Lecture 1

3/11/25

INTRODUCTION TO SUPERSYMMETRY

- Rigid Susy
- Primarily $N=1$ susy in 4D

PLAN

1) Hystorical motivations (also modern) for susy
2) $N=1$ susy in 4D
3) Representations of susy algebra
4) N-extended susy in 4D

5) $N=1$ superspace

6) Wess-Zumino model

7) SUSY Yang-Mills theories

8) Non-renormalization theorems (finiteness of $N=4$ SYM)

References

1) M. Sohnius, Phys.Rep. 128 (1985) (INSPIRE)

2) John TERNING "Modern Supersymmetry"

3) Matteo Bertolini (lectures) → look in internet

4) D. Freedman & T. Van Proeyen

5) Wess-Bagger

6) J. Gates, M. Grisaru, M. Roček, W. Siegel
"Superspace and one thousand lessons in susy"

7) Adel Bilal, hepth/0101005

8) P. Argyres (look for his personal web page)

PREREQUISITES

- QFT
- Gauge theories
- Group theory - Lorentz

- Spinor algebra, conventions –
  Tensorial algebra

Conventions

4D Minkowski spacetime $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$

Spinorial representations of Lorentz group

- Weyl spinors

Left $\quad \psi_L = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} \quad \alpha = 1,2 \quad \psi_L \in \left(\frac{1}{2}, 0\right)$

Right $\quad \psi_R = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad \dot{\alpha} = \dot{1}, \dot{2} \quad \psi_R \in \left(0, \frac{1}{2}\right)$

- Dirac spinor

$$\psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad \in \left(\tfrac{1}{2},0\right) \oplus \left(0,\tfrac{1}{2}\right)$$

Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

Weyl basis we define 2 sets of 2×2 matrices

$$\sigma^\mu_{\alpha\dot{\alpha}} \equiv \left(\mathbb{1}_{2\times 2}, \sigma^i\right)$$

$i = 1,2,3$ (Pauli matrices)

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \equiv \left(\mathbb{1}_{2\times 2}, -\sigma^i\right)$$

We choose for γ^μ the following 4×4 matrices

$$\gamma^\mu = \left(\begin{array}{c|c} 0_\alpha{}^\beta & \sigma^\mu{}_{\alpha\dot\beta} \\ \hline \bar\sigma^{\mu\,\dot\beta\alpha} & 0^{\dot\alpha}{}_{\dot\beta} \end{array}\right)$$

$$\gamma^\mu \psi_D = \left(\begin{array}{c|c} 0_\alpha{}^\beta & \sigma^\mu{}_{\alpha\dot\beta} \\ \hline \bar\sigma^{\mu\,\dot\beta\alpha} & 0^{\dot\alpha}{}_{\dot\beta} \end{array}\right) \begin{pmatrix} \psi_\beta \\ \bar\chi^{\dot\beta} \end{pmatrix}$$

$$= \begin{pmatrix} (\sigma^\mu)_{\alpha\dot\beta} \, \bar\chi^{\dot\beta} \\ (\bar\sigma^\mu)^{\dot\beta\beta} \, \psi_\beta \end{pmatrix}$$

In general: $\xi\eta \equiv \xi^\alpha \eta_\alpha$

$\bar\xi\bar\eta = \bar\xi_{\dot\alpha} \bar\eta^{\dot\alpha}$

What is ξ^α ?

What is $\bar\xi_{\dot\alpha}$?

We can introduce antisymmetric tensors $\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$

$\varepsilon_{\dot\alpha\dot\beta}$ $\varepsilon^{\dot\alpha\dot\beta}$

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \varepsilon_{12} = -1$$

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \varepsilon^{12} = 1$$

$$\varepsilon^{\alpha\beta} \varepsilon_{\beta\gamma} = \delta^{\alpha}{}_{\gamma}$$

The same for $\varepsilon_{\dot\alpha\dot\beta}$, $\varepsilon^{\dot\alpha\dot\beta}$ $\rightsquigarrow \begin{cases} \varepsilon_{\dot1\dot2} = -1 \\ \varepsilon^{\dot1\dot2} = 1 \end{cases}$

