $1/r^2$, instead like $1/r^3$, which is not enough.

So there do exist exact solutions with arbitrarily many aligned Kerr bhs



but there are conical singularities along the axis that appear in order to counterbalance the attraction

Consider $\xi = \frac{\partial}{\partial t}$ observers That follow orbits of ξ remain static relative to ∞

$$\xi^\mu \xi_\mu = g_{tt} = -\left(1 - \frac{2Mr}{\Sigma}\right) \quad \Sigma = r^2 + a^2 \cos^2 \theta$$

ξ is Timelike where $\Sigma > 2Mr$ $r^2 + a^2 \cos^2 \theta - 2Mr > 0$

$$\Rightarrow r > r_e = M + \sqrt{M^2 - a^2 \cos^2 \theta}$$

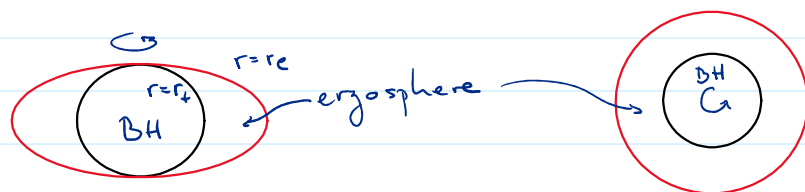
This does not coincide with the event horizon, but is outside of it

$$r_+ = M + \sqrt{M^2 - a^2} \leq r_e$$

They coincide only on the axis $\theta = 0, \pi$

The region $r_+ < r < r_e$ is called the ergosphere (or ergoregion)

from above



In the ergosphere ξ is spacelike: $\nexists$ static observers (relative to ∞)

Inside the ergosphere no observer can follow a trajectory parallel to $\xi = \frac{\partial}{\partial t}$, which would be static w.r.t. infinity. So in the ergosphere no observer can remain at rest relative to infinity. They necessarily move along ϕ , since they are dragged by the rotation of the black hole. We can argue this in a bit more detail:

$g_{tt} > 0$ in the ergosphere

Observers w/ velocity u^μ : $g_{\mu\nu} u^\mu u^\nu = -1$

$$u = u^t \partial_t + u^\phi \partial_\phi + u^r \partial_r + u^\theta \partial_\theta$$

$$g_{tt} (u^t)^2 + 2g_{t\phi} u^t u^\phi + g_{\phi\phi} (u^\phi)^2 + (u^r)^2 g_{rr} + (u^\theta)^2 g_{\theta\theta} = -1$$

since in the ergosphere $g_{tt} > 0$, and outside the horizon $g_{\phi\phi}, g_{rr}, g_{\theta\theta} > 0$.

Then we must have

$$g_{t\phi} u^t u^\phi < 0 \text{ (strictly)}$$

For positive rotation, $g_{t\phi} < 0$. Then $u^t u^\phi > 0$. Future directed is $u^t > 0$
so then $u^\phi > 0$

Then $\Omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t} > 0$: must rotate in the same sense as the black hole

The Killing vector $\xi_t = \partial_t + \Omega_H \partial_\phi$ $\left(\Omega_H = \frac{a}{r_+^2 + a^2} \right)$
is corotating with the horizon

Extracting rotational energy out of The black hole

Particle four-momentum p^μ : energy $E = -\xi_\mu p^\mu$ $\xi = \frac{\partial}{\partial t}$

This is The energy relative To asymptotic static frames.

At ∞ ξ, p are Timelike. No negative energies, $E > 0$.

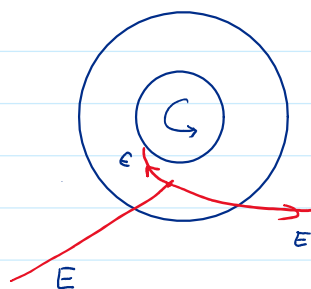
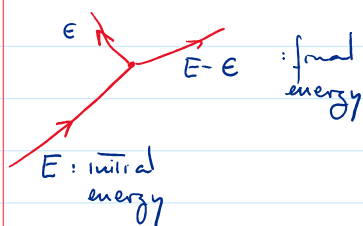
But in The ergosphere ξ is spacelike. Then for a particle there, The component $E = -\xi^\mu p_\mu$ is a spatial component of four-momentum, i.e. a momentum, and momentum can be negative (The locally measured energy of The particle will be $-u^\mu p_\mu$, and This will be positive).

So for a particle within The ergosphere, The energy conjugate To t (asymptotic time) can be negative. i.e., in The ergosphere one can have particles, or excitations, of negative energy (They're outside The horizon, but as long as They have -ve energy They can't escape To ∞ . For this, They must acquire some more energy such That $-\xi^\mu p_\mu$ becomes positive).

The possibility of negative energies in a region outside The horizon allows a mechanism for extracting energy out of The black hole:

particle fission

Fission in The ergosphere

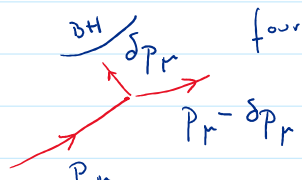


May have $E < 0$

$E - E = E + |E| > E$: particle comes out w/ more energy than it came in!

In more detail :

four-momentum conservation : $p_\mu = \delta p_\mu + (p_\mu - \delta p_\mu)$



contract w/ ξ : $E = E + (E - E)$

$$-\delta p_\mu \xi^\mu = E$$

$$\delta p_\mu (\partial_\mu)^r = j \quad \text{where } j \text{ is the spin of the particle}$$

If the particle w/ δp_μ enters the horizon towards the future, then we must have

$$-\xi^\mu \delta p_\mu \geq 0 \quad \text{w/ } \xi = \partial_t + \Omega_H \partial_\phi \text{ horizon generator}$$

since the locally measured energy is positive

$$\Rightarrow E - \Omega_H j \geq 0 \quad \text{ie } j \leq \frac{E}{\Omega_H} \quad (\text{we assume wlog } \Omega_H > 0)$$

so if $E < 0$ then $j < 0$: The particle must go into the horizon rotating more slowly than the bh (locally counterrotating)

The bound can be written as

$$|E| \leq \Omega_H |j|$$

ie in order to dump a certain amount of negative energy, there must be a minimum of negative angular momentum dumped as well.

When absorbing the particle, the mass and spin of the black hole will change

$$M \rightarrow M + \delta M$$

$$\delta M = E < 0$$

$$J \rightarrow J + \delta J$$

$$\delta J = j < 0$$

: black hole loses energy and spins down

$$\Rightarrow \delta J \leq \frac{\delta M}{\Omega_H} \quad \text{and for } \delta M < 0 \text{ we get}$$

$$|\delta J| \geq \frac{|\delta M|}{\Omega_H} \quad : \text{extracting energy means that there's at least a minimum of spin that must be extracted too.}$$

One can show that there actually exist geodesics that allow to do the job.

If we're reducing the mass of the bh, is this compatible with the area law $\delta A_H \geq 0$?

Yes:

$$r_+^2 + a^2 = 2Mr_+$$

$$A_H = \int d\phi d\theta \sqrt{g_{\theta\theta} g_{\phi\phi}} \Big|_{r=r_+} = 4\pi(r_+^2 + a^2) = 8\pi M r_+ = 8\pi(M^2 + \sqrt{M^4 - J^2})$$

Differentiate:

$$\delta A_H = 8\pi \frac{a}{\sqrt{M^4 - J^2}} \left(\frac{\delta M}{\Omega_H} - \delta J \right) = \frac{8\pi}{\kappa_+} (\delta M - \Omega_H \delta J)$$

$$\text{Area law: } \delta A_H \geq 0 \Rightarrow \delta M \geq \Omega_H \delta J$$

$$\text{but in the ergoregion } \epsilon = \delta M < 0$$

$$|\delta M| \leq \Omega_H |\delta J|$$

Exactly the same bound as above!

So the Penrose mechanism is actually exactly compatible w/ area law.

Spin is extracted at a rate that keeps the area non-decreasing.

How much energy can be extracted, if we try to do it most efficiently, i.e. so that $\delta A_H = 0$? (reversible, adiabatic process)

Irreducible mass of a rotating black hole:

$$\delta A_H = 0$$

when all rotational energy has been extracted there remains an "irreducible mass"

$$M_{\text{irr}}^2 = \frac{A_H}{16\pi} = \frac{r_+^2 + a^2}{4} = \frac{1}{2} (M^2 + \sqrt{M^4 - J^2})$$

Maximum energy that can be extracted is $M - M_{\text{irr}}$

Maximum efficiency of Penrose energy extraction is

$$\epsilon = \frac{M - M_{\text{irr}}}{M} = 1 - \frac{1}{\sqrt{2}} \left(1 + \sqrt{1 - J^2/M^4} \right)$$

$$\text{when } M^2 = J \Rightarrow \epsilon = 1 - \frac{1}{\sqrt{2}} \approx 30\%$$

This was done with matter particles. There's a version of it using waves:

Superradiant scattering (Starobinsky + Zeldovich)

Send a massless scalar wave into a rotating black hole. Modes with a certain frequency ω and angular momentum m can come out with an amplitude higher than the initial.

That is

$$\Phi = f(r, \theta) e^{-i\omega t + im\phi} \quad \omega / \omega > 0 \quad (\text{positive frequency, future-propagating positive energy quanta})$$

It's easy to prove that

if $\omega \leq m\Omega_H$ the wave gets amplified

This is the superradiant amplification bound.

If we identify $|E| = \omega$ and $|J| = m$, it's the same as the Penrose bound.

Superradiant amplification as stimulated emission of radiation

Superradiance can be interpreted as stimulated emission of radiation. It is related to spontaneous emission of radiation (Hawking radiation) by the usual relation between Einstein A and B coefficients, i.e. by detailed balance. Equivalently, if a mode (ω, l, m) has a decay rate $\Gamma_{em}(\omega)$, detailed balance requires that when there's an incident flux of the field F_{in} , we have

$$F_{in} \sigma_{em}(\omega) + \Gamma_{em}(\omega) = 0$$

where $\sigma_{em}(\omega)$ is the absorption cross section of the mode.

Superradiant modes spontaneously decay, so $\Gamma_{em}(\omega) > 0$ and so $\sigma_{em}(\omega) < 0$.

One can prove using flux conservation that

$$F_{out} - F_{in} = C_\ell(\omega) F_{in} \sigma_{em}(\omega)$$

where $C_\ell(\omega) > 0$ is a universal coefficient (essentially a conversion factor between plane waves for the fluxes, and spherical waves for the absorption $\sigma_{em}(\omega)$).

Then, since $\sigma_{em}(\omega) < 0$, we have $F_{out} > F_{in}$: the outgoing wave is amplified.

For large occupation numbers, stimulated emission is a classical process. Spontaneous emission is always quantum.

See hep-th/0712.0791 for a microscopic interpretation of superradiance.

For RN black holes

$$ds^2 = -V(r) dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega^2, \quad V = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

$$A_t = -\frac{Q}{r} + \Phi \quad \Phi = \frac{Q}{r_+} \quad \text{so } A_t(r=r_+) = 0$$

$$A_H = 4\pi r_+^2$$

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}$$

$$\kappa = \frac{\sqrt{M^2 - Q^2}}{r_+^2}$$

$$dM = \frac{1}{8\pi} \kappa dA_H + \Phi dQ$$

1st law

We may use r_+, r_- as parameters

$$Q^2 = r_+ r_-$$

$$M = \frac{r_+ + r_-}{2}$$

$$= \frac{r_+ + Q^2/r_+}{2}$$

$$= \frac{r_- + Q^2/r_-}{2}$$

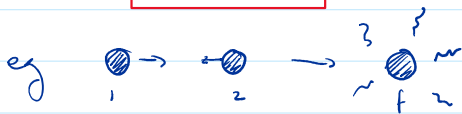
$$\begin{aligned} \left(\frac{\partial M}{\partial A_+}\right)_Q &= \left(\frac{\partial M}{\partial r_+}\right)_Q \frac{\partial r_+}{\partial A_+} + \left(\frac{\partial M}{\partial r_-}\right)_Q \frac{\partial r_-}{\partial A_+} \stackrel{=0}{=} \frac{1}{2} \left(1 - \frac{Q^2}{r_+^2}\right) \frac{1}{2\sqrt{4\pi A_+}} \\ &= \frac{1}{16\pi} \frac{r_+^2 - Q^2}{r_+^3} = \frac{1}{8\pi} \frac{r_+ - r_-}{2r_+^2} = \frac{\kappa_+}{8\pi} \end{aligned}$$

$$\left(\frac{\partial M}{\partial Q}\right)_{A_+} = \left(\frac{\partial M}{\partial r_-}\right)_{A_+} \left(\frac{\partial r_-}{\partial Q}\right)_{A_+} = \frac{1}{2} \frac{2Q}{r_+} = \frac{Q}{r_+} = \Phi_+$$

2nd Law (Hawking)

The area of The horizon(s) cannot decrease in Time

$$\Delta A_H \geq 0$$



$$A_1 + A_2 \leq A_f$$

0th Law: κ is uniform over horizon

Striking analogy:

Laws **Thermodynamics**

BH mechanics

0th $T = \text{const}$ at equilibrium

$\kappa = \text{const}$ over stationary horizon

1st $dE = TdS - pdV + \Omega dJ + \Phi dQ$

$dM = \frac{\kappa}{8\pi G} dA_H + \Omega dJ + \Phi dQ + \dots$

2nd $\Delta S \geq 0$

$\Delta A_H \geq 0$

BUT Their derivations couldn't be more different!

Carnot cycles
Heat reservoirs

Null geodesics
Killing vectors, surface integrals

5

Thermodynamics in the presence of black holes looks bizarre.

Could dump entropy into a bh and lose it from sight, entirely and forever!

[Could also extract the energy of a box w/ radiation (Geroch process) and then dump the entropy into the black hole]

Bekenstein: The area of a black hole is a form of entropy

$$S_{BH} = C \frac{A_H}{G\hbar} \sim \frac{A_H}{l_{Pl}^2} \sim 1 \text{ bit per unit Planck area}$$

Generalized 2nd law: $\Delta(S_{matter} + S_{BH}) \geq 0$

There should be a temperature $T_H = \frac{\hbar \kappa}{8\pi C}$

$$\frac{\kappa}{8\pi G} A_H = TS$$

Puzzle:

If black holes have entropy, they should have a temperature. If they have a temperature, then they should be able to emit thermal radiation.

BUT black holes do not emit anything at all. They're at absolute zero temperature! Or are they?

Hawking set out to prove that Bekenstein was wrong: a quantum system in an equilibrium black hole spacetime should be absolutely cold.

[BUT he actually proved the opposite! When considering a collapsing system, initially a quantum field is parametrically excited by the dynamical geometry. But when it settles down into a stationary equilibrium black hole, the field continues to be excited, and actually there's a continuous

geometry. But when it settles down into a stationary equilibrium black hole, the field continues to be excited, and actually there's a continuous outflux of radiation with a thermal spectrum with a temperature

$$T_H = \frac{\hbar \kappa}{2\pi}$$

This determines $C = \frac{1}{4}$ and therefore fixes

$$S_{BH} = \frac{A_H}{4G\hbar} \quad : \text{Bekenstein-Hawking entropy}$$

This formula combines gravitation, quantum theory, and thermodynamics in a deep and mysterious way. It's our first formula for quantum gravity, and our most important clue into a quantum theory of gravity.

From 2306.11139 Sec.4

4. The black hole that evaporates

In 1974 Hawking studied quantum fields propagating on a geometry that collapses to form a black hole. He found out that during the process of collapse, quantum radiation is emitted, as expected in a time-dependent situation. But more surprisingly, after the black hole has settled into a stationary configuration and the transient effects have died out, there remains a steady outflow of radiation, which observers at a large distance detect as a black body spectrum¹⁷ with temperature

$$T_H = \frac{\hbar\kappa}{2\pi}, \quad (4.1)$$

4.1 Particle production in an external field

We intend to study the production of particles in a given background field, due to fluctuations of a quantum field. Virtual quantum fluctuations of a field with mass m are described by

$$\langle\phi(x)\phi(0)\rangle \sim e^{-x\frac{mc}{\hbar}}. \quad (4.2)$$

This is the probability amplitude that a pair of field-quanta separated by a distance x would spontaneously form. In vacuum, these fluctuations do not materialize in the production of a real pair since that would violate energy conservation. But the energy required for this materialization can be provided if there is an external field to which the quantum field couples.

Let us denote the field strength (force per unit charge) by F , and the coupling (charge) by λ . Assuming the field is uniform, in order to materialize the pair of quanta, we need the equality

$$\lambda Fx = 2mc^2, \quad (4.3)$$

for energy conservation to hold. Then the probability for a pair creation per unit volume and unit time is given by

$$\Gamma \sim |\langle\phi(x)\phi(0)\rangle|^2 \sim e^{-\frac{4m^2c^3}{\hbar\lambda F}}. \quad (4.4)$$

A proper calculation in quantum field theory involves a tunneling process (hence the exponential suppression) which can be evaluated in the WKB approximation, and indeed gives

$$\Gamma \sim A e^{-\gamma\frac{\pi m^2c^3}{\hbar\lambda F}}, \quad (4.5)$$

where A is the quantum one-loop determinant factor, which we will ignore in the following (it is not easy to compute, and yields subdominant corrections), and γ is a numerical factor of order one, which depends on the specific type of particle and its coupling to the field.

This is a process of pair creation by a background field, where the latter is a semi-classical, coherent state involving a large number of quanta in the background. The process is described in terms of a

This is a process of pair creation by a background field, where the latter is a semi-classical, coherent state involving a large number of quanta in the background. The process is described in terms of a non-perturbative instanton bounce. It is different than the perturbative process of pair creation, e.g., e^+e^- creation by photon-photon collision. The non-perturbative e^+e^- production in a background electric field was studied by Schwinger in a classic paper in 1950. In this case $\lambda = e$, $F = E$, and $\gamma = 1$. Schwinger's leading order result yields

$$\Gamma_{e^+e^-} \sim A e^{-\frac{\pi m^2 c^3}{\hbar e E}}. \quad (4.6)$$

The energy for the creation of the pair is provided by the background field, which as a result decays gradually.

A black hole creates a strong gravitational field, so we might also expect particle pairs to form near the horizon. In this case the coupling is the particle's mass, while for the force we take the surface gravity,

$$\lambda = m, \quad F = \kappa. \quad (4.7)$$

With these choices, we find that

$$\Gamma \sim e^{-\gamma \frac{\pi m c^3}{\hbar \kappa}}. \quad (4.8)$$

Observe that the exponent is proportional to m , i.e., to the energy $E = mc^2$ of the particle. Thus we can write it as

$$\Gamma \sim e^{-E/T_H}, \quad (4.9)$$

where

$$T_H = \frac{\hbar \kappa}{\gamma \pi c}. \quad (4.10)$$

This is a *thermal spectrum with temperature* $T_H \propto \kappa$. So, the black hole is expected to radiate like a blackbody. In contrast, the Schwinger production rate (4.6) is not thermal. The reason that it is thermal in the case of a black hole is that gravity couples to the particle's energy. It is very suggestive that the universal character of gravity appears to be related to a universal thermal behavior.

Some caveats about this heuristic argument:

- We have not pinned down the value of γ . For this we need Hawking's proper calculation, which yields $\gamma = 2$.
- The argument was made for massive particles. However, Hawking's result applies as well to massless quanta, with $E = \hbar\omega$.
- The energy required by the creation of the pair is supplied by the black hole. The energy of the black hole decreases, since one of the members of the pair has negative energy (relative to asymptotic observers) and falls inside the black hole. In more detail, if the pair have four-momenta p_1 and p_2 , four-momentum conservation requires that

$$p_1 + p_2 = 0. \quad (4.11)$$

If ∂_t is the generator of asymptotic time-translations, so that t is conjugate to the energy measured by asymptotic observers, then the energy of a particle with four-momentum p is

If ∂_t is the generator of asymptotic time-translations, so that t is conjugate to the energy measured by asymptotic observers, then the energy of a particle with four-momentum p is

$$E = -p \cdot \partial_t. \quad (4.12)$$

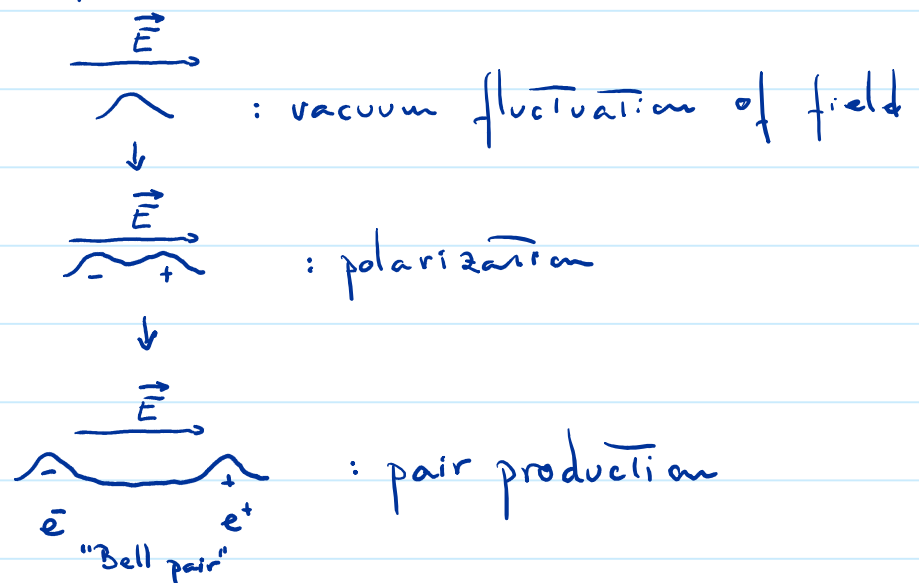
Thus for the particle pair we must have $E_1 + E_2 = 0$. Now, if particle 1 is to escape to infinity, it must have $E_1 > 0$. If particle 2 goes inside the black hole, then in that region ∂_t is spacelike, so E_2 is actually not an energy but a component of momentum, which can be negative. Thus it is consistent to create the pair if one of the particles falls inside the black hole. In this case, the total mass of the black hole will decrease by an amount

$$\delta M = E_2 = -E_1, \quad (4.13)$$

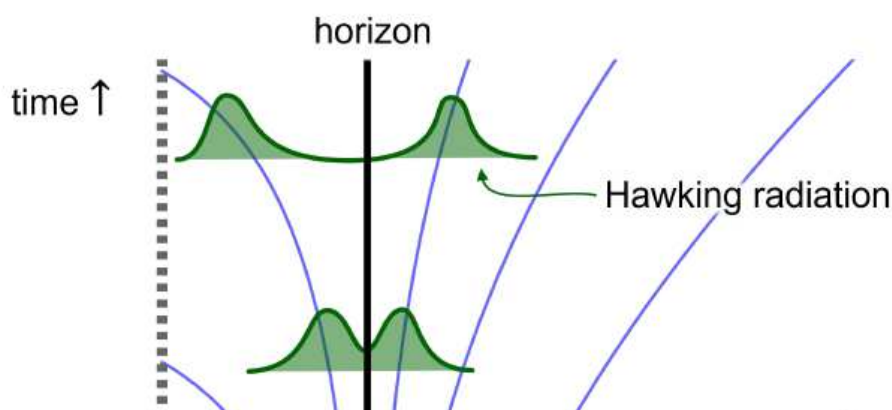
consistent with conservation of the energy as measured by outside observers.

Electron-positron pair production:

Electric field polarizes vacuum:



How does This happen in a black hole?



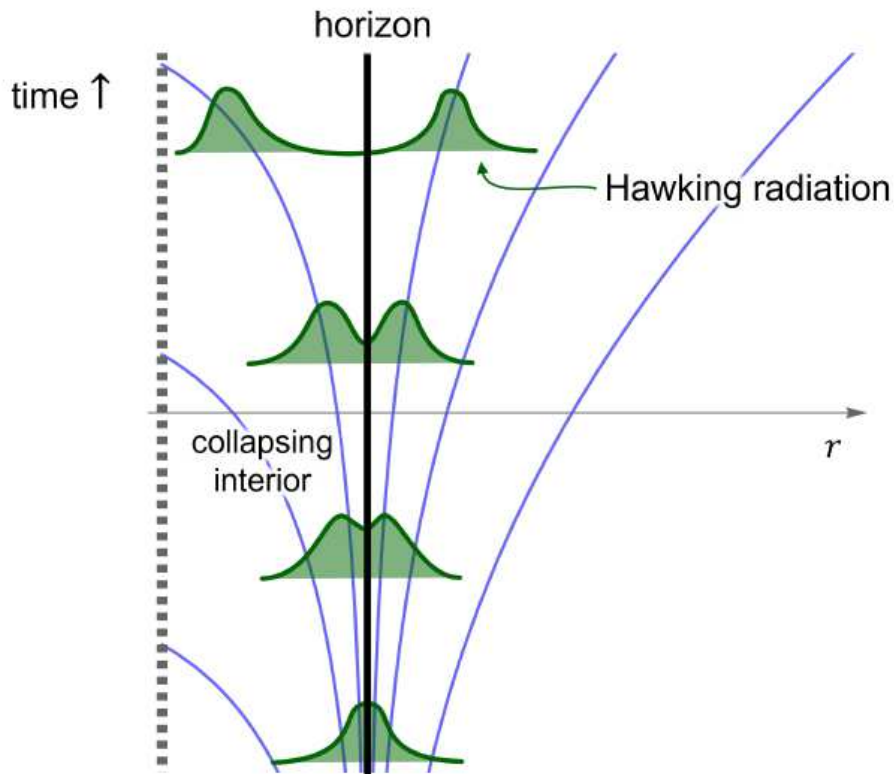


Figure 17: Hawking emission as quantum pair production, in an Eddington-Finkelstein diagram (cf. fig. 2). A quantum wavefunction initially straddling the horizon is stretched by the collapse of the black hole interior, until one of its components escapes away as a Hawking quantum and the other falls to the singularity. The pair is created in an entangled Bell-like state.

Let us provide yet another viewpoint on why quantum physics in the presence of a black hole horizon gives rise to what may seem an unexpected consequence. A classical particle outside a stationary black hole perceives that nothing changes. A quantum field can be excited when the geometry that it lives in is time-dependent, a phenomenon of parametric excitation well-known in cosmology. The exterior of the black hole is static, and so naively one might conclude that no field excitation will occur. However, a quantum wave function can have support simultaneously in the exterior and the interior of the black hole. This is a crucial property, since the interior geometry of a black hole is dynamically collapsing (a Little Big Crunch) in a time scale $\sim \kappa^{-1}$ (in units $c = 1$). A quantum field will be sensitive to the time dependence of the collapsing interior and therefore will be excited. The characteristic time of this collapse implies that the field excitations will have typical frequency $\sim \kappa$, and the energy of the produced quanta will be $\sim \hbar\kappa$, which agrees with (4.10).

The part of the wave function that is in the interior will be dragged in by the collapse, and stretched until it becomes a wave packet separated from its exterior partner (see figure 17). This produces a pair of quanta, with positive and negative energies relative to asymptotic time. The quantum that is radiated away will be entangled with the quantum in the interior, since they are part of the same wave function, so they will be in a maximally entangled Bell-like state.

Observe also that, according to this interpretation, a black hole with a time-independent interior (static or stationary) must not give rise to Hawking radiation. This is indeed what happens in extremal black holes, which have $\kappa = 0$ and therefore $T_H = 0$ too. Nevertheless, extremal black

holes can decay through non-thermal spontaneous emission of superradiant modes (3.18).

4.2 Further aspects of Hawking radiation and black hole evaporation

We can now draw two major consequences: black holes must have entropy, and they will evaporate.

Back to the 19th-century, with a black hole. Rudolf Clausius explains to us that black holes must be assigned an entropy on very general phenomenological grounds: an object with energy E that radiates at a temperature T has an entropy S given by

$$S = \int \frac{dE}{T}. \quad (4.14)$$

Let us see how this works. As the seasoned experimentalists that we are, we can measure the energy (mass) of the black hole and its temperature from far away without even knowing that the radiating object is a black hole. We proceed to collect data using a dynamometer and a bolometer and thus we obtain $T(M)$. Our measurements yield

$$T = \frac{\hbar}{8\pi G M}, \quad (4.15)$$

which, using that $E = M$, we plug into (4.14) to find

$$dS = \frac{8\pi G}{\hbar} M dM, \quad (4.16)$$

$$\Rightarrow S = \frac{4\pi G}{\hbar} M^2, \quad (4.17)$$

When we work out the numbers we are at first surprised to find that this entropy is (as we will see in a moment) enormous. But we attribute it to having a peculiar object that has a very large energy but is radiating at an extremely low temperature.

By now we have also figured out, e.g., by scattering particles or waves, that the radiating object is a black hole whose radius we have measured to be $r_H = 2GM$. Then we realize that we can write

$$S = \frac{4\pi(2GM)^2}{4G\hbar} = \frac{\mathcal{A}_H}{4G\hbar} \quad (4.18)$$

$$= \frac{\text{Horizon area}}{\text{Planck area}}. \quad (4.19)$$

Therefore, the entropy of the black hole is nothing but its area measured in Planck units—truly, an enormous number for any macroscopic area! But then Clausius looks puzzled at us: why should entropy—a quantity that he introduced for understanding the efficiency of heat exchanges—bear any relation with that most basic entity, the geometry of space and time?

The identification of the black hole area with an entropy was first proposed—very boldly, and not without controversy—in 1973 by Bekenstein. It was then put on a firm footing by Hawking's discovery that black holes emit thermal radiation with a precise temperature. For this reason,

$$\boxed{S_{BH} = \frac{\mathcal{A}_H}{4G\hbar}} \quad (4.20)$$

is called the Bekenstein-Hawking Black Hole entropy formula. It contains G and $\hbar$, and over the last half-century it has provided the deepest and most fruitful guidance towards a quantum theory of gravity.

Now we go to interrogate other 19th-century physicists about the implications of this result. Ludwig Boltzmann informs us that, ultimately, it implies the existence of many microscopic states corresponding to the macroscopic system that we characterize by this mass and temperature i.e., the black hole. Indeed, he continues, there must be as many as $e^{S_{BH}}$ states! In other words, a black hole—which is nothing but strongly warped space and time—must somehow be made of a humongous number of microscopic degrees of freedom, even if we do not see them at all.

Boltzmann raises an eyebrow and asks: if we are to think of these degrees of freedom as somehow related to ‘atoms of spacetime’ (an idea he seems to relish), shouldn’t their number scale like the spatial volume, instead of the area? He begins to warn us about the tribulations of applying statistical reasoning to entities that have long been regarded as paradigms of determinism, but it is time that we take leave of him (some of his concerns will reappear later) and continue with other consequences of Hawking’s discovery.

Evaporation rate and black hole lifetime. The black hole will evaporate by emitting radiation like a blackbody. The radiating power of a blackbody of area A and temperature T is

$$\frac{dE}{dt} = \sigma AT^4, \quad (4.21)$$

where σ is the Stefan-Boltzmann factor, which depends on the specific (effectively massless) fields that are being radiated. For a real scalar field, $\sigma = \pi^2/60$. As a first approximation we can take $A \simeq \mathcal{A}_H$. For a Schwarzschild black hole,

$$\mathcal{A}_H = 16\pi M^2, \quad T = \frac{1}{8\pi M}, \quad (4.22)$$

and since $dE = -dM$ we can obtain the evaporation rate of the black hole as

$$\frac{dM}{dt} \simeq -\frac{\sigma}{256\pi^3} \frac{1}{M^2} \propto M^{-2}, \quad (4.23)$$

so that the total evaporation time derived from

$$\int_0^{t_{\text{evap}}} dt \propto -\int_M^0 dM' M'^2, \quad (4.24)$$

is simply

$$t_{\text{evap}} \propto M^3. \quad (4.25)$$

It is then obvious that the black hole evaporates in a finite time. This calculation is of course very rough since it neglects the back-reaction effect that the emission of radiation and loss of mass have on the black hole geometry and on the radiation process itself. These effects should be small when the energy of emitted quanta is much smaller than the mass of the black hole,

$$T_H \ll M, \quad (4.26)$$

i.e., as long as $M \gg M_{\text{Planck}}$. Therefore, for most of the black hole lifetime the approximation is good, and its mass will reach Planck size in a time $\propto M^3$.

Generalized Second Law (Bekenstein 1972). The second law of thermodynamics is extended to include the contributions from black hole entropy so that

$$\Delta S_{\text{total}} = \Delta S_{BH} + \Delta S_{\text{matter}} \geq 0. \quad (4.31)$$

Each of the two contributions can separately be negative, but only as long as their total sum is positive.

There is a 2nd law of black holes, proven by Hawking, that applies in the classical theory:

$$\Delta A_H \geq 0.$$

"BH area Theorem": The total area of the event horizon cannot decrease in time.

The GSL reduces to this in the limit $t_h \rightarrow 0$

since $S_{BH} = \frac{A_H}{4Gt_h} \propto \frac{1}{t_h}$ while $S_{\text{matter}} \propto t_h^0$

so

$$S_{\text{Tot}} = \frac{A_H}{4Gt_h} + S_{\text{matter}} \xrightarrow{t_h \rightarrow 0} \frac{A_H}{4Gt_h}$$

In the process of collapse to a black hole, $\Delta S_{\text{matter}} < 0$ but the entropy S_{BH} that is produced overwhelms this decrease by far. A simple estimate for the collapse of a weakly-gravitating ball of radiation gives

$$\frac{\Delta S_{BH}}{|\Delta S_{\text{matter}}|} \sim \left(\frac{M}{M_{\text{Planck}}} \right)^{1/2} > 1, \quad (4.32)$$

which is in fact $\gg 1$ for any astrophysical black hole. In order to understand the huge increase in the entropy, observe that the ordinary matter before the collapse naturally probes only a small, low-energy subset of all the degrees of freedom of the fundamental theory (e.g., string theory), so it has a low entropy. The black hole, instead, samples all the stringy (fundamental) states available.²⁰

On the other hand, in the process of black hole evaporation we have $\Delta S_{BH} < 0$, but the radiation carries sufficient entropy to compensate for it. Again a simple calculation for the entropy production when massless radiation is emitted can be seen to yield

$$\frac{\Delta S_{\text{rad}}}{|\Delta S_{BH}|} \simeq \frac{4}{3} > 1. \quad (4.33)$$

Observe that in this case the entropy produced is of the same order as the entropy of the black hole. The reason is that the evaporation produces a number of quanta $N \sim S_{BH}$. The factor $4/3$ comes from the relation $s_{\text{rad}} = \frac{4}{3}\epsilon_{\text{rad}}T$ between the entropy and energy density of massless radiation at temperature T (in D spacetime dimensions, $s_{\text{rad}} = \frac{D}{D-1}\epsilon_{\text{rad}}T$). The number is guaranteed to be > 1 for any form of radiation or matter that is thermodynamically stable.

Black holes in imaginary Time

For studying QFT at finite Temperature we consider the QFT in $t \rightarrow i\tau$

$$ds^2 = \underbrace{d\tau^2}_{S^1} + \underbrace{dx^2 + dy^2 + dz^2}_{\mathbb{R}^3}$$

$$\tau \sim \tau + \beta$$

$$S^1 \times \mathbb{R}^3$$



KMS condition: correlation functions in QFT at finite Temperature are periodic in imaginary Time

To prove this

Thermal expect. value

$$\langle O \rangle_\beta = \text{Tr} \left(\frac{e^{-\beta H}}{Z} O \right)$$

Euclidean Heisenberg picture evolution

$$O(\tau_0 + \tau) = e^{H\tau} O(\tau_0) e^{-H\tau}$$

Euclidean Time-ordered Two-point correlation fn at Temperature β^{-1} :

$$\begin{aligned} \zeta_p(\tau) &= \frac{1}{Z} \text{Tr} \left(e^{-\beta H} T(\phi(0) \phi(\tau)) \right) & \tau = \tau_2 - \tau_1 & \quad 0 < \tau < \beta \\ &= \frac{1}{Z} \text{Tr} \left(e^{-\beta H} \phi(\tau) \phi(0) \right) & \phi(\beta) &= e^{\beta H} \phi(0) e^{-\beta H} \\ &= \frac{1}{Z} \text{Tr} \left(\phi(0) e^{-\beta H} \phi(\tau) \right) \\ &= \frac{1}{Z} \text{Tr} \left(e^{-\beta H} \phi(\beta) \phi(\tau) \right) \\ &= \frac{1}{Z} \text{Tr} \left(e^{-\beta H} T(\phi(\beta) \phi(\tau)) \right) \\ &= \zeta_p(\tau - \beta) \end{aligned}$$

What does this have to do with BHs?

$$t = -i\tau$$

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega$$

$\tau \sim \tau + \beta$: we're going to see that not all β are allowed

ζ_0 near the "Euclidean horizon" at $r = 2M$

$$r = 2M + \xi^2 \quad \xi^2 \ll 2M$$

$$ds^2 \approx \frac{\xi^2}{2M} d\tau^2 + 8M d\xi^2 + 4M^2 d\Omega$$

$$8M \xi^2 = x^2$$

$$ds^2 \approx \kappa^2 x^2 d\tau^2 + dx^2 + 4M^2 d\Omega$$

$$\kappa = \frac{1}{4M} : \text{surface gravity}$$

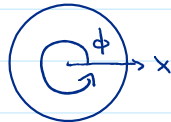
$$\rho^2 d\phi^2 + d\rho^2$$

$$x = \rho$$

$$\kappa\tau = \phi \quad \tau = \frac{1}{\kappa}\phi$$

$$\tau \sim \tau + \beta \quad \phi \sim \phi + \kappa\beta$$

Normally $\phi \sim \phi + 2\pi$. If instead $\kappa\beta < 2\pi$: conical singularity



distributional (delta-fn) curvature

$\kappa\beta = 2\pi$ for absence of conical singularities in the Euclidean geometry

$$\Rightarrow \beta = \frac{2\pi}{\kappa}$$

$$T = \frac{\kappa}{2\pi}$$

$$T = \frac{\hbar\kappa}{2\pi} : \text{Hawking Temperature}$$

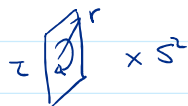
Since the geometry is periodic $\tau \sim \tau + \beta$, all correlation functions of a QFT in this geometry will also exhibit this same periodicity: KMS

$\Rightarrow$ QFT in a bh background is finite-temperature field theory at

$$T = T_H = \frac{\kappa}{2\pi}$$

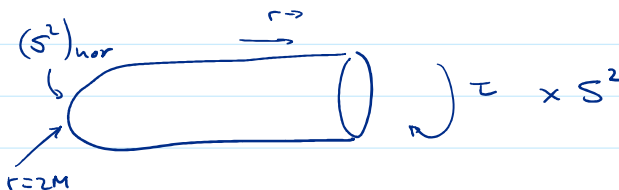
The geometry of Euclidean Schwarzschild:

near $r=2M$



large r $ds^2 \rightarrow d\tau^2 + dr^2 + r^2 d\Omega$

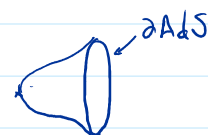
$$\tau \sim \tau + \beta$$



There's nothing behind $r=2M$
Geometry is completely non-singular
and geodesically complete

Topology is $\mathbb{R}^2 \times S^2 \neq S^1 \times \mathbb{R}^3$

This is also valid for black holes in AdS



BH entropy from Euclidean Quantum Gravity (Gibbons + Hawking)

$$Z[\rho] = \int \mathcal{D}g e^{-I_E[g]}$$

$I_E[g]$: classical Euclidean action
sum over geometries that are
periodic in imaginary time $\Delta\tau = \beta$

The Euclidean gravitational action is

$$I_E = -\frac{1}{16\pi G} \int_M d^4x \sqrt{g} R - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{h} K$$

and it is crucial to include the GHY boundary term.

- QPI is ill-defined:
- UV problems: non-renormalizability
 - IR problems: unboundedness under conformal transformations (There are ways around this)
 - what geometries/topologies/signatures... are allowed?

Evaluate it in the saddle-point approximation:

$$Z[\rho] \approx e^{-I_E[g_{cl}]}$$

g_{cl} : classical solution to the field equations of $I_E[g]$
 $\frac{\delta I_E}{\delta g}[g_{cl}] = 0$

with g_{cl} periodic in $\tau \sim \tau + \beta$

This is like a WKB approximation to quantum mechanics, in the semiclassical limit.

There may be more than one saddle contributing. The dominant one is the one with smallest $I_E[g_{cl}]$.

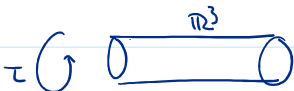
Once we have computed the partition function w/ canonical boundary conditions (fixed temperature), we can apply conventional thermodynamics to it:

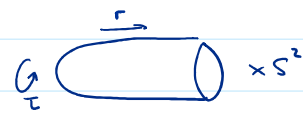
$$F = -\frac{1}{\beta} \log Z \approx -\frac{1}{\beta} I_E \quad (\text{using the saddle-point result})$$

$$E = -\frac{\partial \log Z}{\partial \beta} \approx \partial_\beta I_E$$

$$S = -(\beta \partial_\beta - 1) \log Z \approx (\beta \partial_\beta - 1) I_E$$

For AdS spaces w/ $\tau \sim \tau + \beta$, we have at least two different saddle point geometries:

Euclidean flat space $S^1 \times \mathbb{R}^3$ 

Euclidean Schwarzschild black hole $S^2 \times \mathbb{R}^2$ 

↳ horizon is a fixed point of Euclidean time evolution

When computing $I_E[g_{\text{cl}}]$, the GHY Term is crucial. E.g. for a vacuum solution only the boundary term contributes.

The action I_E is infinite, since space is infinite. To deal with this:

- First, regularize w/ cutoff at $r=R$ (large radius)
- Subtract $I_E - I_E^0$ I_E^0 action of a reference background state ($\sim$ vacuum)

Ensure that boundary conditions are the same for $g_{\mu\nu}$ and $g_{\mu\nu}^0$.

$g_{\mu\nu}^0$: Euc Schwar with $\beta = 8\pi M$

$g_{\mu\nu}^0$: Euc Mink

$\Rightarrow$ finite $I_E - I_E^0 \approx \beta F(\beta)$ for Schwar bh

This can indeed be carried out and one finds $F = \frac{\beta}{16\pi}$

so, using the expressions above, $E = \frac{P}{8\pi} = M$ which are correct.

$$S = 4\pi M^2 = \frac{A_H}{4}$$

In AdS, it is possible to perform the subtraction without reference to a background. One introduces "boundary counterterms", which are of the form $\int_{\partial M} \sqrt{h} F(g, \nabla g, \dots)$ $\hookrightarrow$ local intrinsic invariants of the bdy geometry eg $\sqrt{h}$, $\sqrt{h} R(h)$, ...

(In AdS, counterterms are not local invariants).

There is a general argument to show that in any Einstein-Hilbert Theory of gravity (w/ any matter), the GPI yields an entropy

$$S = \frac{A_h}{4G\hbar}$$

In this sense, there's a very direct relation between Einstein gravity (ie the EHGHY action) and $S \propto A_h$. "Matter = Geometry"

For obtaining only the entropy we can take a shortcut, which avoids the regularization and subtraction.

• For any field in flat space if $\partial/\partial\tau$ is a Killing symmetry

$$I_E = \int d\tau d^3x \mathcal{L} = \beta \int d^3x \mathcal{L} = \beta H \quad H: \text{hamiltonian}$$

$$S = (\beta \partial_\beta - 1) I_E = 0$$

This means that the classical contribution to the entropy in QFT is zero. Entropy in QFT arises at one-loop level

entropy is a sum over microstates = Trace over states

$$\sum_{\text{states}} \text{Tr} \rightarrow \sim \# \text{ of states of the theory}$$

This result is valid for any Euclidean spacetime where $\partial/\partial\tau$ generates a symmetry in which $\partial/\partial\tau$ acts without fixed points

$$\Rightarrow I_E \propto \beta \Rightarrow S = 0$$

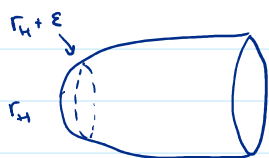
• When there's a black hole horizon (static — can be extended to stationary)

$\frac{\partial}{\partial\tau}$ has a fixed point at the Euclidean horizon



At the horizon the Hamiltonian evolution degenerates

Split the integration of the action into two regions: near the horizon, and the rest
(cap) (cylinder)



The contribution to I_E from $r > r_h + \epsilon$

is $\propto \beta \Rightarrow$ This region doesn't contribute to S (it would contribute to F and to E , and is the region that requires regularization)



(it would contribute to F and to E , and is the region that requires regularization and subtraction, but S is unaffected by that)

So we only need to compute the Euclidean gravitational action:

$$I_E = -\frac{1}{16\pi G} \int_M d^4x \sqrt{g} R - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{h} K$$

in the cap region.

For a solution of $R_{\mu\nu} = 0 \Rightarrow R = 0$: The bulk term vanishes.

Actually, the bulk contribution in the cap vanishes in general, even with matter, since the volume (and any bulk magnitude) vanishes in the limit $\epsilon \rightarrow 0$.