Raising & lowering spinorial indices

$$\psi_\alpha \rightsquigarrow \psi^\alpha \equiv \varepsilon^{\alpha\beta}\psi_\beta$$

$$\bar\chi^{\dot\alpha} \rightsquigarrow \bar\chi_{\dot\alpha} \equiv \varepsilon_{\dot\alpha\dot\beta}\bar\chi^{\dot\beta}$$

$$\zeta\eta \equiv \zeta^\alpha\eta_\alpha \equiv (\varepsilon^{\alpha\beta}\zeta_\beta)\eta_\alpha$$

$$= \zeta_\beta(-\varepsilon^{\beta\alpha}\eta_\alpha) = -\zeta_\beta\eta^\beta$$

$$\bar\zeta\bar\eta \equiv \bar\zeta_{\dot\alpha}\bar\eta^{\dot\alpha} = -\bar\zeta^{\dot\alpha}\bar\eta_{\dot\alpha}$$

$$(\sigma^\mu)_{\alpha\dot\alpha} \rightsquigarrow (\sigma^\mu)^{\beta\dot\beta} \equiv \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}(\sigma^\mu)_{\alpha\dot\alpha}$$

$$= (\bar\sigma^\mu)^{\dot\beta\beta}$$

<u>DO IT</u>

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} \quad \rightsquigarrow \quad (\bar{\sigma}^\mu)_{\dot{\beta}\beta} \equiv \varepsilon_{\beta\alpha}\,\varepsilon_{\dot{\beta}\dot{\alpha}}\,(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}$$

$$= \sigma^\mu{}_{\beta\dot{\beta}}$$

<u>DO IT</u>

<u>Exercise</u>: Prove that

$$\bar{\chi}\,\bar{\sigma}^\mu\psi \equiv \bar{\chi}_{\dot{\alpha}}\,(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}\,\psi_\alpha \overset{!}{=} -\,\psi\,\sigma^\mu\,\bar{\chi}$$

[Remember that $\bar{\chi}_{\dot{\alpha}}\,\psi_\alpha = -\,\psi_\alpha\,\bar{\chi}_{\dot{\alpha}}$]

- <u>Dirac conjugate</u> of a spinor

$$\psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad \underset{\text{Dirac conjugate}}{\rightsquigarrow} \quad \bar{\psi} = \psi^\dagger \gamma^0$$

$$\uparrow$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\bar{\psi} = \left((\psi_\alpha)^* \;\; (\bar{\chi}^{\dot{\alpha}})^* \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left((\bar{\chi}^{\dot{\alpha}})^* \;\; (\psi_\alpha)^* \right)$$

Our conventions :

$$\begin{cases} \psi_\alpha^* = - \bar{\psi}_{\dot{\alpha}} \\ (\psi^\alpha)^* = - \bar{\psi}^{\dot{\alpha}} \\ (\bar{\psi}^{\dot{\alpha}})^* = - \psi^\alpha \\ \bar{\psi}_{\dot{\alpha}}^* = - \psi_\alpha \end{cases}$$

$$\Rightarrow \quad \bar\Psi = \left(-\chi^{\alpha} \quad -\bar\psi_{\dot\alpha}\right)$$

$$(\xi\eta)^{*} = \eta^{*}\xi^{*}$$

- **Charge conjugation** $\rightsquigarrow$ matrix $C = -i\gamma^0\gamma^2$

$$\Psi = \begin{pmatrix}\psi_\alpha \\ \bar\chi^{\dot\alpha}\end{pmatrix} \rightsquigarrow \Psi^c \equiv C\,\bar\Psi^T = -i\gamma^0\gamma^2\left(\Psi^\dagger\gamma^0\right)^T$$

$$= -i\gamma^0\gamma^2\gamma^0\,\Psi^{*} = i\underset{\begin{pmatrix}0 & \sigma^2 \\ -\sigma^2 & 0\end{pmatrix}}{\underset{\uparrow}{\gamma^2}}\Psi^{*}$$

$$= \begin{pmatrix}\chi_\alpha \\ \bar\psi^{\dot\alpha}\end{pmatrix}$$

- **Majorana spinor** $\psi = \psi^c$

$$\Longleftrightarrow \quad \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix} = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot\alpha} \end{pmatrix} \quad \Longleftrightarrow \quad \begin{matrix} \chi_\alpha \equiv \psi_\alpha \\ \bar{\chi}^{\dot\alpha} \equiv \bar{\psi}^{\dot\alpha} \end{matrix}$$

$$\rightsquigarrow \quad \psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot\alpha} \end{pmatrix} \quad \leftarrow \quad \bar{\psi}^{\dot\alpha} = -(\psi^\alpha)^*$$