Only the boundary GHY term will contribute to the entropy.

$$K = \frac{1}{\sqrt{h}} n^\mu \partial_\mu \sqrt{h}$$

n^μ : normal to the boundary

$h_{\mu\nu}$: metric induced at bdy

$$I_E^{(\epsilon)} = -\frac{1}{8\pi G} n^\mu \partial_\mu \int_{\partial M} d^3x \sqrt{h}$$



$$\approx -\frac{1}{8\pi G} n^\mu \partial_\mu \int_{r \approx r_H + \epsilon} d^3x \sqrt{h}$$

$$\int d^3x = \int d\tau d^2\Omega$$

$$ds^2 \approx \kappa^2 x^2 d\tau^2 + dx^2 + r_H^2 d\Omega_2 \quad r - r_H \propto x^2$$

$$\sqrt{h} = \kappa x r_H^2 \omega_2$$

$\omega_2 = \sin\theta$: volume element of S^2 (it could be any transverse space)

Euclidean regularity requires $\kappa = \frac{2\pi}{\beta}$

$$n^x = 1 \quad n^\mu \partial_\mu = \partial_x$$

$$A_H = 4\pi r_H^2$$

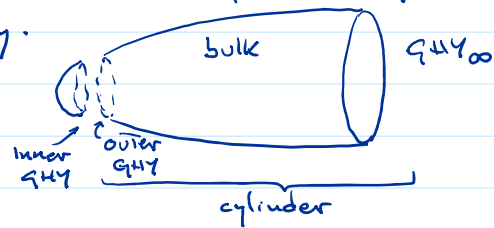
$$I_E^{(\epsilon)} = -\frac{1}{8\pi G} \int_{\beta}^{\infty} \int_{\partial M} 4\pi \kappa r_H^2 \partial_x x = -\frac{1}{4G} 4\pi r_H^2 = -\frac{1}{4G} A_H$$

$$\Rightarrow S = (\beta \partial_\beta - 1) I_E = (\beta \partial_\beta - 1) I_E^{(\epsilon)}$$

$$\Rightarrow \boxed{S = \frac{A_H}{4G}} \quad \text{The correct Bekenstein-Hawking entropy}$$

Let's see more carefully that the ^{entire} contribution from the cylinder doesn't contribute to the entropy.

When we split the geometry we must include another "outer" boundary near the horizon. Then



$$I_E = I_{\text{cylinder}} + I_{\text{GHY (inner)}}$$

$$\text{where } I_{\text{cylinder}} = I_{\text{bulk}} + I_{\text{GHY}(\infty)} + I_{\text{GHY (outer)}}$$

$I_{\text{GHY (outer)}}$ will have sign opposite to $I_{\text{GHY (inner)}}$ since the normal is reversed. Other than this, the calculation is the same, but now β is not fixed by regularity since there's no fixed point in the cylinder. Then, proceeding as before

$$\begin{aligned} I_{\text{GHY (outer)}} &= -\frac{1}{8\pi G} \tilde{n}^\mu \partial_\mu \int_{\partial M} \sqrt{h} \\ &= \frac{1}{8\pi G} \int_0^\beta d\tau \int d\Omega \partial_x (x \kappa r_H^2) \\ &= +\frac{1}{8\pi G} \beta \kappa A_H \end{aligned} \quad \tilde{n}^\mu = -n^\mu$$

where β and κ are now independent.

In an AF space the bulk term vanishes on-shell, while the asymptotic boundary gives, after subtraction $I_{\text{GHY}(\infty)} = \beta M$

In general (eg also in AdS) after subtraction

$$I_{\text{bulk}} + I_{\text{GHY}(\infty)} = \beta M$$

So we find

$$I_{\text{cylinder}} = \beta \left(M + \frac{\kappa A_H}{8\pi G} \right)$$

Since this term is linear in β , then it does not contribute to the entropy.

There's a general formula for the entropy of higher-curvature gravities due to Wald, which generalizes Bekenstein-Hawking.

... — ... 1. To the natural Tolman of the Euclidean black hole

due to Wald, which generalizes Bekenstein-Hawking.

This entropy is due to the non-trivial topology of the Euclidean black hole solution, specifically to the fixed point set of the isometry $\partial/\partial\tau$.

- This is a classical contribution to entropy, which is unusual.

In QFT $\sum_i \int d^4p \mathcal{Q}$: sum over states running in the circle

The action receives contributions in a loop expansion

$$I = \underset{\substack{\uparrow \\ \text{1-loop} \\ \bigcirc}}{t^0} + \underset{\substack{\uparrow \\ \text{2-loop} \\ \bigcirc\bigcirc}}{t^2} + \underset{\substack{\uparrow \\ \text{3-loop} \\ \bigcirc\bigcirc\bigcirc}}{t^3} + \dots$$

In QFT at finite p , we have $F, S \sim \frac{I}{t^h}$
so $S \propto t^0 + t^1 + t^2 + \dots$

BUT here we have found

$$S_{BH} = \frac{A_H}{4G\hbar} : \text{0-loop calculation of entropy?!}$$

what microscopic states are being counted?

How does the gravitational path integral give us the correct result for the entropy, while not performing any sum over microscopic states?

Gravity is some sort of classical collective "mean field", which emerges after statistical averaging over microscopic degrees of freedom, which can capture their number (entropy) w/out revealing their microscopic nature

In string theory, it has been possible to perform countings over microstates of a fundamental theory that correctly reproduce the BH entropy:

$$S_{\text{micro}} = S_{BH} \quad S_{\text{micro}} : \text{stat-mech counting of entropy}$$

Will a similar counting scheme emerge and area $S = \frac{A_H}{4G\hbar}$

$S_{\text{micro}} \sim S_{\text{BH}}$

S_{micro} : stat-mech counting of entropy

Build a susy bh w/ certain charges, mass, and area $S_{\text{BH}} = \frac{A_H}{4G}$

It's very similar to the extremal Reissner-Nordström solution,

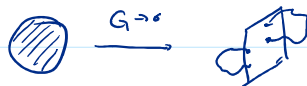
for which $S_{\text{BH}} = \pi Q^2$

Now consider the gravitational coupling G to be variable.

$G \rightarrow 0$: The size of horizon decreases

until it's small enough that it doesn't look like

a black hole, but a configuration of strings (or D-branes)



As we change G : mass changes (renormalization of mass)

Keep charges fixed

susy $\Rightarrow$ entropy as a function of charges doesn't change

In the string side we can do a stat-mech calculation of S_{micro} .

Strömberg+Vafa (1996) found $S_{\text{micro}} = S_{\text{BH}}$

The result has been generalized and extended for large classes of black holes, in different dimensions, with different kinds of charges, different horizon topologies, and sometimes with higher-curvature corrections.

For The Euclidean Schwarzschild solution

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2$$

we want to compute The Euclidean action, relative to That of Minkowski space, ie background subtraction, since it can be regarded as The ground state of AF spacetimes:

$$ds_{(0)}^2 = dt_0^2 + dr^2 + r^2 d\Omega^2$$

The Euclidean action is

$$I_E = -\frac{1}{16\pi G} \int_M d^4x \sqrt{g} R - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{h} K$$

In vacuum $R=0$ so

$$I_E[g] - I_E[g^{(0)}] = -\frac{1}{8\pi G} \int_{\partial M} \sqrt{h} (K - K^0)$$

We must make sure That The geometries induced on ∂M by $g_{\mu\nu}$ and $g_{\mu\nu}^{(0)}$ are The same.

However The extrinsic curvatures are different since They're embedded in different spacetimes, and This is what gives a non-zero renormalized action (it is also The geometric meaning of mass in GR)

Cutoff at $r=R \gg GM$. Expand in $1/R$, Take $R \rightarrow \infty$ at end

Cutoff at $r=R \gg qM$. Expand in $1/R$, Take $R \rightarrow \infty$ at end

$$ds^2(\partial\mathcal{H}) = \left(1 - \frac{2qM}{R}\right) d\tau^2 + R^2 d\Omega \quad \tau \sim \tau + \beta$$

$$ds_o^2(\partial\mathcal{H}) = d\tau_o^2 + R^2 d\Omega \quad \tau_o \sim \tau_o + \beta_o$$

Must be the same.

Angular directions are already the same.

Match time circles: $\int_0^\beta d\tau \sqrt{h_{\tau\tau}} = \int_0^{\beta_o} d\tau_o \sqrt{h_{\tau\tau}^o}$

$$\beta \sqrt{1 - \frac{2qM}{R}} = \beta_o$$

$$\beta \left(1 - \frac{qM}{R} + \dots\right)$$

Now To find $\sqrt{h} K = n^\mu \partial_\mu \sqrt{h}$ we need the unit normal in the radial direction:

$$n^r = \frac{1}{\sqrt{g_{rr}}} = \sqrt{1 - \frac{2qM}{R}} \approx 1 - \frac{qM}{R} + \dots$$

$$\text{and } \sqrt{h} = \sqrt{1 - \frac{2qM}{r}} r^2 \sin\theta \approx \left(1 - \frac{qM}{r}\right) r^2 \sin\theta$$

Then

$$\sqrt{h} K = \left(1 - \frac{qM}{R}\right) \partial_r \left(\left(1 - \frac{qM}{r}\right) r^2 \sin\theta \right) \Big|_{r=R}$$

$$= \sin\theta \left(1 - \frac{qM}{R}\right) (2R - qM) \approx 2R \sin\theta \left(1 - \frac{3}{2} \frac{qM}{R}\right) \quad : \text{diverges as } R \rightarrow \infty$$

$$\sqrt{h} K^0 = 2R \sin\theta$$

$$\int \underbrace{\sqrt{h} (K - K^0)}_{\text{4d}} = \int d\theta d\phi \sin\theta \left[\int d\tau 2R \left(1 - \frac{3}{2} \frac{qM}{R}\right) - \int d\tau_o 2R \right]$$

$$= \beta 8\pi R \left(1 - \frac{3}{2} \frac{qM}{R}\right) - \beta_o 8\pi R$$

$\hookrightarrow \beta \left(1 - \frac{qM}{R}\right)$

$$\begin{aligned} & \rightarrow \beta \left(1 - \frac{GM}{R}\right) \\ & = -4\pi \beta GM \end{aligned}$$

$$\begin{aligned} \Rightarrow \beta F &= I_E[g] - I_E[g^0] = -\frac{1}{8\pi G} (-4\pi \beta GM) \\ &= \frac{1}{2} \beta M = \frac{\beta^2}{16\pi G} \end{aligned} \quad \beta = 8\pi GM$$

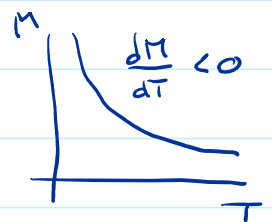
$$E = \partial_\beta(\beta F) = \frac{\beta}{8\pi G} = M \quad \checkmark$$

$$S = (\beta \partial_\beta - 1)(\beta F) = \frac{\beta^2}{16\pi G} = 4\pi GM^2 = \frac{4\pi (2GM)^2}{4G} = \frac{A_H}{4G}$$

Notice The specific heat is negative:

$$\begin{aligned} C &= \frac{dE}{dT} = \frac{dM}{dT} \\ &= -\frac{1}{8\pi G} \frac{1}{T^2} \end{aligned}$$

$$T = \frac{1}{8\pi GM}$$



This is Typical of gravitating systems eg stars:

as The radiate energy, There's less outward pressure,
Then contract and Temperature raises. Eg neutron
stars have inner Temperatures $10^{11}-10^{12}$ K, outer 10^6 K
vs Sun 10^7 K, outer 10^3-10^4 K

BHs are also hotter The smaller They are, in AdS space

$C < 0 \Rightarrow$ They can't be in stable Thermal equilibrium
w/ radiation and Tend to evaporate.

Canonical ensemble is not well defined in AdS gravity

Canonical ensemble is not well defined in AF gravity

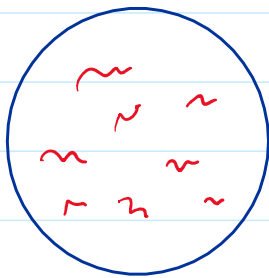
$$\rho(E) \sim e^{(E/E_{pe})^2}$$

$$Z[\beta] = \int dE e^{-\beta E} \rho(E) \sim \int dE e^{-\beta E} e^{E^2/E_{pe}^2}$$

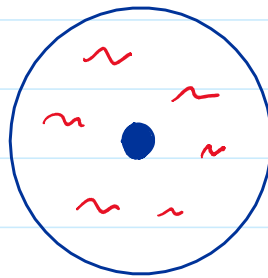
$\downarrow$
suppresses states
with $E \gg 1/\beta$

$\underbrace{\hspace{10em}}$
not suppressed!
Diverges as $E \rightarrow \infty$

PJT The system in a box (Gibbons + Perry 1976)
with Thermal boundary conditions: fixed temperature
Three phases in equilibrium at a given T

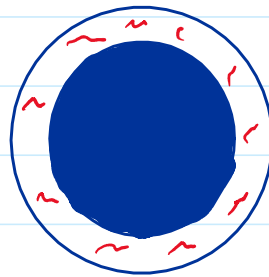


Thermal
radiation
gas



small bh
+ radiation

$$C < 0$$



large bh
+ radiation

$$C > 0$$

This is what also happens in AdS

Thermodynamics of AdS black holes

viernes, 16 de septiembre de 2022 20:26

BH Thermodynamics began w/ The Schwarzschild and Kerr black holes. These exhibit The peculiarity that The specific heat is negative (and other problems due to The ergosphere).

Their Thermodynamics (canonical ensemble) is not properly defined.

The situation improves if we put The system in a box. AdS provides a covariant box, and also allows for an interpretation in terms of a dual quantum field Theory in one less dimension, Through The AdS/CFT correspondence.

Black holes in AdS exhibit much richer and cleaner Thermodynamic structure.

Also The calculation of Thermodynamic quantities can be done w/ elegant methods motivated by holographic renormalization: counterterm subtraction.

Consider AdS₄ gravity $R_{ij} = -\frac{n}{L^2} g_{ij}$

Solved by

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-1}$$

$$f(r) = 1 - \frac{\mu}{r^{n-2}} + \frac{r^2}{L^2}$$

$$\left[\begin{array}{l} \text{could have } k = \frac{\mu}{r^{n-2}} + \frac{r^2}{L^2} \\ k = \pm 1, 0 \\ d\Omega_{n-1}, d\vec{x}_{n-1}^2, dt_{n-1} \\ \text{will see } k=0 \text{ but not } k=-1 \end{array} \right]$$

If $\mu=0$ This is AdS in global coordinates

Boundary at $r \rightarrow \infty$ is $\mathbb{R}_t \times S^{n-1}$

For $n=2$ we get The BTZ black hole (if $\mu > 1$)

This case is different than $n > 2$, which are all qualitatively the same. Henceforth we'll only do $n=4$, i.e. AdS_5 black holes.

$$f = 1 - \frac{\mu}{r^2} + \frac{r^2}{L^2}$$

Horizon at $f(r_+) = 0$

$$\mu = \frac{r_+^2 (r_+^2 + L^2)}{L^2} \quad \left(\begin{array}{l} \text{convenient to} \\ \text{use } r_+ \text{ as parameter} \\ \text{instead of } \mu \end{array} \right)$$

AdS "compresses" The size of black holes:

$$r_+^2 = \mu - \frac{r_+^4}{L^2} < \mu$$

AdS has a length scale L , so we'll get different classes of bhs depending on whether

classes of bhs depending on whether

$r_+ \gtrless L$ small and large AdS bhs
(in $D=3$ only large AdS bhs)

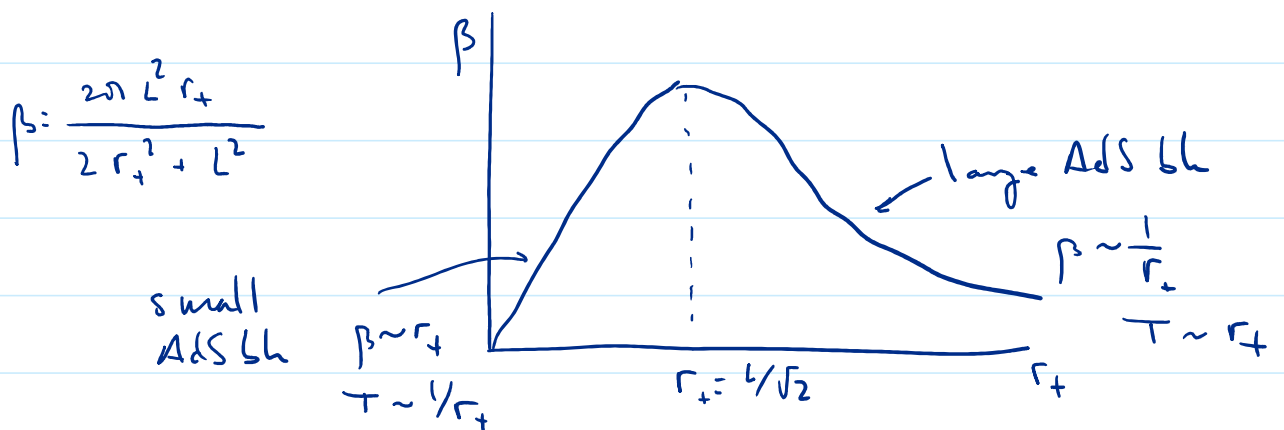
Small: $r_+ \ll L \Rightarrow r_+^2 \approx \mu$ like in AF space

Large: $r_+ \gg L \Rightarrow r_+^4 \approx \mu L^2$ $f(r) \approx \frac{r^2}{L^2} - \frac{\mu}{r^2}$
 $\approx$ planar black brane in AdS

Temperature (from surface gravity, or from Euclidean regularity)

$$T = \frac{1}{\beta} = \frac{f'(r_+)}{4\pi} = \frac{2r_+^2 + L^2}{2\pi L^2 r_+}$$

Phases of AdS black holes



Specific heat: $C = T \frac{\partial S}{\partial T} = -\beta \frac{\partial S}{\partial \beta} = -\frac{3\pi^2}{2G} r_+^2 \frac{\partial r_+}{\partial \beta}$

$\frac{\partial r_+}{\partial \beta} > 0 \Rightarrow C < 0$: small bhs Thermo unstable

$$\frac{\partial r_+}{\partial p} < 0 \Rightarrow c > 0 : \text{large bls thermo stable}$$

We now want to compute the free energy.

There are two ways:

- Quick, pedestrian:

Mass M from asymptotic formula

$$M = \frac{3\pi}{8G} \mu = \frac{3\pi}{8G} \frac{r_+^2(r_+^2 + L^2)}{L^2} \quad \left(\begin{array}{l} \text{can be derived from} \\ \text{linearized analysis} \\ \text{around AdS} \end{array} \right)$$

Entropy from Area:

$$S = \frac{A_+}{4G} = \frac{\pi^2}{2G} r_+^3$$

$$\text{Then } F = M - TS = \frac{\pi}{8GL^2} (-r_+^4 + r_+^2 L^2)$$

- Sophisticated, "ab-initio" derivation from Euclidean path integral, w/ counterterm subtraction

In this case we compute the Euclidean action for the solution w/ $t \rightarrow i\tau$ $\tau \sim \tau + \beta$

$$I = \underbrace{I_{EH}}_{\text{bulk}} + \underbrace{I_{YGH} + I_{\text{counterterm}}}_{\text{boundary}}$$

$$I_{EH} = -\frac{1}{16\pi G} \int_M d^5x \sqrt{g} \left(R + \frac{12}{L^2} \right)$$

$$I_{YGH} = -\frac{1}{8\pi G} \int_{\partial M} d^4x \sqrt{h} K : \text{well-defined variational problem w/ fixed bdy metric}$$

$$\frac{1}{4G_H} \int_{\partial M} 8\pi G \sqrt{\gamma} \tilde{R}$$

problem w/ fixed bdy metric
NOT a counterterm

$$I_{\text{counterterm}} = \frac{1}{8\pi G} \int_{\partial M} d^4x \sqrt{h} \left(\frac{3}{L} + \frac{L}{4} R \right)$$

↳ boundary scalar curvature

↳ depends only
on intrinsic bdy geometry

Removes divergences that arise from
having an infinite space w/ ∞ volume

E.g. bulk Term, use $R = -\frac{20}{L^2}$ $R + \frac{12}{L^2} = -\frac{8}{L^2}$

$$I_{EH} = + \frac{1}{16\pi G} \int_M d^5x \sqrt{g} \frac{8}{L^2} = \frac{\text{Vol}(M)}{2\pi G L^2}$$

Introduce a long-distance regulator $r = \hat{R} \gg r_+, L$
expand for large $\hat{R}$. Divergences cancel out
and finally Take $\hat{R} \rightarrow \infty$

$$\Omega_3 = 2\pi^2$$

One finds

$$\bar{I} = \frac{\beta \pi}{8G L^2} \left(-r_+^4 + r_+^2 L^2 + \frac{3}{4} L^4 \right) = \beta F$$

$$F = \underbrace{\frac{\pi}{8G L^2} \left(-r_+^4 + r_+^2 L^2 \right)}_{\text{like above}} + E_0$$

$$E_0 = \frac{3\pi}{32G} L^2 : \text{independent of } r_+ (\beta)$$

Casimir energy of
CFT in S^3

Doesn't affect Thermodynamics

CFI in S
Doesn't affect Thermodynamics

Apply conventional Thermodynamics:

$$E = \partial_\beta \bar{I} = \frac{3\pi}{8G} \mu + \bar{E}_0 = M + \bar{E}_0$$

$$S = (\beta \partial_\beta - 1) \bar{I} = \frac{\pi}{2G} r_+^3 = \frac{\Delta_4}{4G}$$

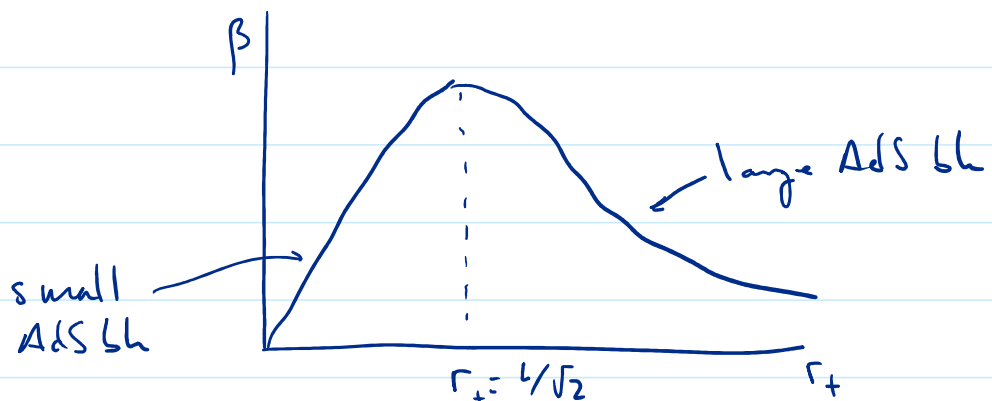
Note: at high T , w/ $r_+ \gg L$ we get

$$T \approx \frac{r_+}{\pi L^2}$$

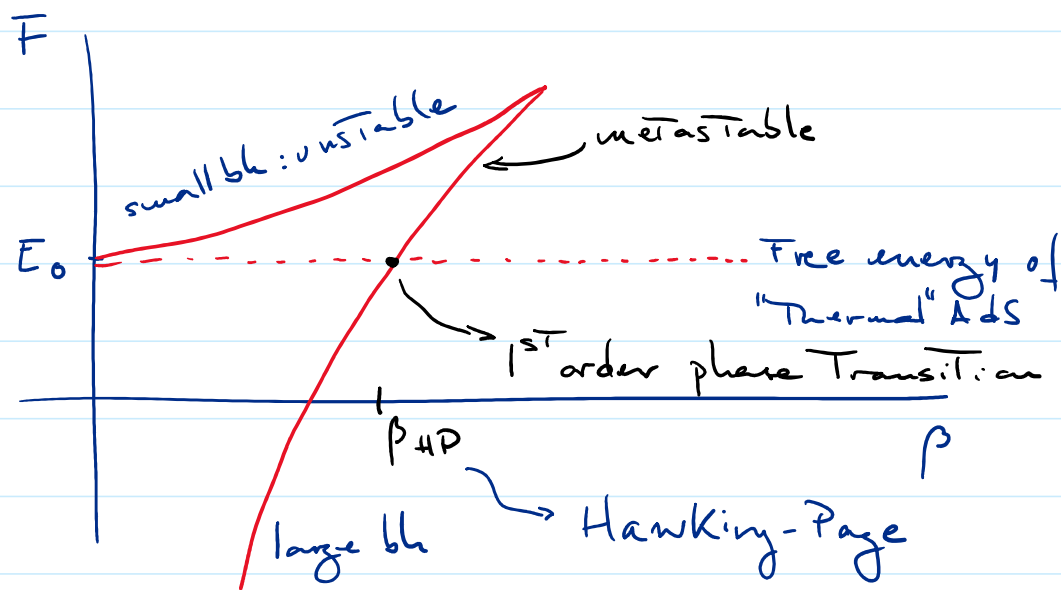
$$\left. \begin{array}{l} F \sim T^4 \\ E \sim T^4 \\ S \sim T^3 \end{array} \right\} \text{like a conformal (massless)}$$
 They at finite Temperature
in 4 dimensions

Phase diagram

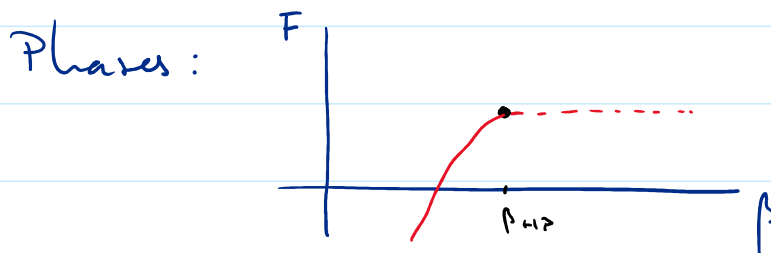
Recall



In canonical ensemble, F is minimized:



Phase Transition for $r_+ = L$ $\beta_{HP} = \frac{2\pi}{3} L$



low T : $0 \leq T < T_{HP}$: Thermal AdS_5
(w/ Thermal radiation gas, whose energy and entropy are negligible)

$T > T_{HP}$: large black hole
Represents deconfined phase of CFT at finite temperature

At large T , The entropy of the black hole overwhelms the suppression $e^{-\beta E}$.

overwhelms the suppression $e^{-\beta E}$.

Latent heat at transition is $T_{HP} S_{HP}$

• More complex phase structures are possible:

- add charge: RN-AdS : chemical potential in the CFT
finite "baryon" density

- vary Λ : "extended Thermodynamics"

"black hole chemistry"

$$\text{Enthalpy } H = E + PV$$

$\uparrow$
from Λ

AdS black brane Thermodynamics

Let us now study the properties of very large AdS black holes.

Starting from the solution above, we scale

$$r \rightarrow \lambda r \quad \mu \rightarrow \lambda^4 \mu$$

$$t \rightarrow t/\lambda$$

$$l^2 d\Omega_3 \rightarrow \frac{1}{\lambda^2} d\vec{x}_{(3)}^2 \quad (\text{stereographic coords around a point})$$

and send $\lambda \rightarrow \infty$: very large black hole.

We obtain a solution with a planar horizon:

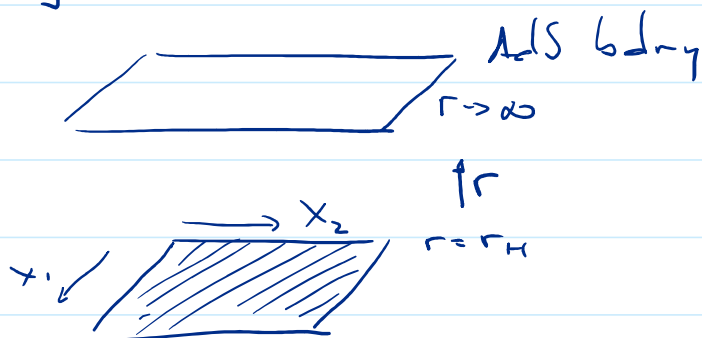
$$ds^2 = -V(r) dt^2 + \frac{dr^2}{V(r)} + \frac{r^2}{L^2} d\vec{x}_{(3)}^2$$

$$V(r) = \frac{r^2}{L^2} - \frac{\mu}{r^2}$$

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\vec{\Sigma}_{n-1}^{(k)} \quad k=1,0$$

$$f(r) = k - \frac{\mu}{r^{n-2}} + \frac{r^2}{L^2}$$

This is The AdS₅ black brane.



The horizon is at $r_+ = (\mu L^2)^{1/4}$

and The Temperature $T = \frac{r_+}{\pi L^2}$

The entropy and energy now become infinite because the planar horizon has infinite extent.

BUT we can define an energy density and entropy density:

$$S = \frac{1}{4G} \frac{r_+^3}{L^3} = \frac{\pi^3 L^3}{4G} T^3$$

(like a "photon gas" in 3+1 dimensions)

$$p = \frac{3}{16\pi G} \frac{r_+^4}{L^3} = \frac{3}{16\pi G} \frac{\mu}{L^3} = \frac{3\pi^3 L^3}{16G} T^4$$

Note $dp = T ds$

Since This black brane is an extended object,

we expect not only an energy density, but

1. The ...

we expect not only an energy density, but also pressure components along the spatial directions x^i .

Energy density is read from T_{tt}

Pressure " " " T_{ii}

Momentum " " " T_{ti}

From Thermodynamics we expect

$$E + PV = TS \quad \text{ie} \quad \rho + P = Ts$$

so we can obtain $P = Ts - \rho$

$$P = \frac{\rho}{3} = \frac{\pi^3 L^3}{16G} T^4 : \text{equation of state of radiation fluid in } 3+1 \text{ (conformally invariant)}$$

These are the Thermodynamic properties of the quantum state that is dual to the AdS black brane: a thermal radiation, scale-invariant plasma.

• AdS₅/CFT₄ dictionary for CFT₄ = $N=4$ SU(N) SYM

$$N^2 = \frac{\pi L^3}{2G} \quad (\text{obtained eg from properties of vacuum or states around vacuum})$$

When the quantum theory is free, one has, from standard Thermodynamic analysis, that the thermal equilibrium state has

$$p = \sigma T^4 \quad \text{w/} \quad \sigma = \frac{\pi^3}{30} \left(\overset{\substack{\uparrow \\ \text{scalars}}}{n_0} + \frac{7}{4} \overset{\substack{\uparrow \\ \text{fermions}}}{n_{1/2}} + 2 \overset{\substack{\uparrow \\ \text{vectors}}}{n_1} \right)$$

$$\begin{aligned} \text{for } N=4 \text{ SU}(N) \text{ SYM} \quad n_0 &= 6N^2 \\ N \gg 1 \quad n_{1/2} &= 4N^2 \\ n_1 &= N^2 \end{aligned}$$

$$\text{so } p_{\text{free}} = \frac{\pi^3}{30} N^2 15 T^4 = \frac{\pi^3 N^2}{2} T^4$$

The gravity calculation gives

$$p_{\text{grav}} = \frac{3\pi^3}{16\pi} L^3 T^4 = \frac{3\pi^3 N^2}{8} T^4$$

$$\text{so } p_{\text{free}} = \frac{4}{3} p_{\text{grav}} \quad : \text{ almost the same.}$$

The $\frac{4}{3}$ difference is because the gravitational calculation is dual to a strongly coupled regime of the quantum theory.

3D Gravity

martes, 6 de agosto de 2024

11:10

Lower-D gravity?

$$\Phi_N \sim -\frac{GM}{r^{D-3}}$$

$$D < 4$$

Gravity is weaker at small r : improved UV behavior
good for QFT fluctuations

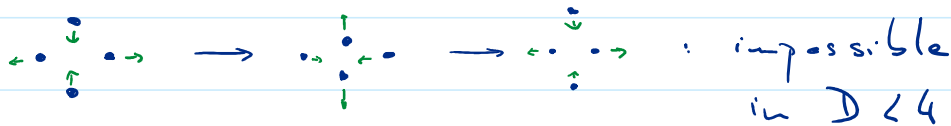
not decaying at large r : very sensitive to
long distance

May lose black holes!

- Gravitational d.o.f.'s: more polarizations in $D > 4$

No propagating d.o.f.'s already in $D=3$:

no room for quadrupolar (shear) motion!



[gauge fields can oscillate in $D=3$
not in $D=2$]

[scalars in all $D \geq 2$]

Scalars and dimensions

In $D=4$ (with $c=1$) GM is a length

In arbitrary D , $(GM)^{1/D-3}$ is a length

In arbitrary D , $(GM)^{1/D-3}$ is a length

In $D=3$ GM is dimensionless

The mass does not determine a length scale,

Thus there can't be a black hole horizon solely determined by M . A particle coupled to gravity creates a conical defect (dimensionless).

$$\left[\begin{array}{l} \text{Mass in } D \geq 4 \text{ is measured from "extrinsic curvature defect" of large-radius spheres} \\ M \text{ in } D=3 \text{ it's measured from conical angle defect of large-radius circles} \\ \\ M=0 \quad \cdot \quad \frac{\Delta \text{Circumference}}{\Delta \text{Radius}} = 2\pi \quad \quad \quad \cdot \quad M>0 \quad 2\pi - \frac{\Delta \text{Circumference}}{\Delta \text{Radius}} \propto GM \end{array} \right]$$

- Need a **length scale** to have a black hole.

Eg a cosmological radius L , from a cosmological constant $\Lambda \sim 1/L^2$

BUT having a length scale is necessary, not sufficient:
We also need attraction.

Mass in 3D doesn't attract by itself; conical defect is global effect

$\Lambda > 0$ won't do: expansion

$\Lambda < 0$ may do: collapse

So we'll consider $\Lambda \sim -1/L^2$. The black holes

will have size $\propto L$ and thus will be "of AdS size",
ie "large AdS bls". $\nexists$ small AdS bls in $D=3$

- Gravity in 3D has no propagating dof's

$$3D \text{ Weyl} = 0 \text{ identically} \quad \left[\begin{array}{l} \text{in 3D Weyl} = 0 \nrightarrow \text{conf flat} \\ \text{Cotton} = 0 \Leftrightarrow \text{conf flat} \end{array} \right]$$

Riemann determined by Ricci.

BUT Ricci is determined by Einstein equations: once
a 3D spacetime is required to satisfy the Einstein eqs,
there doesn't remain any freedom in it (no grav waves).

With cosmological constant we'll have

$$R_{ij} = -\frac{2}{L^2} g_{ij} \quad \text{ie Ricci} = \text{constant}$$

so then Riemann = constant. If $\Lambda < 0$

$\Rightarrow$ All solutions are locally equivalent to AdS_3

[3D gravity is topological, only boundary dof's]

Can only differ:

- (i) in global structure (like plane and cylinder differ)
- (ii) in distributional singularities (like plane and cone differ)
Sources: conical points
line sources

So we know the local geometry: AdS_3

So we know the local geometry: AdS_3
 and we're going to see how to find
geometries that differ globally.

AdS_3 :

BT2 '92

BHT2 '93

We know S^3 is described as a surface in $\mathbb{R}^4$
 $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$ in $ds^2 = dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$
 (unit radius $L=1$)

In order to change $S^3 \rightarrow AdS_3$ we flip two signs:
 To go Lorentzian (have Time) and to have -ve curvature

$$ds^2 = -dx_0^2 - dx_1^2 + dx_2^2 + dx_3^2$$

$$-x_0^2 - x_1^2 + x_2^2 + x_3^2 = -1$$

Can solve for surface

$$\underbrace{-x_0^2 - x_1^2}_{\text{polar}} + \underbrace{x_2^2 + x_3^2}_{\text{polar}} = -1$$

$$(t, r) \quad (r, \phi)$$

$$x_2 = r \sin \phi$$

$$x_3 = r \cos \phi$$

$$x_0 = \sqrt{r^2 + 1} \sinh t$$

$$x_1 = \sqrt{r^2 + 1} \cosh t$$

: periodic t ?

No: "unwrap" to global
 covering space $-\infty < t < \infty$

$$X_1 = \sqrt{r^2 + 1} \cosh t$$

covering space $-\infty < t < \infty$

(but there remains a memory of this periodicity)

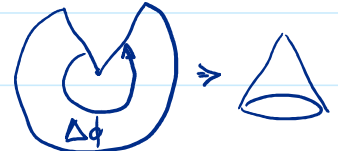
$$\Rightarrow ds^2 = -(r^2 + 1) dt^2 + \frac{dr^2}{r^2 + 1} + r^2 d\phi^2$$

Take $\phi \sim \phi + \Delta\phi$

If $\Delta\phi \neq 2\pi$: conical singularity at $r=0$

Naked: No horizon

$$-g_{tt} > 0 \quad \forall r$$



BTZ:

Now solve the surface equation by pairing up the coordinates in a different way, in boost-like fashion:

$$\underbrace{-x_0^2 - x_1^2}_{\text{boost}} + \underbrace{x_2^2 + x_3^2}_{\text{boost}} = -1$$

$$x_1 = r \cosh \phi \quad -x_1^2 + x_3^2 = -r^2$$

$$x_3 = r \sinh \phi$$

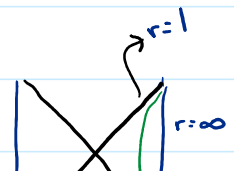
$$x_0 = \sqrt{r^2 - 1} \sinh t \quad -x_0^2 + x_2^2 = r^2 - 1$$

$$x_2 = \sqrt{r^2 - 1} \cosh t$$

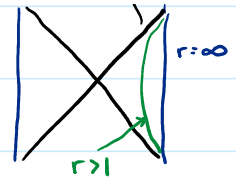
$$\Rightarrow ds^2 = -(r^2 - 1) dt^2 + \frac{dr^2}{r^2 - 1} + r^2 d\phi^2$$

but now in principle $-\infty < t < \infty$

$$-\infty < \phi < \infty$$



but now in principle $-\infty < t < \infty$
 $-\infty < \phi < \infty$



This is Rindler-AdS₃

Horizon at $r=1$ is non-compact acceleration horizon.

There is The AdS version of Rindler space:

$$r = 1 + \xi^2/2 \quad \xi^2 < 1 \Rightarrow ds^2 = -\xi^2 dt^2 + d\xi^2 + d\phi^2 \quad \text{3D Rindler}$$

$$= -dT^2 + dX^2 \quad T = \xi \sinh t \quad X = \xi \cosh t$$

∂_t and ∂_ϕ are boost vectors

$$\partial_t = x_2 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_2} \quad \text{Timelike for } -x_0^2 + x_2^2 > 0 \quad r^2 > 0$$

$$\partial_\phi = x_1 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_1} \quad \text{" " } -x_1^2 + x_3^2 > 0 \quad \text{" } r^2 < 0 \text{"}$$

Spacelike " $-x_1^2 + x_3^2 < 0 \quad r^2 > 0$

Rindler-AdS₃ is just a patch of AdS₃.

We now introduce a global difference with AdS₃:

Identify along orbits of ∂_ϕ

$$\phi \sim \phi + \Delta\phi \quad (\text{OK where } r^2 > 0)$$

$\Delta\phi$ is arbitrary: need not be 2π since ϕ is a boost coordinate, and $r=0$ is hidden behind the horizon (see below for more on $r=0$)

Horizon now is compact: black hole horizon

$$ds^2 = -(r^2 - 1) dt^2 + \frac{dr^2}{r^2 - 1} + r^2 d\phi^2 \quad \text{with } \phi \sim \phi + \Delta\phi$$

$$ds^2 = -(r^2 - 1) dt^2 + \frac{dr^2}{r^2 - 1} + r^2 d\phi^2 \quad \text{with } \phi \sim \phi + \Delta\phi$$

is The BTZ black hole with horizon at $r=1$, "area" (length) $A_H = \Delta\phi$

The form above is not the "canonical metric".

Rescale $\phi = \frac{\Delta\phi}{2\pi} \tilde{\phi}$ so $\Delta\tilde{\phi} = 2\pi$

and absorb factors in $r = \frac{2\pi}{\Delta\phi} \tilde{r}$
 $t = \frac{\Delta\phi}{2\pi} \tilde{t}$

Then

$$ds^2 = -(\tilde{r}^2 - 8GM) d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 - 8GM} + \tilde{r}^2 d\tilde{\phi}^2$$

with $8GM = \left(\frac{\Delta\phi}{2\pi}\right)^2$ $\tilde{\phi} \sim \tilde{\phi} + 2\pi$

M is The black hole mass (from detailed analysis)

$$M = \frac{(D-2)\Omega_{D-2}}{16\pi G} \mu = \frac{\mu}{8G} \quad \begin{matrix} D=3 \\ \uparrow \end{matrix}$$

Notice global AdS_3 has $M = -1/8G$

BHs have $M > 0$.

$-1/8G < M < 0$ are conical singularities (particles)

["M=0 black hole" is mildly singular. π is obtained by identifying along a null boost.]

[$r=0$: null hypersurface : $\partial/\partial\phi$ is null There no curvature singularity]

no curvature singularity
metric can be analytically continued

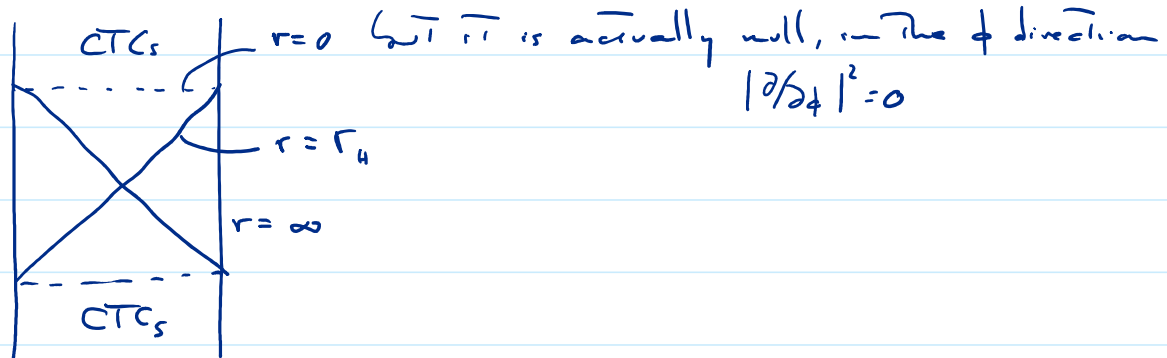
There are CTCs beyond: points

identified along Timelike directions (boost)

some null geodesics are incomplete

(similar to Lorentzian Taub-NUT and Misner space)

Keep only $r > 0$



So:

AdS₃ with points identified along orbits of a boost vector is the BTZ black hole

- Global definition, indep of coordinates!

- Mass is given by period of identification

Other forms of the metric:

from $ds^2 = -(r^2 \pm 1) dt^2 + \frac{dr^2}{r^2 \pm 1} + r^2 d\phi^2$ AdS₃
BTZ

proper radius $dp^2 = \frac{dr^2}{r^2 \pm 1}$

+ : $r = \sinh p$ $ds^2 = -\cosh^2 p dt^2 + dp^2 + \sinh^2 p d\phi^2$ AdS₃

$\downarrow$
 no zero
 no horizon

$\downarrow$
 zero at $p=0$:
 origin of rotation

$$-: r = \cosh p \quad ds^2 = -\sinh^2 p \, dt^2 + dp^2 + \cosh^2 p \, d\phi^2 \quad \text{BTZ}$$

$\downarrow$ zero at $p=0$ horizon $\downarrow$ no zero finite area horizon

[Observe Euclidean $t \rightarrow i\tau$ are equivalent $\tau \leftrightarrow \phi$
 Euclidean Thermal AdS_3 and BTZ only
 differ in The choice of Euclidean Time]

Rotation: identify ϕ after a shift in Time
(equivalently, along a vector $\partial/\partial\phi + \alpha\partial/\partial t$, $|\alpha| < 1$)

In This case There are no incomplete geodesics. Exercise by hand region w/ CTCs

Easy To find That

$$G = 1/8$$

$$ds^2 = - \left(r_+^2 - M + \frac{J^2}{4r^2} \right) dt^2 + \frac{dr^2}{r_+^2 - M + J^2/4r^2} + r^2 \left(d\phi - \frac{J}{2r^2} dt \right)^2$$

$$= - \frac{(r_+^2 - r_+^2)(r_-^2 - r_-^2)}{r^2} dt^2 + \frac{r^2 dr^2}{(r_+^2 - r_+^2)(r_-^2 - r_-^2)} + r^2 \left(d\phi - \frac{r_+ r_-}{r^2} dt \right)^2$$

$$r_+^2 r_-^2 = \frac{J^2}{4}$$

$$r_+^2 + r_-^2 = M$$

Usual pattern: $g_{tt} \propto J$

$$-8GM + \frac{J^2}{4r^2}$$

$\hookrightarrow$ centrifugal repulsion

$\exists$ extremal limit $r_+ = r_-$ $M = |J|$

Regular w/ finite area

This is also obtained from identifications along a null boost

The BTZ black hole plays a central role in one of the best understood and most detailedly studied

The BTZ black hole plays a central role in one of the best understood and most detailedly studied instances of holography: AdS_3/CFT_2

Both sides of the duality are eminently solvable due to the large amount of symmetry, and allow precise computations and checks.

BTZ also underlies many precise calculations of the Bekenstein-Hawking entropy in string theory, including the first one by Strominger + Vafa.

BTZ black hole entropy from CFT_2

$$ds^2 = -\left(\frac{r^2}{l^2} - 8GM\right) dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - 8GM} + r^2 d\phi^2 \quad (\text{remove } \sim) \\ l = AdS_3 \text{ radius}$$

$$r_H = l \sqrt{8GM}$$

$$A_H = 2\pi r_H = 2\pi l \sqrt{8GM}$$

$$S = \frac{A_H}{4G} = \frac{\pi l}{2} \sqrt{\frac{8M}{G}} = \pi l \sqrt{\frac{2M}{G}}$$

We want to reproduce this as the degeneracy of a CFT_2 with central charge $c = \frac{3l}{2G}$ at high energy / Temperature.

at high energy / temperature.