Double index convention for tensorial algebra

Take a vector $V_\mu = (V_0, V_1, V_2, V_3)$

We define $$V_{\alpha\dot\alpha} = \frac{1}{\sqrt{2}} (\sigma^\mu)_{\alpha\dot\alpha} V_\mu \qquad (1)$$

$$\equiv \left(V_{1\dot{1}}, V_{1\dot{2}}, V_{2\dot{1}}, V_{2\dot{2}} \right)$$

$$\begin{cases} V_{1\dot{1}} = \frac{1}{\sqrt{2}} (\sigma^{\mu})_{1\dot{1}} V_{\mu} = \frac{1}{\sqrt{2}} (V_0 + V_3) \\ V_{2\dot{2}} = \frac{1}{\sqrt{2}} (V_0 - V_3) \\ V_{1\dot{2}} = \frac{1}{\sqrt{2}} (V_1 - i V_2) \\ V_{2\dot{1}} = \frac{1}{\sqrt{2}} (V_1 + i V_2) \end{cases}$$

$$V_{\alpha\dot{\alpha}} \rightsquigarrow \quad V^{\beta\dot{\beta}} = \varepsilon^{\beta\alpha} \varepsilon^{\dot{\beta}\dot{\alpha}} V_{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} (\bar{\sigma}^{\mu})^{\dot{\beta}\beta} V_{\mu}$$

From (1), multiplying by σ^ν we find

$$V_\mu = \frac{1}{\sqrt{2}} (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}}$$

For coordinates : $$\begin{cases} x^{\alpha\dot{\alpha}} = \frac{1}{2} (\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} x_\mu \\ x^\mu = (\sigma^\mu)_{\alpha\dot{\alpha}} x^{\alpha\dot{\alpha}} \end{cases}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \rightsquigarrow \partial_{\alpha\dot{\alpha}} = (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu$$

s.t. $$\partial_{\alpha\dot{\alpha}} x^{\beta\dot{\beta}} \underset{\uparrow}{=} \delta_\alpha^{\ \beta} \delta_{\dot{\alpha}}^{\ \dot{\beta}}$$

$$\partial_\mu x^\nu = \delta_\mu^{\ \nu}$$

$$\partial^{\alpha\dot\alpha}\,\partial_{\alpha\dot\alpha} = (\bar\sigma^\mu)^{\dot\alpha\alpha}\,\partial_\mu\,(\sigma^\nu)_{\alpha\dot\alpha}\,\partial_\nu$$

$$= \frac{1}{2}\underbrace{\left[(\bar\sigma^\mu)^{\dot\alpha\alpha}(\sigma^\nu)_{\alpha\dot\alpha} + (\bar\sigma^\nu)^{\dot\alpha\alpha}(\sigma^\mu)_{\alpha\dot\alpha}\right]}_{\substack{\shortparallel \\ 4\eta^{\mu\nu}}}\partial_\mu\partial_\nu$$

$$= 2\,\partial_\mu\partial_\nu\,\eta^{\mu\nu} = 2\,\partial^\mu\partial_\mu = 2\,\Box$$

$$\boxed{\Box = \frac{1}{2}\,\partial^{\alpha\dot\alpha}\,\partial_{\alpha\dot\alpha}}$$

- <u>Fierz identities</u>

Take 2 Weyl spinors A_α, B_α

* $$\underbrace{A_\alpha B_\beta}_{} - \underbrace{A_\beta B_\alpha}_{} = \varepsilon_{\alpha\beta} A^\gamma B_\gamma$$