There exists a general formula for the degeneracy of states $e^{S_{\text{CFT}}}$ of a CFT_2
"Cardy formula"

$$S_{\text{CFT}} = 2\pi \sqrt{\frac{cE}{3} \frac{L}{2\pi}}$$

L = length of spatial
circle of CFT_2
 E = Total energy

Now plug in here $\begin{cases} E = M \\ L = 2\pi l \end{cases} \quad c = \frac{3l}{2g}$

$$S_{\text{CFT}} = 2\pi \sqrt{\frac{Ml^2}{2g}} = \pi l \sqrt{\frac{2M}{g}} = S_{\text{BH}}$$

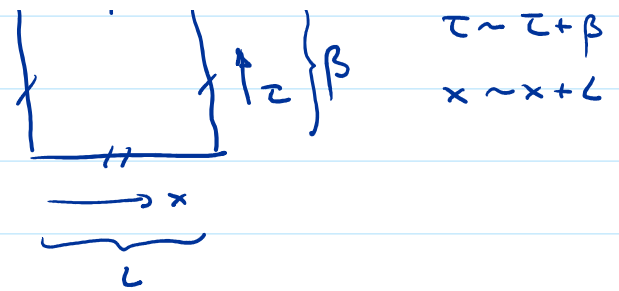
$S_{\text{CFT}} = S_{\text{BH}}$

 : it works exactly!

Cardy formula for the degeneracy of CFT_2 states
at high energy / Temperature

$Z(\beta) = \text{Tr } e^{-\beta H}$: compute it from Euclidean
path integral on Torus





$Z(\beta, L)$ is dimensionless, and the only scales are β and L , so it must be that

$$Z(\beta, L) = Z(\beta/L, 1) = Z(\beta/L)$$

Now we use modular invariance, which is a property of CFT_2 at one loop, and which implies that $Z(\beta, L) = Z(L, \beta)$

$$= Z\left(L \times \frac{L}{\beta}, \beta \times \frac{L}{\beta}\right) = Z\left(\frac{L^2}{\beta}, L\right)$$

Then for fixed L we can write

$$Z(\beta) = Z\left(\frac{L^2}{\beta}\right)$$

often one writes
 $L = 2\pi R$ and then
 $Z(\beta) = Z\left(\frac{4\pi^2 R^2}{\beta}\right)$

This relates the density of states at high-temp $\beta \rightarrow 0$, to the density at low-temp, $\beta \rightarrow \infty$

At low temperatures, Z is dominated by the vacuum state of the CFT_2 on a circle, with

energy $E_0 = -\frac{c}{12} \frac{2\pi}{L} = -\frac{\pi c}{6L}$ (Casimir energy)

energy $E_0 = -\frac{c}{12} \frac{2\pi}{L} = -\frac{\pi c}{6L}$ (Carnivir energy on a circle)

so

$$Z(L^2/\beta) = Z(\tilde{\beta}) \approx e^{-\tilde{\beta} E_0} \quad \tilde{\beta} = L^2/\beta$$

$$= e^{\frac{L^2}{\beta} \frac{\pi c}{6L}}$$

ie for $\beta \rightarrow 0$ $Z(\beta) \approx e^{\frac{\pi L c}{6\beta}} = e^{-\beta F}$

$$\Rightarrow F = -\frac{\pi L}{6} c T^2 \quad T = 1/\beta$$

Using standard Thermodynamics we find that

$$\left. \begin{aligned} E &= \frac{\pi L}{6} c T^2 \\ S &= \frac{\pi L}{3} c T \end{aligned} \right\} \Rightarrow \boxed{S = 2\pi \sqrt{\frac{cE}{3} \frac{L}{2\pi}}}$$

In $D=3$ we had $Weyl=0$ so all solutions of Einstein's eqs $R_{ij} = \Lambda g_{ij}$ ($\Lambda \geq 0$) were locally equivalent to AdS_3 , $Mink_3$, dS_3 and only differed in global properties.

In $D=2$ we have $R_{ij} = \frac{1}{2} g_{ij} R$ identically, so we can't find any solutions since there are no equations!

The reason is that $I_{\text{Euler}} = \frac{1}{16\pi G} \int d^2x \sqrt{-g} R$ is a Topological invariant (Euler number) and its variation only yields boundary terms

In order to have non-trivial equations (although not necessarily local propagating d.o.f.'s) we can introduce a scalar field (dilaton):

$$I = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Phi(x) R + \dots \quad (\text{review Grumiller et al '02})$$

- In $D>2$ Φ can be absorbed in a change of conformal frame
but not in $D=2$ $g_{\mu\nu} \rightarrow F(\Phi) g_{\mu\nu}$

- Observe that $\Phi(x)$ acts to give an x -dependent gravitational coupling $\frac{1}{G_{\text{eff}}} = \frac{\Phi(x)}{G}$

- If we have a 2D BH, its horizon is a point. We expect its entropy to be

$$S = \frac{1}{4G_{\text{eff}}} = \frac{\Phi(x_H)}{4G}$$

This can (and should) be derived from a proper analysis of the complete Euclidean action, but it is generally valid if there are no more curvatures in the action.

It is also well motivated when the 2D bh is viewed as the result of a spherical reduction of a higher-D black hole.

Eg

$$ds^2 = \underbrace{-f(r)dt^2 + \frac{dr^2}{g(r)}}_{\text{2D bh}} + \underbrace{r^2 d\Omega_2}_{\Phi(t,r)}$$

If we only consider spherically symmetric configurations, we can integrate the angles (θ, ϕ) in the action and obtain an effective 2D gravity theory of the form above.

They of the form above.

We see that $\Phi(t, r)$ measures the area of the S^2 , so $A(r_4) = 4\pi \Phi(r_4)$

and the formula above gives the correct entropy

$$\left[\begin{array}{l} \text{The factor } 4\pi \text{ appears from the integration of} \\ \int \sin \theta d\theta d\phi \text{ in the action: } \frac{1}{16\pi G_4} \int d\Omega^2 d^2x R^{(4)} = \underbrace{\frac{4\pi}{16\pi G_4}}_{\frac{1}{16\pi G_2}} \int d^2x \Phi R^{(2)} + \dots \\ \frac{A^{(4)}}{4G_4} = \frac{4\pi \Phi_H}{4G_4} = \frac{\Phi_H}{4G_2} \quad G_2 = \frac{G_4}{4\pi} \end{array} \right]$$

2D dilaton Theories differ in the choice of terms "... " in the action.

We'll only consider two simple but important cases:

JT gravity

Jackiw '83

Terileborn '83

$$I = \frac{1}{16\pi G} \int d^2x \sqrt{-g} \Phi (R - \Lambda) \quad \text{with } \Lambda = -\frac{2}{L^2} = -2 \quad L=1$$

2D effective string Theory ("CGHS")

$$I = \frac{1}{16\pi G} \int d^2x \sqrt{-g} (\Phi R - \Lambda) \quad \text{with } \Lambda = -4\lambda^2 \text{ (conventionally)}$$

This is not the most usual way of presenting it, rather one performs a Weyl Transformation to the "string metric"

$$g_{ij} = e^{-2\phi} \hat{g}_{ij} \quad \text{with} \quad \hat{g} = e^{-2\phi}$$

and then (verify!), removing hats from $\hat{g}_{ij}$

$$I = \frac{1}{16\pi G} \int d^2x \sqrt{g} e^{-2\phi} (R + 4(\nabla\phi)^2 + 4\lambda^2)$$

"ghost" sign? no problem since there are no propagating dofs : "Kinetic Term" $(\nabla\phi)^2$ can always be removed w/ a conformal change of frame as we have seen

These Two Theories are important because they admit classical black hole solutions, but also because, when coupled to 2D conformal field matter, the quantum CFT can be solved and its backreaction on the black hole can be computed.

In particular, JT gravity has played a central role in recent computations of the "Page curve" for evaporating black holes - a test of unitary evolution. It has also attracted a lot of attention as a highly solvable model of holographic quantum gravity, since its dynamics describes the low-energy physics of the SYK model, a quantum-mechanical theory that captures many highly non-trivial properties of the statistical mechanics of black hole microscopics.

They also appear as the spherical reduction of relevant higher-D black holes

JT bh from:

- BTZ
- near-extremal charged bhs

CGHS bh from:

- near-extremal dilaton magnetic bhs
- near-extremal NS5 branes
- $D \rightarrow \infty$ neutral black holes (also from BTZ but Trickier)

Black holes in JT gravity

martes, 6 de agosto de 2024 11:10

$$I = \frac{1}{16\pi G} \int_M d^2x \sqrt{-g} \Phi(x) (R+2) + \frac{1}{8\pi G} \int_{\partial M} dx \sqrt{|h|} \Phi \downarrow_{GHY} \downarrow_{\text{counterterm}} (K-1)$$

GHY boundary Terms are important when one computes the action, when quantum effects are introduced, or when a dynamical boundary ω_{off} is introduced ("near-AdS" "Schwarzschild particle")

Can add a Topological Term (Euclideanized)

$$I_{\text{Topo}} = -S_0 \left(\frac{1}{4\pi} \int_M d^2x \sqrt{g} R + \frac{1}{2\pi} \int_{\partial M} dx \sqrt{|h|} K \right) = -S_0 \chi$$

Euler characteristic
 $\chi = 2(1-g) - b$

This gives a constant contribution to the action and to the entropy.

When $S_0 \gg 1$ it suppresses the contributions from higher topologies: $e^{-I_{\text{Topo}}} = e^{S_0 \chi} : e^{2S_0} (1, e^{-2S_0}, e^{-4S_0}, \dots)$
sphere Torus

This is a parameter that is different than the coupling $\frac{1}{16\pi G}$

$G_{\text{th}} \ll 1$ suppresses quantum gravity corrections in a given topology eg in the Euclidean disk $(\chi=1)$ The classical limit is

$G_{\text{th}} \rightarrow 0$. One can quantize the theory for G_{th} finite

e^{-S_0} suppresses contributions from higher topologies.

Classical Theory:

Vary $\delta \Phi \Rightarrow R = -2$: in 2D, R determines all the curvature, so all the

solutions must be locally
equivalent to AdS_2

- No dynamics in AdS_2

If we put an excitation in $AdS_{D \geq 4}$, its effect at large distances dies away as $\frac{1}{r^{D-3}}$: can have finite energy spectrum of excitations.

In $D=2$ The lines of force don't dilute since there are no directions in which to expand

$\Rightarrow$ any excitation will have a large effect at asymptotic infinity : "non-normalizable"
= no finite-energy excitations in AdS_2

We can have finite dynamics ("normalizable excitations") if we introduce a cutoff near the AdS_2 boundary, thus regularizing the infinite energy : near- AdS_2 boundary dynamics.

- This has a counterpart in the properties of a putative CFT_1 dual:

a CFT has $T_c^i = 0$

In 1+0 dimensions this means $T_c^t = 0$: zero energy!

The theory can only describe ground states, which is consistent (eg BPS ground states) but not very interesting for black evaporation.

which is consistent w/ the previous result,
not very interesting for black evaporation.

Moreover, consider the possible form of the
density of states $\rho(E) = \frac{dn}{dE}$: dimension $\frac{1}{E}$

In 1+0, the only scale is E (in higher D there
is the spatial volume) so it must be

$$\rho(E) = A \delta(E) + \frac{B}{E}$$

$A \delta(E)$: ground-state degeneracy

$\frac{B}{E}$: divergent in infrared $E \rightarrow 0$

must introduce an IR cutoff in order to have
finite energy states.

The cutoff breaks conformal symmetry

Near-conformal Theory

This is the dual counterpart of near- AdS_2 dynamics

AdS_2 geometries and JT solutions

Consider, as we did in 3D, the
hyperboloid embedding of AdS_2 :

$$ds^2 = -dx_0^2 - dx_1^2 + dx_2^2 \quad -x_0^2 - x_1^2 + x_2^2 = -1$$

Depending on what we choose to be the time
coordinate, we'll get different patches of AdS_2 .

Take t a polar angle in the "Time plane" (x_0, x_1)

$$\left. \begin{aligned} x_0 &= \cosh p \sinh t \\ x_1 &= \cosh p \cosh t \\ x_2 &= \sinh p \end{aligned} \right\} \text{Global AdS}_2$$

or

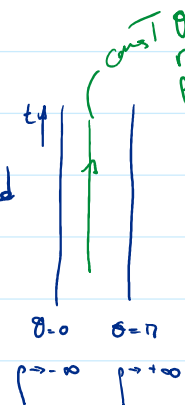
$$\left. \begin{aligned} x_0 &= \sqrt{r^2+1} \sinh t \\ x_1 &= \sqrt{r^2+1} \cosh t \\ x_2 &= r \end{aligned} \right\}$$

	<u>Boundary</u>	<u>Center</u>
$ds^2 = -\cosh^2 p dt^2 + dp^2$	$p \rightarrow \pm \infty$	$p = 0$
$r = \sinh p$		
$= -(r^2+1) dt^2 + \frac{dr^2}{r^2+1}$	$r \rightarrow \pm \infty$	$r = 0$
$r = \cot \theta$		
$= \frac{-dt^2 + d\theta^2}{\sin^2 \theta}$	$\theta = 0, \pi$	$\theta = \pi/2$
$t = u+v$ $\theta = u-v$		
$= -4 \frac{du dv}{\sin^2(u-v)}$	$u-v = 0, \pi$	$u-v = \pi/2$

There are no horizons, $-g_{tt} > 0$ everywhere.

The Two asymptotic boundaries are disconnected

(This is then like a 1+1 wormhole)



Now Take t To be a boost coordinate

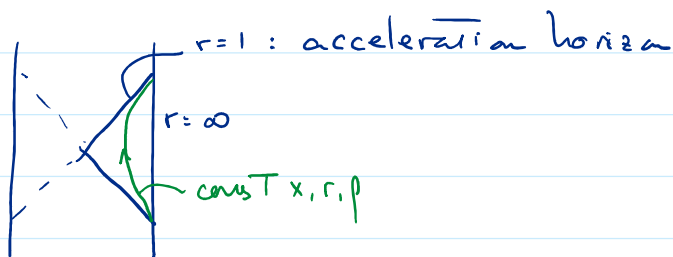
$$\left. \begin{aligned} x_0 &= \sinh p \sinh t \\ x_1 &= \cosh p \\ x_2 &= \sinh p \cosh t \end{aligned} \right\} \text{Thermal (Rindler) AdS}_2$$

or

$$\left. \begin{aligned} x_0 &= \sqrt{r^2-1} \sinh t \\ x_1 &= r \\ x_2 &= \sqrt{r^2-1} \cosh t \end{aligned} \right\}$$

We'll have a Rindler horizon where $x_2^2 - x_0^2 = 0$

	<u>Body</u>	<u>Horizon</u>
$ds^2 = -\sinh^2 p dt^2 + dp^2$	$p \rightarrow \infty$	$p = 0$
$r = \cosh p$		
$= -(r^2 - 1) dt^2 + \frac{dr^2}{r^2 - 1}$	$r \rightarrow \infty$	$r = 1$
$r = \cosh x$		
$= \frac{-dt^2 + dx^2}{\sinh^2 x}$	$x \rightarrow 0$	$x \rightarrow \infty$
$t = u+v$ $x = u-v$		
$= -4 \frac{du dv}{\sinh^2(u-v)}$	$u-v \rightarrow 0$	$u-v \rightarrow \infty$



Like in BTZ , we can rescale

$$r \rightarrow r/r_H \quad t \rightarrow r_H t$$

To find

$$ds^2 = -(r^2 - r_H^2) dt^2 + \frac{dr^2}{r^2 - r_H^2} \quad r_H = \frac{2\pi}{\beta}$$

BUT, unlike in BTZ , here we cannot make global identifications that make them different!
(There is no angular direction to identify)

A final form is **AdS₂ in Poincaré coordinates:**

$$x_0 = \frac{1}{2z} (1 - t^2 + z^2)$$

$$-x_0^2 - x_1^2 + x_2^2 = -1$$

$$x_1 = -t/2$$

$$x_2 = \frac{1}{2} (1 + t^2 - z^2)$$

$$x_1 = -z$$

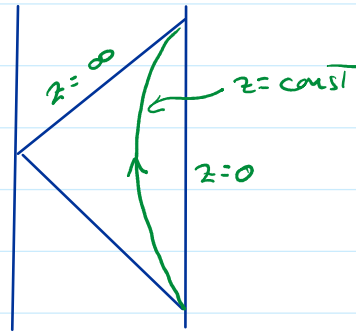
$$x_2 = \frac{1}{2z}(1+t^2-z^2)$$

$$ds^2 = -\frac{dt^2 + dz^2}{z^2}$$

$$= -4 \frac{du dv}{(u-v)^2}$$

$$t = u+v$$

$$z = u-v$$



$$= -r^2 dt^2 + \frac{dr^2}{r^2} \quad z = 1/r$$

When $r \gg 1$, The other forms of AdS_2 reduce to this one.

All of These are locally and globally AdS_2
What makes The difference is The dilaton.

Vary g_{ij} (The equation is $T_{ij}^{\Phi} = 0$)

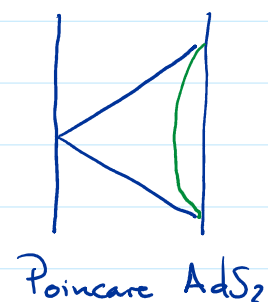
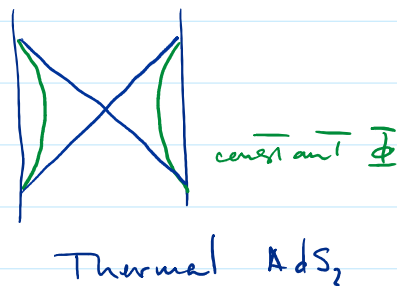
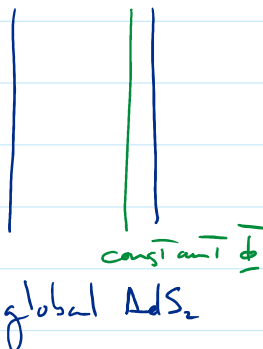
$$\Rightarrow \frac{1}{2} g_{ij} \square \Phi = \nabla_i \nabla_j \Phi = g_{ij} \Phi$$

$$\text{For } ds^2 = -(r^2 + \kappa) dt^2 + \frac{dr^2}{r^2 + \kappa} \quad \kappa = \pm 1, 0$$

The dilaton eqns are solved for all κ by

$$\Phi = ar \quad (a = \text{constant})$$

but notice That r is different in each case:

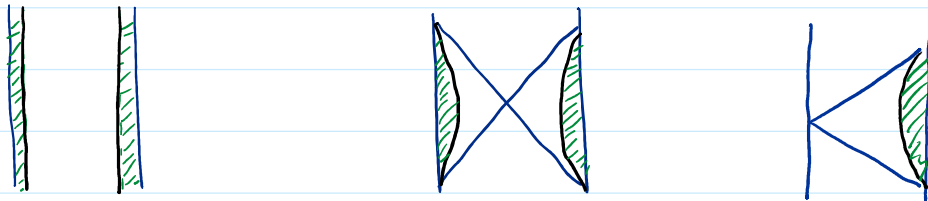


$\Phi \rightarrow \infty$ at the boundary $r \rightarrow \infty$

Near-AdS₂ and Schwarzian dynamics

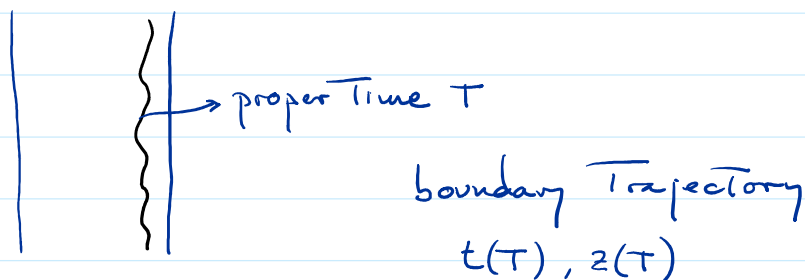
Introduce a cutoff at $\bar{\Phi} = \text{const} \gg 1$ ^{"renormalized"}
 $r = 1/\epsilon$ $\Phi|_{\text{cutoff}} = \Phi_b / \epsilon$

This gives near-AdS₂ geometries which are inequivalent and which can have a dynamical boundary:



The dynamics of near-AdS₂ JT gravity is

The dynamics of a "particle" moving near the boundary of AdS₂:
 (or 2)



$$ds^2|_b = -\frac{dT^2}{\epsilon^2} = -\frac{dt^2 + dz^2}{z^2} = -\frac{t'^2 + z'^2}{z(T)^2} dT^2 \quad ' = \frac{d}{dT}$$

$$t'^2 - z'^2 = \frac{z^2}{\epsilon^2} \quad z(T) = \epsilon t'(T) + O(\epsilon^2)$$

Compute extrinsic curvature of boundary

Compute extrinsic curvature of boundary

$$K = 1 + \varepsilon^2 \{t, T\} + O(\varepsilon^4)$$

$$\{t, T\} \equiv \frac{t'''}{t'} - \frac{3}{2} \frac{t''^2}{t'^2} \quad \text{Schwarzian derivative}$$

Symmetric under $t \mapsto \frac{at+b}{ct+d} \quad SL(2, \mathbb{R})$

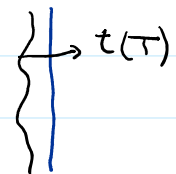
$$\left[\text{Then } t(T) = \frac{aT+b}{cT+d} \Leftrightarrow \{t, T\} = 0 \right]$$

Gravitational action: bulk term vanishes

$$I_{ST} \Big|_{\text{on-shell}} = \frac{1}{8\pi G} \int_{\partial M} dT \sqrt{|h|} \Phi (K-1)$$

$$= \frac{\phi_b}{8\pi G} \int_{\partial M} dT \{t, T\} : \text{Schwarzian action for the boundary particle}$$

$t(T)$: diffeomorphisms of boundary
= trajectory of bdy particle



Euclidean $T \mapsto -i\tau$

$$I_{ST} = - \frac{\phi_b}{8\pi G} \int_0^\beta d\tau \{t, \tau\} \quad \text{periodic in } \tau \sim \tau + \beta$$

Equations of motion obtained by varying $t \mapsto t + \delta t$

$$\text{use } \int d\tau \delta \{t, \tau\} = - \int d\tau \frac{\{t, \tau\}'}{t'} \delta t$$

$$\Rightarrow \frac{d}{d\tau} \{t, \tau\} = 0 \quad : \{t, \tau\} \text{ is conserved}$$

$$\Rightarrow \frac{d}{d\tau} \{t, \tau\} = 0 \quad : \{t, \tau\} \text{ is conserved} \\ (\text{by } SL(2, \mathbb{R}) \text{ symmetry})$$

Solved by $t(\tau) = \tan \frac{\pi \tau}{\beta}$: Thermal AdS_2

If we now compute the action we get

$$I_{\text{cl}}^{\text{Sch}}(\beta) = -\frac{1}{4\pi G} \frac{\pi^2}{\beta} \phi_b$$

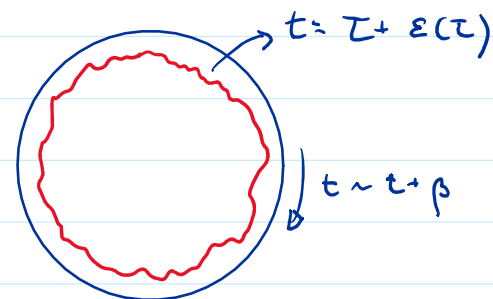
Notice $\beta F = I_{\text{cl}} \Rightarrow F \propto T^2$ not $F \propto T$ is not CFT₁

Quantization of The Schwarzian action

The Schwarzian action can be exactly quantized: Stanford + Witten

We'll do perturbative quantization around The Thermal solution:

$$\tan \frac{\pi}{\beta} (\tau + \epsilon(\tau)) \quad \text{with small } \epsilon(\tau)$$



$$\text{Sch} \left(\tan \frac{\pi}{\beta} (\tau + \epsilon(\tau)) \right) = \frac{2\pi^2}{\beta^2} + \frac{2\pi^2}{\beta^2} \epsilon'^2 + 2 \frac{2\pi^2}{\beta^2} \epsilon' + \epsilon''' - \epsilon' \epsilon''' - \overset{? \text{ check}}{\epsilon''} + O(\epsilon^3)$$

Periodic ϵ : Terms linear in ϵ will cancel when integrated

$$\text{By parts} \quad \int -\epsilon''' \epsilon' = -\epsilon'' \epsilon' + \int \epsilon''^2$$

Then

$$I = -\frac{\phi_b}{8\pi G} \int_0^\beta d\tau \left(\frac{2\pi^2}{\beta^2} + \frac{2\pi^2}{\beta^2} \epsilon'^2 - \frac{1}{2} \epsilon''^2 \right)$$

Now expand ε in Fourier modes:

$$\varepsilon(\tau) = \frac{\beta}{2\pi} \sum_{n>1} \left(e^{-i\frac{2\pi}{\beta} n\tau} \varepsilon_n + e^{i\frac{2\pi}{\beta} n\tau} \varepsilon_n^* \right) \quad \begin{array}{l} \varepsilon_n^* = \varepsilon_{-n} \\ \text{real } \varepsilon \end{array}$$

We exclude $n=0, \pm 1$ because these modes would generate infinitesimal $SL(2, \mathbb{R})$ transformations, which are symmetries of the system. Including them would be overcounting.

Use $\int_0^\beta d\tau e^{-i\frac{2\pi}{\beta}(n+m)\tau} = \delta_{nm}$

so that $\int_0^\beta d\tau \varepsilon'^2 = 2\beta \sum_{n>1} n^2 |\varepsilon_n|^2$

$$\int_0^\beta d\tau \varepsilon''^2 = \frac{8\pi^2}{\beta} \sum_{n>1} n^4 |\varepsilon_n|^2$$

Then

$$\mathcal{I} = -\frac{1}{4\pi\alpha} \frac{\pi^2}{\beta} \phi_b + \frac{1}{4\pi\alpha} \phi_b \frac{2\pi^2}{\beta} \sum_{n>1} n^2 (n^2 - 1) |\varepsilon_n|^2$$

To perform the one-loop integral we need the integration measure

$$\mathcal{D}\varepsilon = \prod_{n>1} 4\pi n (n^2 - 1) d^2 |\varepsilon_n|$$

Stanford-Witten
+ others

Then

$$\mathcal{Z}[\beta] = e^{S_0 + \frac{\phi_b}{4\pi\alpha} \frac{\pi^2}{\beta}} \int \prod_{n>1} 4\pi n (n^2 - 1) d^2 |\varepsilon_n| \exp \left[\frac{\phi_b}{4\pi\alpha} \frac{2\pi^2}{\beta} \sum_{n>1} n^2 (n^2 - 1) |\varepsilon_n|^2 \right]$$

The integrals are gaussian $\int_{-\infty}^{\infty} dx e^{-cx^2} = \sqrt{\frac{\pi}{c}}$

so

$\frac{1}{n^2}$

$\Rightarrow$ ζ -fn (below)

$\frac{3}{2} - \phi_b \pi^2$

$$Z(p) = e^{S_0 + \frac{\phi_b}{4\pi\epsilon} \frac{\pi^2}{\beta}} \prod_{n>1} \frac{4\pi\epsilon}{\phi_b} \frac{2\beta}{n} = \frac{1}{4\pi^2} \left(\frac{\phi_b}{4\pi\beta} \right)^{3/2} e^{S_0 + \frac{\phi_b}{4\pi\epsilon} \frac{\pi^2}{\beta}}$$

$\rightarrow \zeta\text{-fn (below)}$

$$\Rightarrow E = \frac{3}{2\beta} + \frac{\phi_b}{4\pi\epsilon} \pi^2 T \quad T = 1/\beta$$

$$S = S_0 + \frac{\phi_b}{4\pi\epsilon} \pi^2 T + \underbrace{\frac{3}{2} \log T}_{\text{gets large } T \text{ small } T} + \text{const}$$

$$\prod_{n>1} \frac{a}{n} = \frac{a^{-3/2}}{\sqrt{2\pi}} \quad \text{by zeta-function regularization}$$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\log \prod_{n=2}^{\infty} \frac{a}{n} = \sum_{n=2}^{\infty} (\log a - \log n)$$

$$\sum_{n=1}^{\infty} 1 = \zeta(0) = -\frac{1}{2} \quad \Rightarrow \quad \sum_{n=2}^{\infty} 1 = -\frac{1}{2} - 1 = -\frac{3}{2}$$

$$\sum_{n=1}^{\infty} \log n = -\zeta'(0) = \frac{1}{2} \log 2\pi = \sum_{n=2}^{\infty} \log n$$

$$\Rightarrow \log \prod_{n=2}^{\infty} \frac{a}{n} = -\frac{3}{2} \log a - \frac{1}{2} \log 2\pi \Rightarrow \prod_{n=2}^{\infty} \frac{a}{n} = \frac{a^{-3/2}}{\sqrt{2\pi}}$$

2D black holes from Hi-D gravity

martes, 6 de agosto de 2024

18:14

Many 10-D black holes appear in the spherical reduction of a higher-D black hole, near the horizon, and often in a low-energy / low-temperature (near-extremal or not) limit.

The simplest is AdS_2 from near-horizon (near-) extremal Reissner-Nordström (in any D)

$$ds^2 = - \frac{f(r)}{\left(1 + \frac{q}{r}\right)^2} dt^2 + \left(1 + \frac{q}{r}\right)^2 \left(\frac{dr^2}{f(r)} + r^2 d\Omega_2 \right)$$

$$f(r) = 1 - \frac{r_0}{r}$$

Charge $\propto q$ $r_0 =$ non-extremality parameter

This is not the usual area-radius gauge:

$$\tilde{r} = r + q$$

$$\frac{f(r)}{\left(1 + \frac{q}{r}\right)^2} = \frac{r(r-r_0)}{(r+q)^2} = \frac{(\tilde{r}-q)(\tilde{r}-q-r_0)}{\tilde{r}^2} = 1 - \frac{2M}{\tilde{r}} + \frac{Q^2}{\tilde{r}^2}$$

$$M = q + r_0/2$$

$$r_+ = M + \sqrt{M^2 - Q^2} = r_0 + q$$

$$Q^2 = q(q+r_0)$$

$$M^2 - Q^2 = r_0^2/4$$

Horizon at $r = r_0$. Area = $4\pi(r_0 + q)^2$

Extremal limit: $r_0 = 0$, horizon at $r = 0$, finite area

1-D case takes $r \ll a$. near-horizon limit:

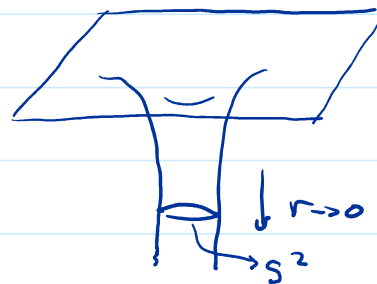
extreme limit: $r \rightarrow 0$, $r \rightarrow \infty$

In This case Taking $r \ll q$, near-horizon limit:

$$ds^2 \approx -\frac{r^2}{q^2} dt^2 + q^2 \frac{dr^2}{r^2} + q^2 d\Omega_2$$

$$= q^2 \left(-\frac{dt + \frac{2}{r} dr}{2} + d\Omega_2 \right)$$

Poincaré-AdS₂ $\times$ S^2



$$\text{spatial distance} \propto \int \frac{dr}{r} = \log r \xrightarrow{r \rightarrow 0} -\infty$$

Now consider near-extremal, near-horizon

Take $r, r_0 \ll q$

$$ds^2 = -\frac{r(r-r_0)}{q^2} dt^2 + q^2 \frac{dr^2}{r(r-r_0)} + q^2 d\Omega_2$$

change $r = \frac{r_0}{2} (\tilde{r} + 1)$ $t = \frac{2q^2}{r_0} \tilde{t}$. Then

$$ds^2 = q^2 \left(-(\tilde{r}^2 - 1) d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 - 1} + d\Omega_2 \right)$$

Rindler-AdS₂ $\times$ S^2

$$\text{or } r = r_0 \cosh^2 p/2 \quad t = \frac{2q^2}{r_0} \tilde{t} \quad p = 2 \operatorname{arctanh} \sqrt{\frac{r-r_0}{r}}$$

$$ds^2 = q^2 \left(-\sinh^2 p d\tilde{t}^2 + dp^2 + d\Omega_2 \right)$$

To get Global $AdS_2 \times S^2$ we should start w/ a wormhole

Global AdS_2 is wormhole-like

$$-g_{tt} = r^2 + \kappa \quad \kappa = -1 : \text{gravitational attraction: +ve energy}$$

$$\kappa = +1 \quad \text{"} \quad \text{repulsion: -ve energy}$$

Thermodynamics:

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}$$

$$S_{BH} = \frac{A_H}{4G} = \frac{\pi r_+^2}{G}$$

$$T = \frac{r_+ - r_-}{4\pi r_+^2}$$

$$M = \frac{r_+ + r_-}{2}$$

Expand for small T :

$$M_0 = Q$$

$$S_0 = \frac{\pi Q^2}{G}$$

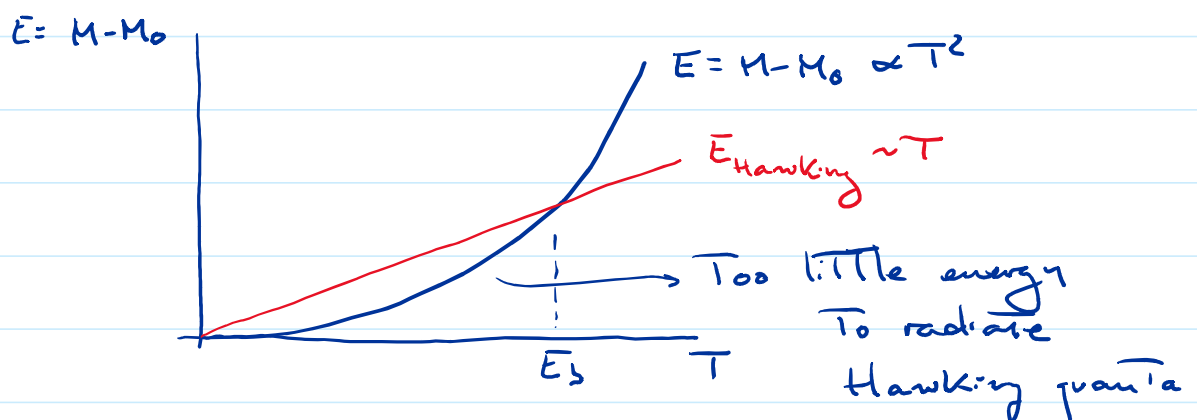
$$M(T) = M_0 + \frac{2\pi^2}{\bar{E}_b} T^2 + O(T^3)$$

$$S(T) = S_0 + \frac{4\pi^2}{\bar{E}_b} T + O(T^2)$$

$$\bar{E}_b = \frac{G}{Q^3}$$

The Terms $M - M_0 \propto T^2$ are reproduced by The
 $S - S_0 \propto T$ classical Schwarzschan

Observe paradox at extremely low T :



Thermodynamics breaks down: fluctuations of radiation are a large perturbation of system.

radiation are a large perturbation of system.

- Semiclassical QFT in curved background breaks down

Backreaction important

- Ensemble inequivalence

- Quantum physics (quantum gravity) important at very low T

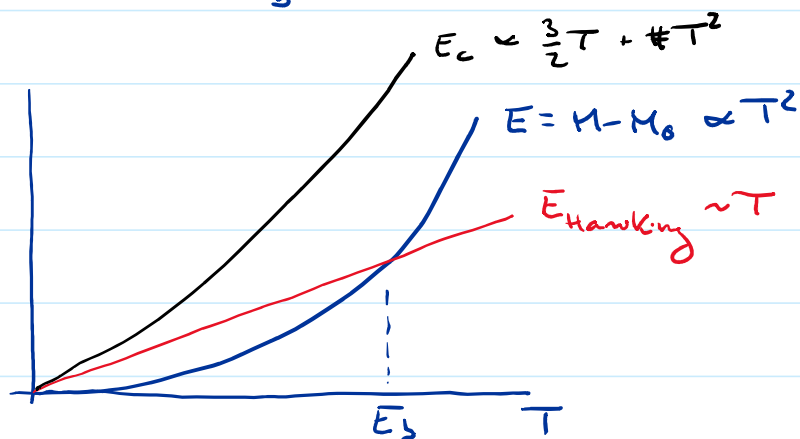
Scale: $E_b = \frac{G}{Q^3} \sim \frac{\ell_{Pl}^2}{r_+^3} \sim \frac{1}{r_+ S_0} \ll \frac{1}{r_+}$

For solar mass size $E_b \sim 10^{-83} \text{ K}$, $\lambda_b = r_+ S_0 \sim 10^{38} \text{ mm}$

One can obtain JT gravity as the theory that describes the near-horizon dynamics of near-extremal charged black holes. (TO DO)

The quantum Schwarzschild corrects this

$$E_c = M - M_0 = \frac{2\pi^2}{E_b} T^2 + \frac{3}{2} T$$



Microstate counting of D1-D5-P black holes

It is a striking irony that the classically simplest black holes—such as the Schwarzschild and Kerr solutions—are the hardest quantum mechanically; and, viceversa, the black holes best understood in quantum gravity—such as the D1-D5-P or MSW (i.e., M5³-P) systems—are very strange classically.¹

In these notes we sketch of how microstate counting works for the archetypal D1-D5-P black hole. We will not give the full derivation, but we will outline the main ideas.

The black hole arises as a solution of a five-dimensional (super)gravity theory that describes the low-energy limit of string theory, with metric

$$ds_{(5)}^2 = -\frac{1}{(H_1 H_5 H_p)^{2/3}} dt^2 + (H_1 H_5 H_p)^{1/3} (dr^2 + r^2 d\Omega_{(3)}) . \quad (1)$$

It carries charges under three $U(1)$ gauge fields, labeled by indices $1, 5, p$. The solution also involves a dilaton field, which is regular on and outside the horizon but will not play a direct role here. This geometry is closely related to the 4D extremal Reissner–Nordström (RN) solution discussed earlier. The functions H_i , $i = 1, 5, p$, are harmonic in $\mathbb{R}^4$,

$$H_i = 1 + \frac{q_i}{r^2} , \quad (2)$$

and if we set them equal $q_i = q$ we obtain

$$ds_{(5)}^2 = -\frac{1}{H^2} dt^2 + H (dr^2 + r^2 d\Omega_{(3)}) , \quad H = 1 + \frac{q}{r^2} . \quad (3)$$

Up to dimension-dependent exponents, this is indeed analogous to the 4D extremal RN black hole. From g_{tt} we see that there is a horizon at $r = 0$, and from $g_{\Omega\Omega}$ we find that the horizon has finite area,

$$\mathcal{A}_H = \Omega_{(3)} \sqrt{q^3} = 2\pi^2 \sqrt{q^3} . \quad (4)$$

The solution exhibits a throat region similar to RN, with near-horizon geometry $\text{AdS}_2 \times S^3$ when $H \rightarrow q/r^2$. For general, unequal but nonzero q_i , the entropy is is

$$S_{BH} = \frac{\mathcal{A}_H}{4G_5} = \frac{\pi^2}{2G_5} \sqrt{q_1 q_5 q_p} . \quad (5)$$

This is the entropy we aim to reproduce through microstate counting in string theory. The q_i are proportional to the charges of the three gauge fields. Before discussing their quantization, it is useful to reformulate the solution.

A common trick is to relate a black hole in D dimensions to a black string in $D+1$ dimensions, wrapped on a compact circle $z \sim z + 2\pi R$. In the same way, our five-dimensional black hole can be uplifted to a six-dimensional black string,²

$$ds_{(6)}^2 = \frac{H_p}{H_1^{3/4} H_5^{1/4}} (dz + (1 - H_p^{-1}) dt)^2 - \frac{1}{H_p H_1^{3/4} H_5^{1/4}} dt^2 + H_1^{1/4} H_5^{3/4} (dr^2 + r^2 d\Omega_{(3)}) . \quad (6)$$

¹Similarly, de Sitter space is very simple classically but notoriously elusive in quantum gravity.

²This is the Einstein-frame metric.

We omit the details of the uplift and how the H_i are rearranged. The solution is translationally invariant along z , and hence string-like, but it is not static: the off-diagonal components g_{tz} indicate frame-dragging along z , much as angular momentum causes dragging in Kerr. Thus, the black string carries linear momentum along z . Asymptotically,

$$g_{tz} = \frac{q_p}{r^2} + O(r^{-4}) \quad (7)$$

so the linear momentum is proportional to q_p . If we set $q_{1,5} = 0$, this reduces to a black string boosted to the speed of light.

The geometry behaves smoothly near $r = 0$: the S^3 and the compact S^1 both remain finite due to cancellations among the H_i . The horizon geometry is $S^1 \times S^3$ with finite size, and its entropy,

$$S_{BH} = \frac{\mathcal{A}_H^{(string)}}{4G_6} \quad (8)$$

agrees with the 5D result (5), once we use $G_6 = 2\pi R G_5$. This consistency is expected: both descriptions refer to the same physical object.

Quantized charges. The advantage of writing this 5D black hole as a 6D black string is that one charge becomes momentum along the compact circle. In quantum mechanics, this momentum is quantized in units of $1/R$. In gravity, conserved quantities like mass, angular momentum, or momentum are extracted from asymptotic behavior of the metric³ so from (7) we can read off the momentum P , and thus the quantization of q_p :

$$q_p = \frac{4G_5}{\pi} P = \frac{4G_5}{\pi} \frac{N_p}{R}. \quad (9)$$

Here N_p is an integer counting momentum units. In the classical solution discreteness is invisible, since validity of the semiclassical approximation requires $N_p \gg 1$.

The other charges $q_{1,5}$ are quantized similarly. Their precise identification depends on the embedding of the solution in string theory, but U-dualities guarantee that if q_p is quantized, then so are $q_{1,5}$. In the embedding of interest to us, q_1 and q_5 correspond to D-brane charges, and one finds⁴

$$q_{1,5} = \sqrt{\frac{4G_5 R}{\pi}} N_{1,5} \quad (10)$$

with integer $N_{1,5}$. Thus the entropy becomes

$$S_{BH} = 2\pi \sqrt{N_1 N_5 N_p}. \quad (11)$$

Remarkably, this expression depends only on integer charges, not on G_5 , R , or other moduli. Supersymmetry guarantees that this structure is preserved as couplings are varied.⁵

³e.g., in 4D $g_{tt} = -1 + 2GM/r + O(r^{-2})$ and $g_{t\phi} = -2J \sin^2 \theta / r + O(r^{-2})$.

⁴We take the internal T^4 to have volume $(2\pi \ell_{string})^4$ (see below).

⁵Strictly, what is protected is a supersymmetric index, whose gravity computation has been clarified in recent years.

Thus we may imagine beginning from this black hole, and then decreasing the strength of the gravitational interactions while keeping the N_i fixed. All the ‘charge radii’ q_i then decrease and at some point the horizon size becomes smaller than the string length and the system is more appropriately described as a state in perturbative string theory, rather than as a black hole⁶. Which state in string theory this is, can be determined by looking at string theory states with the same charges. In some cases we may be able to count their degeneracy, at least for large values of the charges. Before we identify the charges, we anticipate how the counting may be performed.

Counting microscopic states. We have constructed a six-dimensional black string that carries momentum along its length. To reproduce its entropy, we model it as a relativistic circular string wrapped on $z \sim z + 2\pi R$ and carrying quantized excitations.

The oscillation energies are quantized in units of $1/R$, and can be described by a 1+1-dimensional field theory with one bosonic degree of freedom for each polarization. Taking the oscillators to be thermally distributed, extensivity and scale invariance imply an energy of the form

$$E = \sigma RT^2 \quad (12)$$

where σ is a numerical constant that measures the number of degrees of freedom (polarizations) of the string. A standard calculation in statistical mechanics gives

$$\sigma = \frac{\pi^2}{6} \quad \text{for a boson,} \quad \sigma = \frac{\pi^2}{12} \quad \text{for a fermion.} \quad (13)$$

The entropy follows from $dS = dE/T$,

$$S = 2\sigma RT = 2\sqrt{\sigma ER}. \quad (14)$$

Since we are considering massless (and chiral) excitations along the string, they have momentum $P = E$, which is quantized, so

$$E = \frac{N_p}{R} \quad (15)$$

and

$$S = 2\sqrt{\sigma N_p}. \quad (16)$$

This already shows a feature of the entropy of the black hole we seek: $S \propto \sqrt{N_p}$.

For a theory with N_B bosonic and N_F fermionic degrees of freedom, weakly interacting, the contributions simply add:

$$\sigma = \frac{\pi^2}{6} \left(N_B + \frac{1}{2} N_F \right). \quad (17)$$

⁶This is the case when the black hole is embedded as a D1-D5-P solution, but not when it is F1-NS5-P, which has the same Einstein metric and the same expression (11) for the entropy. The reason is that in this case $q_1 \sim G_5 N_1$, $q_5 \sim G_5^0 N_5$, so the size (charge-radius) of the NS5 does not become small as the gravitational coupling is sent to zero.

Thus

$$S = 2\pi \sqrt{\frac{N_B + \frac{1}{2}N_F}{6}} N_p = 2\pi \sqrt{\frac{c}{6}} N_p \quad (18)$$

where $c = N_B + \frac{1}{2}N_F$ corresponds to the central charge of the theory⁷. In a supersymmetric theory $N_B = N_F$, so

$$c = \frac{3}{2}N_B. \quad (19)$$

Therefore, if the effective 1+1 theory has

$$N_B = 4N_1N_5 \quad (20)$$

its entropy matches the black hole formula (11). What remains for us is to obtain this number from a microscopic string picture.