Take 2 Weyl spinors $\bar{A}_{\dot\alpha}, \bar{B}_{\dot\alpha}$

* $$\bar{A}_{\dot\alpha} \bar{B}_{\dot\beta} - \bar{A}_{\dot\beta} \bar{B}_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta} \bar{A}^{\dot\gamma} \bar{B}_{\dot\gamma}$$

$$= - \varepsilon_{\dot\alpha\dot\beta} \bar{A}_{\dot\gamma} \bar{B}^{\dot\gamma} = \varepsilon_{\dot\beta\dot\alpha} \bar{A}_{\dot\gamma} \bar{B}^{\dot\gamma}$$

Consistency check : multiply the first one by $\varepsilon^{\beta\alpha}$

" the second one by $\varepsilon^{\dot\beta\dot\alpha}$

SYMMETRIES

FT based on a symmetry principle. Symmetries allow to

1) Classify elementary particles
2) Describe fundamental interactions

1) Noether theorem $\rightsquigarrow$ symmetry $\Longrightarrow$ conserved current J_μ

$$\begin{array}{c}\partial_\mu J^\mu = 0 \\ \wr\end{array}$$

0-form symmetry

Conserved charge $Q = \int d^3x \, J^0$ $\qquad$ $\partial_0 Q = 0$

$\Downarrow$

Q eigenvalues are constant quantities that can be used to classify elementary particles

- A symmetry built in a FT ~ Poincaré symmetry

$\swarrow$ $\qquad$ $\searrow$

translations $\qquad$ Lorentz transfs

P_μ $\qquad$ $M_{\mu\nu}$

Noether currents: P_μ , $W_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}$

(Pauli-Lubanski)

$$\partial_\mu P^\mu = 0 \quad , \quad \partial_\mu W^\mu = 0$$

We construct 2 casimir ops : $\begin{cases} P^2 \rightsquigarrow m^2 \quad (m^2 \neq 0) \\ W^2 \rightsquigarrow m^2 s(s+1) \end{cases}$

$m^2 \equiv$ square mass of the particle

$s \equiv$ spin of the particle

$$m^2 = 0 \rightsquigarrow W_\mu \sim P_\mu \Longrightarrow W_\mu = \underset{\uparrow}{\lambda} P_\mu$$

helicity

$\lambda = \pm s$

- We can have internal global symmetries generated by $(T^i)_{i=1,\dots}$ (Lie algebra)

Examples:

$$\begin{cases} SU(2) & \text{isospin} \\ SU(N) & \text{flavor symmetry} \\ U(1) & \text{many } U(1)\text{'s} \end{cases}$$

2) Global internal symmetry α^i $\rightsquigarrow$ Gauging the symmetry $\alpha^i(x)$

$$\partial_\mu \rightsquigarrow D_\mu = \partial_\mu - ig \underbrace{A_\mu^a}_{} T_a$$

↑

Gauge fields

Comment

P_μ, $M_{\mu\nu}$ act on local fields as derivative ops

→ infinitesimal transl. a_μ $\quad \delta\phi = a^\mu P_\mu \phi$

→ Infinitesimal Lorentz transf $\varepsilon_{\mu\nu}$ $\quad \delta\phi = \varepsilon_{\mu\nu} M^{\mu\nu}\phi \qquad (2)$

→ Global internal $\quad \delta\phi = i\alpha^a T_a \phi$

Symmetry α^i

$P_\mu, M_{\mu\nu}, T^a$ satisfy the following algebra

ALGEBRA OF COMMUTATORS

$$[P_\mu, P_\nu] = 0$$

$$[M_{\mu\nu}, P_\rho] = i\left(\eta_{\rho\nu} P_\mu - \eta_{\rho\mu} P_\nu\right) \qquad (3)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho}\right)$$

$$[T^a, T^b] = i f^{abc} T^c \qquad (4)$$

$$[P_\mu, T^a] = [M_{\mu\nu}, T^a] = 0 \qquad (5)$$

$\Rightarrow$ Bosonic generators, bosonic groups $\Rightarrow$ in transf (2) also the parameters a_μ, $\varepsilon_{\mu\nu}$, α^a are bosons (spin 0 objects)

Since the transf parameters are 0-spin objects we cannot change the spin of ϕ by acting with a Poincaré or internal transf

$$\Updownarrow$$

if $\mathcal{L} = \mathcal{L}_{bos} + \mathcal{L}_{ferm}$

then $\mathcal{L}_{bos}$ and $\mathcal{L}_{ferm}$ have to transform separately and independently

From commutators (5) it follows that the symmetry group is of the form

$$\text{Poincaré} \times G \qquad (G = \text{internal group})$$

Basic question: Can we generalise the symmetry group to a larger group where Poincaré generators no longer commute with internal group generators?