D1-D5 bound state. To identify the underlying string theory degrees of freedom, we uplift the solution to ten dimensions,⁸

$$ds_{(10)}^2 = \frac{H_p}{\sqrt{H_1H_5}} (dz + (1 - H_p^{-1})dt)^2 - \frac{1}{H_p\sqrt{H_1H_5}} dt^2 + \sqrt{\frac{H_1}{H_5}} ds^2(T^4) + \sqrt{H_1H_5} (dr^2 + r^2 d\Omega_{(3)}) . \quad (21)$$

We have added four compact directions forming a T^4 . At $r = 0$, their size is finite since $H_1/H_5 \rightarrow q_1/q_5$. Setting $q_p = 0$ isolates the other charges: with $q_1 = 0$, the solution has translation invariance along z and T^4 , so q_5 corresponds to a five-brane. Conversely, with $q_5 = 0$, q_1 corresponds to a one-brane (a string) along z . Thus, the system is a bound state of a string and a five-brane, aligned along z , carrying momentum in that direction.

Taking the gravitational coupling to zero while keeping N_i and R finite sends all charge-radii q_i to zero, flattening the spacetime. The entropy (11), however, remains finite.⁹ We must therefore have some perturbative string configuration in Minkowski space that gives rise to this entropy. We can identify it from the charges of the state. In string theory, the objects that carry the charges that source the RR fields of the supergravity solution are D-branes. So the supergravity solution must turn, at weak coupling, into a bound state of D1 and D5 branes. Their dynamics is well captured in perturbative string theory via open strings ending on the branes.

To connect with the effective string picture, we take the T^4 to be much smaller than the z -circle. The bound state then appears string-like in six dimensions. The compact T^4 provides polarization directions for oscillations, but modes with momentum along T^4 are heavy and decouple. Supersymmetry ensures fermionic partners as well.

⁷The concept of central charge remains meaningful even in the strongly coupled regime.

⁸This is the string-frame metric, not suitable for computing horizon areas.

⁹The D1 and D5 contributions diverge in mass, but momentum excitations remain finite, as appropriate for viewing the system as excitations above the D1-D5 ground state. Excitations that create D-anti-D brane pairs are suppressed.

For $N_5 = 1$, the N_1 D1-branes bound to the D5 can oscillate freely inside it: these are massless modes.¹⁰ Each D1 has four transverse directions inside the D5, yielding

$$N_B = 4N_1 \quad (N_5 = 1). \quad (22)$$

With N_5 D5-branes, each D1 can oscillate inside all of them.¹¹ Thus

$$N_B = 4N_1N_5, \quad (23)$$

which is precisely the number (20) needed to reproduce the black hole entropy.

Remarks:

- We have described an extremal, *zero-temperature* black hole in terms of a *thermal distribution* of string oscillations. How can this be consistent? The key is that the string supports both left- and right-moving excitations. Suppose each sector is thermally populated, with temperatures $T_{L,R}$, and that their interaction is weak so that $S = S_L + S_R$, $E = P_L + P_R$, and the net momentum is $P = P_L - P_R$, with $P_{L,R} = N_{L,R}/R$. Since $T_{L,R}^{-1} = (\partial S_{L,R}/\partial P_{L,R}) = 2(\partial S_{L,R}/\partial E)_P$, the actual temperature of the full system, $T^{-1} = (\partial S/\partial E)_P$, is

$$T^{-1} = \frac{1}{2} (T_L^{-1} + T_R^{-1}). \quad (24)$$

Thus, if only one type of excitation is present (say $T_R = 0$), the overall temperature vanishes even though the entropy remains finite. This explains how a black hole can carry entropy at zero temperature.

- Extremality and supersymmetry. The microscopic derivation applies strictly to extremal, supersymmetric black holes. Supersymmetry guarantees that the degeneracy of states is invariant under changes of the string coupling g , so the entropy computed at weak coupling continues to hold at strong coupling.¹² Without supersymmetry, levels could shift or lift as g varies, spoiling the agreement.
- Near-extremal black holes. When the system is slightly excited above extremality, both left- and right-moving modes are populated. The ‘effective string’ then emits radiation at temperature (24), via the annihilation of a left- and a right-mover into a bulk closed string quantum. If only one type of excitation is present, no such emission occurs—precisely as expected for a black hole with zero temperature.

¹⁰Oscillations in directions transverse to the D5 brane are instead massive: separating a D1 and D5 brane involves stretching open strings between them.

¹¹More precisely, the oscillations are described by $(1,5)$ and $(5,1)$ open strings, giving N_1N_5 species with four polarizations each. Equivalently, the D1s appear as N_1 instantons in a $U(N_5)$ gauge theory, and c is the dimension of their moduli space.

¹²See footnote 5.

- Rotating black holes. The same picture applies when angular momentum is included, in both extremal and near-extremal cases. The rotation corresponds to an R -charge of the oscillators. The resulting entropy again matches the black hole formula.
- Four-dimensional black holes. Analogous constructions reproduce the entropy of four-dimensional extremal black holes. In those cases, the effective string picture continues to provide the correct microscopic account.
- Near-horizon geometry and AdS/CFT. The near-horizon (or decoupling) limit of the D1–D5 black string is locally $\text{AdS}_3 \times S^3$, much as the extremal Reissner–Nordström solution develops an $\text{AdS}_2 \times S^2$ throat. The ‘effective string’ we have discussed above is in fact a 2D conformal field theory, which appears as the holographic dual of the AdS_3 region. This observation became one of the earliest and most compelling pieces of evidence for the AdS/CFT correspondence.

Now we obtain AdS_3 and BTZ from a charged black string in 6D (could also do in 5D)

Black D1-D5 string:

$$ds^2 = \frac{1}{\sqrt{h_{1,5}}} (-f(r) dt^2 + dx^2) + \sqrt{h_{1,5}} \left(\frac{dr^2}{f(r)} + r^2 d\Omega_3 \right)$$

$$h_{1,5} = 1 + \frac{r_{1,5}^2}{r^2} \quad f(r) = 1 - \left(\frac{r_0}{r} \right)^2$$

x : direction along string, $x \sim x + L$

r_1, r_5 : $\propto$ D1, D5-brane charges

r_0 : non-extremality parameter

$$\left[\begin{array}{l} \text{Might consider boosting it along } x \text{ to get D1-D5-P} \\ ds^2 = \frac{1}{\sqrt{h_{1,5}}} \left(-dt^2 + dx^2 + \frac{r_0^2}{r^2} (\cos \alpha dt + \sin \alpha dx)^2 \right) \\ + \sqrt{h_{1,5}} \left(\frac{dr^2}{f(r)} + r^2 d\Omega_3 \right) \\ \alpha: \text{boost parameter along } x \end{array} \right]$$

Horizon at $r=r_0$

$$\text{Near-extremal: } r, r_0 \ll r_1, r_5 \quad h_{1,5} \approx \frac{r_{1,5}^2}{r^2}$$

Set $\alpha=0$ for simplicity

$$ds^2 \approx \frac{r^2}{r_1 r_5} \left(-\left(1 - \frac{r_0^2}{r^2}\right) dt^2 + dx^2 \right) + r_1 r_5 \frac{dr^2}{r^2 r_0^2} + r_1 r_5 d\Omega_3$$

$$\text{rescale } (t, x) = r_1 r_5 (\tau, \phi) \quad \Delta \phi = \frac{Lx}{r_1 r_5}$$

$$ds^2 = r_{15} \left[\underbrace{-(r^2 - r_0^2) dt^2 + \frac{dr^2}{r^2 - r_0^2} + r^2 d\phi^2}_{\text{static BTZ}} + \underbrace{d\Omega_3}_{S^3} \right]$$

If we have $\alpha \neq 0$ then we get rotating BTZ $\times S^3$

If $\alpha = 0$ $r_0 = 0$ we get Poincaré-AdS₃ $\times S^3$

To get Global AdS₃ $\times S^3$ we must start from a "supertube"

Observe that energy of black string above extremality $\propto r_0^2 \sim$ BTZ mass

momentum of black string $\propto$ BTZ angular momentum

entropy of black string $\propto$ entropy of BTZ $\times$ area of S^3

Let's do the extremal D1-D5-P

$$ds_6^2 = \frac{H_P}{\sqrt{H_1 H_5}} \left(dz + \left(1 - \frac{1}{H_P}\right) dt \right)^2 - \frac{1}{H_P \sqrt{H_1 H_5}} dt^2 + \sqrt{H_1 H_5} (dr^2 + r^2 d\Omega_{(3)}^2)$$

$$H_i = 1 + \frac{Q_i}{r^2}$$

$$\text{Take } Q_{1,5} \gg Q_P, r^2 \text{ so } H_{1,5} \rightarrow \frac{Q_{1,5}}{r^2}$$

$$ds_6^2 \rightarrow \frac{1}{\sqrt{Q_1 Q_5}} r^2 \left[H_P \left(dz + \left(1 - \frac{1}{H_P}\right) dt \right)^2 - \frac{1}{H_P} dt^2 \right] + \sqrt{Q_1 Q_5} \frac{dr^2}{r^2} + \underbrace{\sqrt{Q_1 Q_5} d\Omega_{(3)}^2}_{S^3}$$

Define $\ell^2 = \sqrt{Q_1 Q_5}$ and identify $x \sim x + 2\pi R$

Change $x = R\phi$ so $\phi \sim \phi + 2\pi$

$$\text{Redefine } r^2 + Q_P = \frac{\ell^2}{R^2} \rho^2 \quad Q_P = \frac{\ell^2}{R^2} \rho_+^2 \quad t \rightarrow \frac{R}{\ell} t$$

Then

$$ds^2 = \frac{(\rho^2 - \rho_+^2)^2}{\ell^2} dt^2 - \frac{\rho^2 d\rho^2}{\rho^2 - \rho_+^2} - \frac{2\rho d\rho dt}{\rho^2 - \rho_+^2} - \frac{\rho^2 d\phi^2}{\rho^2 - \rho_+^2} - \frac{\rho^2 d\Omega^2}{\rho^2 - \rho_+^2}$$

Then

$$ds_6^2 = - \underbrace{\frac{(p_-^2 - p_+^2)^2}{\ell^2 p^2} dt^2 + \frac{\ell^2 p^2}{(p_-^2 - p_+^2)^2} dp^2 + p^2 \left(d\phi + \frac{p_+}{\ell^2 p^2} dt \right)^2}_{\text{extremal rotating BTZ bh}} + \ell^2 d\Omega_{(3)}^2$$

locally equiv to AdS_3 : solves $\int dx \sqrt{g} (R + 2/\ell^2)$

Extremal because g_{tt} has double zero at $p = p_+$

More general: non-extremal D1-D5-P

$$ds_6^2 = \frac{H_p}{\sqrt{H_1 H_5}} \left(dz + \frac{r_0^2 \sinh^2 \alpha_p}{2 r^2 H_3} dt \right)^2 - \frac{f}{H_p \sqrt{H_1 H_5}} dt^2 + \sqrt{H_1 H_5} \left(\frac{dr^2}{f} + r^2 d\Omega_{(3)}^2 \right)$$

$$H_i = 1 + \frac{q_i}{r^2} \quad \text{often write } q_i = r_0^2 \sinh^2 \alpha_i \quad i=1,5,p$$

$$f = 1 - \frac{r_0^2}{r^2} \quad r_0 = \text{non-extremality parameter}$$

Take $q_{1,5} \gg r_0^2, r^2, q_p$

$$\text{Define } p_+^2 = \frac{R^2}{\ell^2} r_0^2 \cosh^2 \alpha_p \quad p_-^2 = \frac{R^2}{\ell^2} r_0^2 \sinh^2 \alpha_p$$

Then

$$ds_6^2 = - \frac{(p_-^2 - p_+^2)(p_-^2 - p_-^2)}{\ell^2 p^2} dt^2 + \frac{\ell^2 p^2}{(p_-^2 - p_+^2)(p_-^2 - p_-^2)} dp^2 + p^2 \left(d\phi + \frac{p_+ p_-}{\ell^2 p^2} dt \right)^2 + \ell^2 d\Omega_{(3)}^2$$

4.3 The black hole information loss problem

Shortly after his discovery that black holes emit quantum thermal radiation, Hawking realized a surprising consequence: black hole evaporation implies that classical general relativity and quantum mechanics are incompatible.

Proposed 50 years ago

It was already known that the merger of the two theories was marred by the non-renormalizability of the quantum theory of the massless spin-two field (the graviton). But the latter is a problem of the ultraviolet, high-energy structure of the theory and becomes manifest only when trying to probe the nature of spacetime close to the Planck scale, where the curvatures are so large that the very notion of spacetime geometry becomes dubious. It is therefore a situation where we naturally expect that general relativity, regarded as an effective theory, breaks down.

The problem that Hawking identified is much more surprising, since it appears in situations where the geometry of spacetime is expected to be smooth. The curvature near the horizon of a black hole scales inversely to its size, so for a very large black hole the region around the horizon can be as weakly curved as we wish. Hawking's problem arises for semiclassical black holes with sizes arbitrarily larger than the Planck length. Since there is no diff-invariant quantity that diverges in these regions, why should the effective theory break down?

The problem is: are black holes conventional quantum systems, and if so, how is this compatible with their being perfect absorbers (in the classical limit)

This has turned out to be a surprisingly deep and subtle question—and a very confusing one too. We will not be able to do proper justice to all its intricacy, but we will attempt to make the reader aware of its gist, without going into the resolutions that have been proposed.

It has many facets, many aspects---sometimes referred to as "versions". Some people, lines of work, address only one of them and claim to have solved it (often they have not). This is not enough to claim that the BHIP is solved. We want to understand all its aspects.

4.3.1 Classical prelude: Black holes have no (or too little) hair

A main property of black holes, mentioned in the previous sections, is that they are characterized by very few parameters. The black hole phase space is therefore of low dimensionality.

We referred to this by saying that 'black holes have no hair': the space of black hole solutions is completely characterized by the conserved charges of the black hole, namely its mass, angular momentum, and charges associated to local conservation laws such as electric charges. Nowadays we know of the existence of many kinds of black hole hair. Nevertheless, the phase space of black hole solutions is still characterized by a number of parameters that, for all we know, is much smaller than the huge number $\mathcal{A}_H/\ell_{\text{Planck}}^2$ that the Bekenstein-Hawking formula suggests.

A few years ago one of these aspects was largely solved: recovering the Page curve. But the solution left many questions unanswered eg does an infalling observer after the Page time experience a smooth horizon or not? So there are still very important aspects of the BHIP that remain unsolved.

Therefore, when a system collapses to form a black hole, it appears that almost all of the initial information of the configuration of collapsing matter is lost in the process. The final state only knows about a few aggregate, macroscopic quantities. Classically, there is no way that this information can be retrieved. Therefore the black hole appears to be a sink where phase-space volume is destroyed.

An important question is: Is it possible to find a resolution to all the main aspects of the BHIP that relies only on IR physics, like the effective theory of the GPI? Or do we need more detailed UV info? It'd look like it's the former

4.3.2 Basic formulation of the information loss problem

One may still regard this as not truly problematic, because the information about the state of the matter that formed the black hole may somehow be preserved inside the black hole. However, if the black hole evaporates and disappears then the problem comes back²¹. We will formulate the puzzle in different manners, each at a different level of sophistication and detail. The first formulation contains the core of the problem. The next one makes it more precise. Further formulations are possible that highlight other subtleties of the problem, but we will not discuss them here.

Fundamental irreversible evolution. The original configuration of matter may have been arbitrarily complicated, needing a huge number of parameters to describe it. All but a handful of these data are lost to the outside observers when the black hole forms. The black hole evaporates by emitting Hawking radiation, but this radiation, having a Boltzmann spectrum, only carries information about the total mass of the black hole, and about the conserved charges conjugate to 'chemical potentials' such as angular velocities, electric potentials etc. Thus, when the black hole finally disappears, the initial information is lost.

²¹There is a version of the information loss problem that does not involve the collapse and evaporation of a black hole, but instead deals with the very-long-time behavior of fluctuations in the system of a black hole in equilibrium with its

disappears, the initial information is lost.

²¹There is a version of the information loss problem that does not involve the collapse and evaporation of a black hole, but instead deals with the very-long-time behavior of fluctuations in the system of a black hole in equilibrium with its radiation. Significant progress has recently been made in this direction, but we will not discuss it in these notes.

This is a problem of **fundamental irreversibility** at a deeper level than statistical thermodynamic irreversibility: from the final radiation state **we cannot reconstruct, even in principle, the initial matter state.**

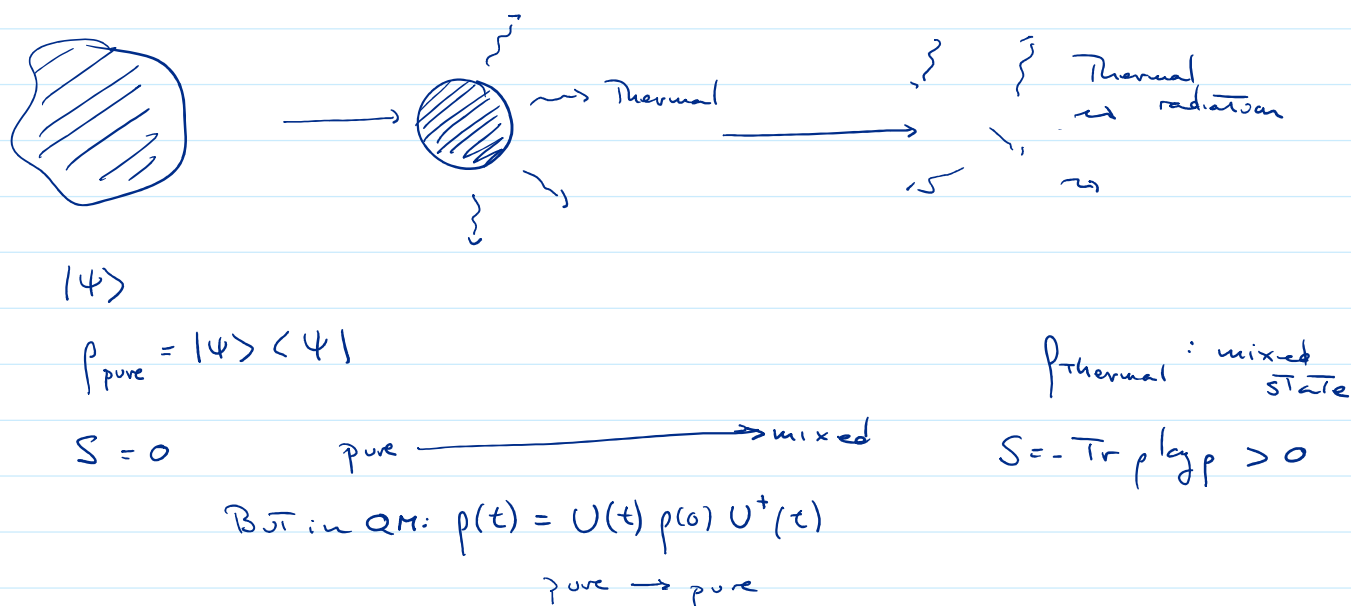
Non-unitary quantum evolution. In the previous paragraph we used 'information' in a rather loose sense. Let us now make it more precise by phrasing the previous problem in quantum mechanical parlance. States in quantum mechanics can be generically described using density matrices ρ . For pure states, which correspond to vectors $|\Psi\rangle$ (properly, rays) in a Hilbert space, the matrix $\rho_{\text{pure}} = |\Psi\rangle\langle\Psi|$ has von Neumann entropy $-\text{Tr}\rho_{\text{pure}} \log \rho_{\text{pure}} = 0$. Mixed states are characterized by having instead $-\text{Tr}\rho_{\text{mixed}} \log \rho_{\text{mixed}} > 0$. Thermal ensembles $\rho_{\text{th}} = Z^{-1} \sum_i e^{-E_i/T} |i\rangle\langle i|$ are of this kind and in fact maximize the entropy. Unitary evolution in quantum mechanics takes a pure state into another pure state

$$|\Psi\rangle \rightarrow U(t)|\Psi\rangle, \quad \rho_{\text{pure}} \rightarrow \rho_{\text{pure}}, \quad (4.34)$$

($U^{-1} = U^\dagger$) and therefore no (fine-grained) entropy can be generated. This is the quantum-mechanical counterpart of Liouville's theorem that Hamiltonian evolution conserves phase space volume, i.e., there are no sinks in phase space. **Unitary evolution is also the quantum-mechanical statement that the initial state can be reconstructed out of the final state.** So unitarity can be regarded as implementing the notion of **fundamental reversibility of quantum-mechanical evolution.**

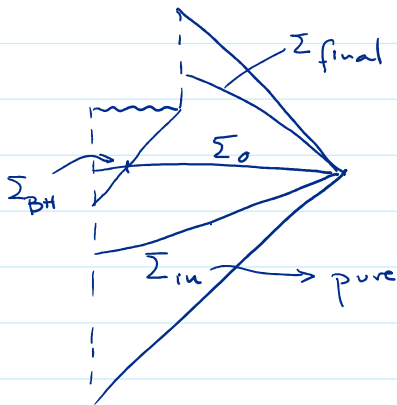
We may take the initial state of the matter that will collapse into a black hole to be a pure state. During most of the evaporation the radiation is thermal to a very good approximation. When the evaporation is completed, only this radiation remains. Thus the evolution of the system violates unitarity, which is one of the basic assumptions of quantum mechanics. The 'black hole information problem' is now stated as the 'black hole unitarity problem'.

Imagine that we manage to collect all the radiation and all the final products of the evaporation of the black hole, isolating them so well as to prevent any losses of coherence to external systems. We **pass these black hole remains through a quantum super-computer** (still under construction) which attempts to reconstruct the initial state, that is, it tries to **find a unitary transformation that, acting on the final state, would yield back the initial one.** **Hawking affirmed that the quantum computer will fail.**



Hawking (1976): evolution in quantum gravity is not unitary

"Nice slice" argument:



State in Σ_0 is entangled with the state in Σ_{BH}

↓
Mixed state in Σ_0 is purified w/ system in Σ_{BH}

State in Σ_{final} is determined only by the state in Σ_0

⇒ State in Σ_{final} is mixed

Bags of gold and the info capacity of a BH

No-cloning Theorem:

$$U|\psi\rangle|0\rangle = |\psi\rangle|\psi\rangle \quad : \text{cloning}$$

$$U(|\psi\rangle + |\phi\rangle)|0\rangle = (|\psi\rangle + |\phi\rangle)(|\psi\rangle + |\phi\rangle) = |\psi\rangle|\psi\rangle + |\phi\rangle|\phi\rangle + |\phi\rangle|\psi\rangle + |\psi\rangle|\phi\rangle$$

$$U|\psi\rangle|0\rangle + U|\phi\rangle|0\rangle = |\psi\rangle|\psi\rangle + |\phi\rangle|\phi\rangle \neq$$

With an ancilla A:

Cloning: $U|\psi 0 A\rangle = |\psi \psi A'\rangle$: impossible if
(reverse: no-deleting theorem) U is linear

$$U(|\psi 0 A\rangle + |\phi 0 A\rangle) = |\psi \psi A'\rangle + |\phi \phi A''\rangle \quad (1)$$

$$\begin{aligned} U(|\psi\rangle + |\phi\rangle)|0A\rangle &= (|\psi\rangle + |\phi\rangle)(|\psi\rangle + |\phi\rangle)|A'''\rangle \\ &= |\psi\psi A'''\rangle + |\phi\phi A'''\rangle \\ &\quad + |\psi\phi A'''\rangle + |\phi\psi A'''\rangle \quad (2) \end{aligned}$$

(1) $\neq$ (2)

Black Hole Information Problem (II): Page curve and entanglement islands

sábado, 3 de agosto de 2024 17:42

Corrections due to backreaction. A better-looking alternative is to imagine that black hole evaporation is not essentially different from the process of burning a chunk of coal, or, conceptually equivalently, gradually annihilating a lump of matter by throwing antimatter at it. We know that this process is governed by quantum mechanics and thus the evolution is unitary. Although in practice it is very difficult to reconstruct the initial state from the final radiation, we know that the information must be there in the form of extremely subtle correlations among radiation quanta. Since we know that Hawking's calculation must at any stage receive small corrections due to the backreaction effects of the radiation on the black hole, it could be that these corrections serve the purpose of encoding the information on the outgoing quanta, in the form of small deviations from the exactly thermal spectrum.