The answer is **NO** and it is the content of

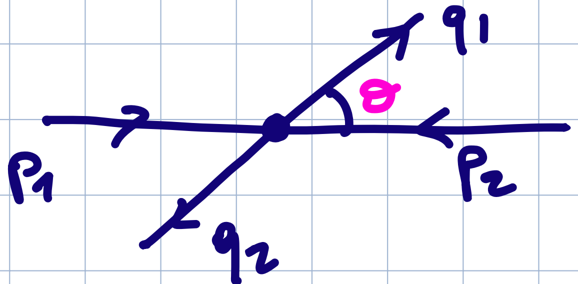
No-go Coleman-Mandula theorem:

Suppose we have a 4D field theory (local, causal, with a finite # of particles) for massive particles with a non-trivial S-matrix. If we require invariance under a general group $\tilde{G}$ which contains Poincaré as a subgroup, the only way to preserve the analyticity of the S-matrix is to take $\tilde{G} =$ Poincaré $\times G$

$$\Updownarrow$$

$$[P_\mu, T^a] = [M_{\mu\nu}, T^a] = 0$$

(Simple proof by Witten in Erice School 1981)



2 → 2 scattering

⇝ $S(\theta)$ analytic function

Some other symmetry generated by $Q_{\mu\nu}$ (of course $[M_{\rho\sigma}, Q_{\mu\nu}] \neq 0$, so this would correspond to a more general $\tilde{G}$) $\Rightarrow$ $\theta = 2n\pi$ $\Rightarrow$ S loses analyticity

Intuition: no-go theorem holds inside the set of ordinary groups of symmetries.

What happens if we include the possibility to have symmetries associated to SUPERGROUPS?

Supergroup has set of bosonic generators $\rightarrow [\,,\,]$

$+$

set of fermionic generators $\rightarrow \{\,,\,\}$

HANDWAVING CONSTRUCTION OF SUSY ALGEBRA

FT allows for the introduction of supersymmetry without any further assumption.

Consider FT for a free complex scalar $\varphi, \bar\varphi$ (massless)

$+$

free Majorana fermion $\begin{pmatrix} \psi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix}$

$$\mathcal{L} = \partial_\mu \varphi \, \partial^\mu \bar\varphi + \frac{i}{4} \bar\Psi \gamma^\mu \partial_\mu \Psi$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$$

$$= \frac{1}{2} \partial_{\alpha\dot\alpha} \varphi \, \partial^{\alpha\dot\alpha} \bar\varphi - \frac{i}{2} \bar\psi^{\dot\alpha} \partial_{\alpha\dot\alpha} \psi^\alpha \qquad (6)$$

$$\partial_{\alpha\dot\alpha} = (\sigma^\mu)_{\alpha\dot\alpha} \partial_\mu$$

Poincaré invariant + U(1) internal symm.

$$\begin{cases} \varphi \to \varphi' = e^{i\alpha}\varphi \\ \psi \to \psi' = e^{i\beta}\psi \end{cases} \qquad \alpha, \beta \in \mathbb{R} \text{ (constant)}$$

Can we find one extra symmetry generated by a spinorial parameter

$$\varepsilon = \begin{pmatrix} \varepsilon_\alpha \\ \bar{\varepsilon}^{\dot\alpha} \end{pmatrix} \qquad \text{Majorana spinor}$$

Claim: $\mathcal{L}$ in (6) is invariant under the following transfs

$$\begin{cases} \delta\varphi = -\varepsilon^\alpha \psi_\alpha & \delta\bar{\varphi} = -\bar{\psi}_{\dot\alpha}\bar{\varepsilon}^{\dot\alpha} \\ \delta\psi_\alpha = -2i(\partial_{\alpha\dot\alpha}\varphi)\bar{\varepsilon}^{\dot\alpha} & \delta\bar{\psi}_{\dot\alpha} = 2i\,\varepsilon^\alpha\partial_{\alpha\dot\alpha}\bar{\varphi} \end{cases} \qquad (7)$$

Check the invariance of $\mathcal{L}$ in (6)

this is a new kind of symmetry since the transfs are driven by spinorial parameters $\equiv$ SUPERSYMMETRY

We should be able to generalise Noether theorem to this new symmetry.