Appealing though this may seem, it can be argued that small deviations in the Hawking emission of each quantum cannot add up to the required total information. In the case of the annihilation of a lump of matter by throwing antimatter, or any initially pure system emitting radiation in a manner governed by a Hamiltonian, the entanglement between the emitted radiation and the burning system initially grows, peaking at the moment when half the total energy has been radiated. From that point on, the entanglement decreases, meaning that the radiation carries more and more information of the total state, and finally the entanglement disappears: all the information is stored in the radiation (although in a highly scrambled manner).

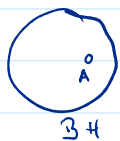
This is not what Hawking emission appears to be doing. The evaporation of a black hole proceeds by creating pairs of quanta. Each member of the pair is entangled with the other: they can be regarded as an EPR pair. One of the two quanta falls into the black hole; the other escapes to infinity. Thus, the entanglement between the interior and the exterior of the black hole grows monotonically along the evaporation. If backreaction results in a small modification in the individual emission process, the entanglement will still keep growing, only now at a slightly lower rate. So it does not seem possible that backreaction, which must be small for large black holes, can effect the reversal in the growth of the entanglement at mid-evaporation that is required to maintain unitarity of the entire process.

Page curve and black hole unitarity

Consider a system made of N Bell pairs

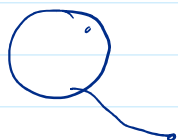


It is "evaporating" by emitting members of Bell pairs



$$\Delta S = \log 2$$

radiation



$$S = \frac{N}{2} \log 2$$



$$\Delta S = -\log 2$$

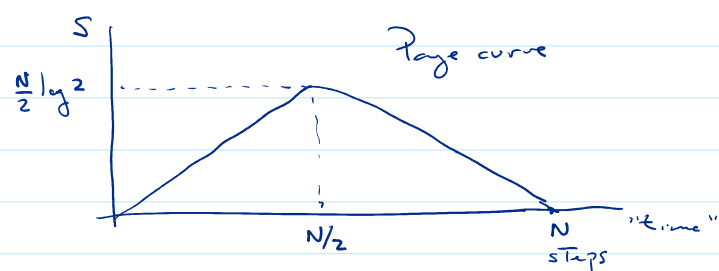
A B

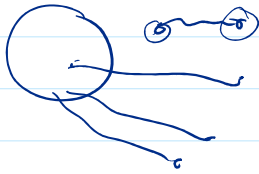
$$|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B)$$

$$\rho_A = \text{Tr}_B |\psi\rangle \langle \psi| = \frac{1}{2} (|0\rangle_A \langle 0| + |1\rangle_A \langle 1|)$$

$$p_0 = \frac{1}{2} \quad p_1 = \frac{1}{2}$$

$$S_A = -\sum p_i \log p_i = \log 2$$

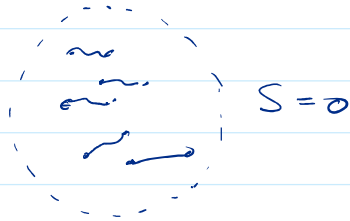




$$\Delta S = -\lg 2$$

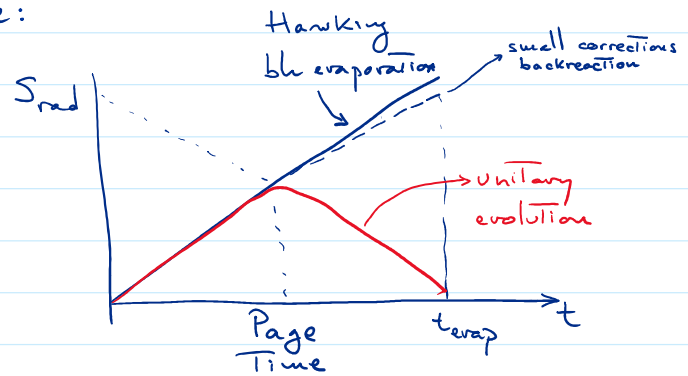
$N/2$

steps

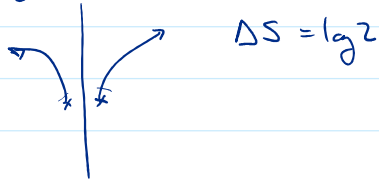


In an evaporating black hole:

Instead of Bell pairs, we have thermally entangled states, but they're analogous.



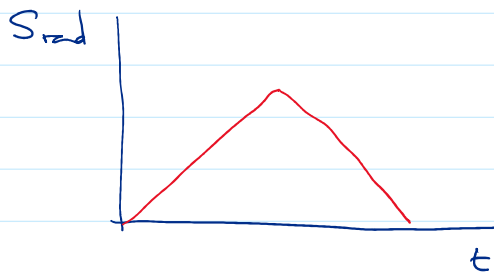
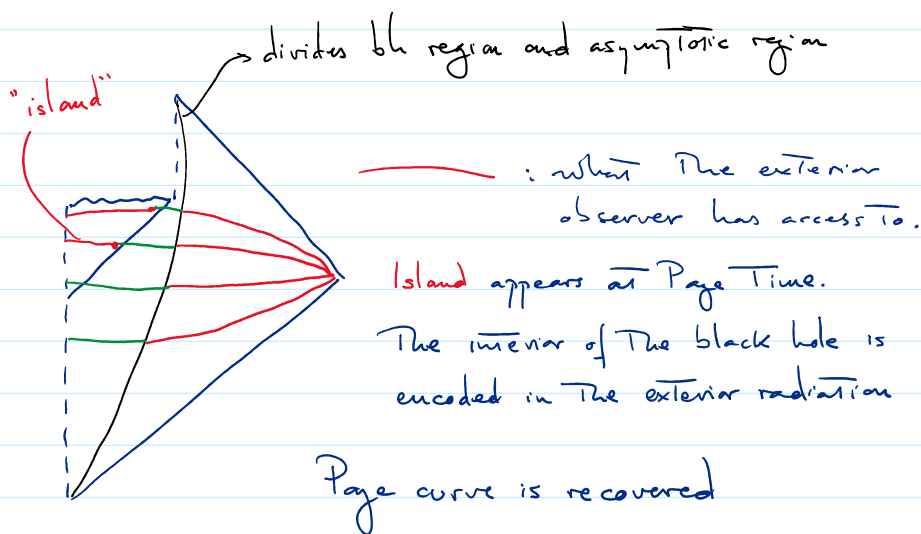
Hawking evaporation:



After the Page Time, this picture can't hold any more.

The created quantum must be entangled w/ early radiation.

Entanglement islands



This has been computed using the euclidean gravitational path integral

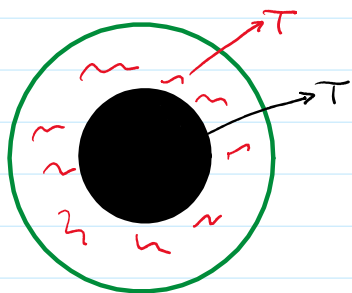
Page curve for eternal black holes

Take a black hole with Temperature T , and at $t=0$ put it in contact with a "radiation bath" at the same Temperature. Initially the bath is in a pure state that is very approximately Thermal. Say they're initially separated by a wall that is then removed.

(The entire system is in a box, or in AdS , in such a way that stable Thermal equilibrium can be achieved).



The bh and the bath exchange radiation



The bh and the bath exchange radiation in detailed equilibrium, in such a way that the bh mass does not change:

radiation flux in = radiation flux out.

The geometry doesn't change (good simplification!)

but the system is evolving in a subtle way:

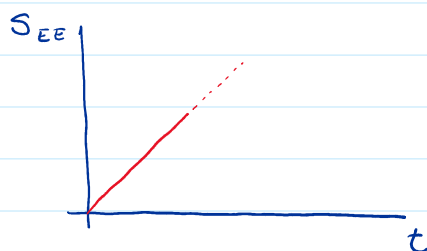
The two subsystems, bh and bath, are becoming increasingly entangled.

The quanta that the bh emits are entangled with interior partners, and the quanta of the bath that go into the bh are also entangled with the other bath radiation.



$$\Delta S = 2 \ln 2$$

$$\Delta E = 0$$

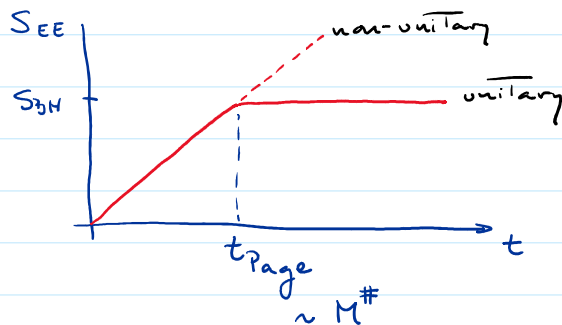


entanglement between bath and bh grows linearly in time

If Hawking's picture of evaporation remains correct, then S_{EE} keeps growing w/out bound.

BUT if S_{BH} measures "The number of Bell pairs that the bh consists of", then S_{EE} should saturate when $S_{EE} = S_{BH}$: external radn

saturate when $S_{EE} = S_{BH}$: external radn can only be entangled with a number $e^{S_{BH}}$ of internal bh states:



A Page curve of This Type has been recovered by analyzing the problem with the gravitational path integral, incorporating spacetime wormholes. The number of independent internal states of the black hole, constructed and computed w/ the QPI w/ wormholes, is not ∞ but $e^{S_{BH}}$.

This doesn't require string Theory, or a UV-complete quantum Theory of gravity. The QPI does it in a very universal way!

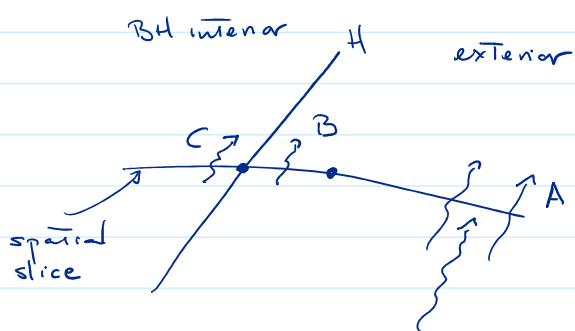
Entanglement structure and strong subadditivity in black hole evaporation

Consider that the black hole has been evaporating for at least as long as half of its lifetime, i.e. it's past the Page time. We want this because, past this time, if evolution is unitary, the information must be coming out of the black hole, i.e. the emitted radiation must be entangled with the early radiation, so "we recover a full Bell pair outside the black hole" (after distillation)

(In the following we'll always be talking about fine-grained, von Neumann entropies to quantify entanglement between disjoint subsystems)

Consider a BH that has been radiating for at least as long as the Page time (it could be an evaporating black hole, or one in equilibrium w/ radiation)

Take a Cauchy slice and consider the quantum radiation system in it, divided into three disjoint regions:



A: first half of the early radiation

B: Hawking modes that are just being produced near the horizon

C: inside partner modes near horizon (partners of B and A)

$R = ABC$ is pure, complete system

A, B, C are disjoint subsystems

B is a region within $\sim R_{\text{BH}}$ of the horizon where a Hawking quantum is being produced. It could be up to the photon ring

The main problem is that according to Hawking's calculation

... that is ...

The main problem is that according to Hawking's calculation the pair that is produced is like a Bell pair, i.e. maximally entangled (or almost) so

$$S(BC) < S(B) : B \text{ is significantly entangled w/ } C$$

On the other hand, after the Page time, if the system is evolving unitarily, information should be coming out of the black hole and outgoing Hawking quanta should have significant entanglement w/ the earlier radiation, i.e.

$$S(BA) < S(B) : B \text{ is significantly entangled w/ } A$$

and also $S(BA) < S(A)$ for the same reason: adding B to A reduces its entanglement entropy

This is a problem due to "monogamy of entanglement": There's a limit to how much a given system can be entangled w/ several other systems.

"Monogamy" says that a given qubit cannot be maximally entangled w/ two different qubits

Then:

If B is highly entangled w/ C as Hawking says it always is,

Then

B cannot be highly entangled with A, as unitary evolution requires after the Page time

Indeed we expect from Hawking that when we add the quantum of radiation B to A, the combined system AB will be more entangled with C (ie have more entropy) than A alone was, ie we expect $S(AB) > S(A)$

This contradicts the condition from unitary evolution that adding B to A after the Page time should reduce its entropy so $S(AB) < S(A)$

In QM there's a general result that formalizes this:

STRONG SUBADDITIVITY (aka weak monotonicity)

Subadditivity: for a bipartite system

$S(AB) \leq S(A) + S(B)$: splitting the system AB increases the entropy

Equality iff there's no entanglement between A and B

(Notice that above we say we expect $S(AB) > S(A)$ but this is because there's a third system C with which A and B are entangled, ie the system is not bipartite)

If A, B, C are strictly disjoint systems:

$$S(AB) + S(BC) \geq S(A) + S(C) \quad (1)$$

The idea is that a given qubit in B can be entangled with A, reducing $S(AB)$, or entangled with C, reducing $S(BC)$, but it cannot be too entangled w/ both (monogamy)

Equivalent formulations of SSA:

$$S(AB) + S(BC) \geq S(B) + S(ABC) \quad (1b)$$

purify w/ D

$$\begin{aligned} S(ABCD) &= 0 \\ S(ABC) &= S(D) \\ S(AB) &= S(CD) \end{aligned}$$

$$\text{Then } S(CD) + S(BC) \geq S(B) + S(D)$$

Now rename $C \rightarrow B, D \rightarrow A, B \rightarrow C$

$$S(AB) + S(BC) \geq S(A) + S(C) \quad (1)$$

Also: $S(B|A) + S(B|C) \geq 0$

where $S(B|A) = S(AB) - S(A)$ is the conditional entropy

It can also be expressed in terms of the mutual information between two systems, defined as

$$I(A, B) = S(A) + S(B) - S(AB)$$



It quantifies how much you can know about B by knowing only A, and viceversa.

Then: subadditivity is $I(A, B) \geq 0$ (This is conventional subadditivity of Thermodynamic entropy)

Strong " is $I(A, BC) \geq I(A, B)$: Knowing both B and C will tell you about A at least as much as knowing only B

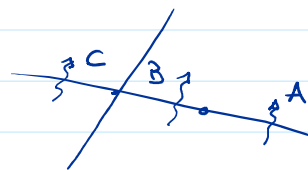
$$\text{gives } I(C, AB) \geq I(C, B)$$

(This may seem obvious for classical information, but it is also true for quantum information)

Returning to our radiating black hole system,

we'll use these notions to sharpen the different

assumptions in the BHP and their consequences.



Consider the following assumptions:

(1) Quantum Theory applies to the entire system
("The rules of quantum field theory in curved spacetime apply")

Then SSA must hold:

$$S(AB) + S(BC) \geq S(A) + S(C) \quad (1)$$

(2) Unitary evaporation requires that after the Page Time, the emitted radiation B is entangled w/ the early radiation A

$S(AB) < S(A)$ (2) : $\exists$ entanglement between B and A, so that information is collected in the radiation in the asymptotic region
strict since we want non-zero entanglement

Eventually we want $S(AB) \approx 0$ but we won't require it

(3) Smoothness of the horizon requires that B is created in an entangled Bell-like pair with a quantum in the interior C, so that

$S(BC) < S(C)$ (3)
strict since we want non-zero entanglement

This is the structure of the state of a quantum field that is regular across a horizon. We saw it for a Rindler horizon: The Minkowski vacuum (regular across the horizon) is fully entangled when the system is divided in two by the horizon. And this structure is locally valid for all smooth horizons. If we break the entanglement between the two sides of the horizon, the state of the quantum field will not be regular: The stress tensor will diverge at the horizon, and this won't be smooth anymore.

The paradox is that (1), (2), (3) are incompatible.

(2) + (3) : $S(A) + S(C) > S(AB) + S(BC) \Rightarrow \Leftarrow (1)$

It then seems like we must give up one of the three assumptions:

~~(1)~~ Firewall proposal: Keep SSA + Unitarity. Then the horizon can't be smooth

~~(2)~~ Information loss: Keep SSA + smooth horizon

~~(3)~~ Breakdown of QM: Keep unitary evaporation and smooth horizon

The ISLANDS resolution says that A, B, C are not disjoint:

after the Page Time, $A \supset C$, so SSA does not apply: (X)

This would seem to allow to preserve unitary evaporation and a smooth horizon.

However, the mechanism that makes $A \supset C$ is unclear and not based on conventional QFT in curved spacetime, it'd be a "subtle violation of (1)" that seems to require an unusual

type of non-locality, beyond the harmless non-locality of conventional entanglement, but hopefully not as drastic and risky) as a Hamiltonian non-locality, like non-local interactions in the action, which in general we don't know how to make sense of in conventional QFT.

There are other aspects of The B*IP
for a black hole in Thermal equilibrium
That appear at much later Times, $t \sim e^S$

Late Time correlations and information loss

Study how Two-point correlation functions
behave in a system initially in Thermal equilibrium:

$$\begin{aligned} \langle \mathcal{O}(t) \mathcal{O}(0) \rangle_\beta &= \text{Tr}(e^{-\beta H} \mathcal{O}(t) \mathcal{O}(0)) \quad \text{perturb at } t=0, \\ &\quad \text{measure again at } t \\ &= \sum_{n,m} e^{-\beta E_n} |\langle n | \mathcal{O} | m \rangle|^2 e^{-it(E_m - E_n)} \end{aligned}$$

Sum over discrete spectrum of energies E_n .

Initially, correlations decay $\sim e^{-t/\tau}$

$\tau \propto \beta$: Thermalization Time

Later

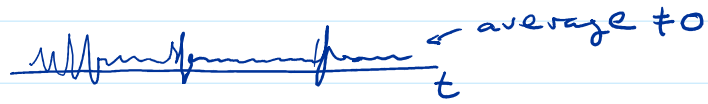
$e^{-it(E_n - E_m)}$: erratic oscillations in Time

Very late Times: oscillations average out and
 $n=m$ dominates

$$\langle \mathcal{O}(t) \mathcal{O}(0) \rangle_\beta \rightarrow \sum_n e^{-\beta E_n} |\langle n | \mathcal{O} | n \rangle|^2$$

Correlations initially decay, but never completely
disappear:

Correlations initially decay, but never completely disappear:



This does not happen in a black hole.

Initial decay $e^{-t/\tau}$: quasinormal decay

Massless field propagation in bh:

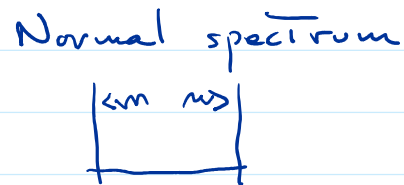
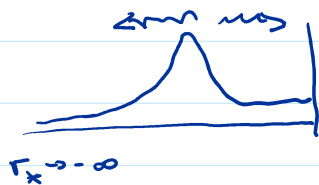


$$-\frac{\partial^2 \psi}{\partial r_*^2} = (\omega^2 - V) \psi(r_*)$$

Study The energy spectrum of bh oscillations

Put black hole in a box at $r_* = R$.

Spectrum is not discrete, but continuous and with absorptive part



Absorption by horizon

Quasinormal spectrum $\omega = \omega_0 - i/\tau$

$$e^{-i\omega t} = e^{-i\omega_0 t - t/\tau}$$

↑ decay

In a black hole

$$\langle \theta(t) \theta(0) \rangle_p \sim e^{-t/\tau} \xrightarrow{t \rightarrow \infty} 0$$

Correlations disappear completely.

Correlations disappear completely.

Black hole in box does not behave according to QM. Info about initial perturbation is completely lost.

Simpler observable: Spectral form factor

For $\langle \mathcal{O}(t) \mathcal{O}(0) \rangle_\rho$ we need to know not only the energy spectrum but also the matrix elements of $\mathcal{O}$. We don't need to be so specific.

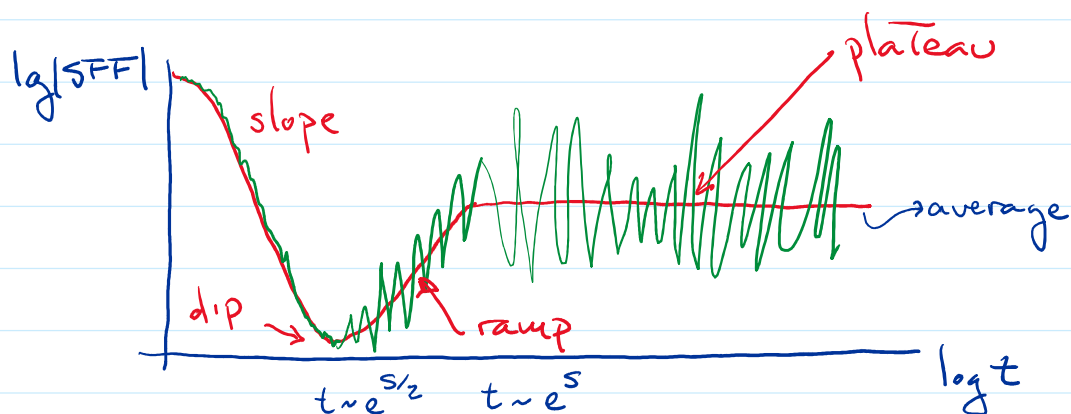
$|\langle n | \mathcal{O} | m \rangle|^2$ is generally expected (ETH)
To be a smooth function of energy.

Ignore it and focus on the energy spectrum:

$$\sum_{n,m} e^{-(\beta+it)E_n} e^{-(\beta-it)E_m} = \underbrace{Z(\beta+it) Z(\beta-it)}_{\text{"Spectral form factor"}}$$

In a chaotic system with a random energy spectrum (eigenvalue statistics given by random matrix distribution)

The SFF behaves like



slope: Thermal relaxation

slope: Thermal relaxation

dip: structure of energy levels begins to matter

ramp: eigenvalue repulsion. Very generally $\propto t$

plateau: discreteness of spectrum

$t \sim e^S$ corresponds to separation between energy levels

$$\Delta \omega \sim \frac{1}{e^S}$$

The GPI w/ wormholes allows to reproduce
The ramp.

The Black Hole Guide to the Quantum Theories of Gravity

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