In the ordinary case, if $\delta\mathcal{L} = \partial_\mu V^\mu$ then Noether current is

$$J_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \underbrace{\delta\phi}_{\substack{\uparrow \\ \text{is linear in the parameters}}} - V_\mu$$

⇒ J_μ is linear in the parameters

$$J_\mu = \underbrace{\alpha^a}_{\text{parameters}} J^a_\mu \qquad \text{with} \quad \partial^\mu J^a_\mu = 0$$

$$\forall\, a = 1, \dots, p$$

In this new case we expect $\tilde{J}_\mu$ to be linear in $\varepsilon_\alpha, \bar{\varepsilon}_{\dot{\alpha}}$ ⇒

$$\tilde{J}^\mu = \varepsilon^\alpha J^\mu_\alpha + \bar{\varepsilon}^{\dot{\alpha}} \bar{J}^\mu_{\dot{\alpha}}$$

$$\text{with} \quad \partial_\mu \boxed{J^\mu_\alpha} = 0 \quad , \quad \partial_\mu \bar{J}^\mu_{\dot{\alpha}} = 0$$

$$\alpha = 1, 2 \qquad \dot{\alpha} = \dot{1}, \dot{2}$$

SUPER CURRENTS

Conserved currents are now spinors - the corresponding conserved charges are

$$\underbrace{Q_\alpha}_{\text{left-Weyl spinorial charge}} = \int d^3x \, J^0_\alpha \qquad\qquad \underbrace{\bar{Q}_{\dot\alpha}}_{\text{right-Weyl spinorial charge}} = \int d^3x \, \bar{J}^0_{\dot\alpha}$$

SUPERCHARGES

these are the generators of susy transfs (7)

$$\delta\varphi = -\varepsilon^\alpha \psi_\alpha \equiv i\left[\varepsilon^\alpha Q_\alpha + \bar{\varepsilon}_{\dot\alpha} \bar{Q}^{\dot\alpha}, \varphi\right]$$

$$\Rightarrow \boxed{\begin{array}{ll} [\bar{Q}_{\dot\alpha}, \varphi] = 0 & [Q_\alpha, \bar{\varphi}] = 0 \\ [Q_\alpha, \varphi] = i\psi_\alpha & [\bar{Q}_{\dot\alpha}, \bar{\varphi}] = i\bar{\psi}_{\dot\alpha} \end{array}}$$

$$\delta\psi_\beta \underset{(7)}{=} -2i(\partial_{\beta\dot\beta}\varphi)\bar\varepsilon^{\dot\beta} = i\left[\varepsilon^\alpha Q_\alpha + \bar\varepsilon_{\dot\alpha}\bar Q^{\dot\alpha}, \psi_\beta\right]$$

$$= i\varepsilon^\alpha\{Q_\alpha, \psi_\beta\} + i\bar\varepsilon_{\dot\alpha}\{\bar Q^{\dot\alpha}, \psi_\beta\}$$

$$\Rightarrow \boxed{\begin{aligned} &\{Q_\alpha, \psi_\beta\} = 0 && \{\bar Q_{\dot\alpha}, \bar\psi_{\dot\beta}\} = 0 \\ &\{\bar Q_{\dot\alpha}, \psi_\beta\} = 2\partial_{\beta\dot\alpha}\varphi && \{Q_\alpha, \bar\psi_{\dot\beta}\} = 2\partial_{\beta\dot\alpha}\bar\varphi \end{aligned}}$$

Main question is: $\{Q_\alpha, Q_\beta\} = ?$

$\{Q_\alpha, \bar Q_{\dot\alpha}\} = ?$

$[Q_\alpha, P_\mu] = ?$ $\quad [Q_\alpha, M_{\mu\nu}] = ?$

Lecture 2

4/11/25

Construct the $Q, \bar{Q}$ algebra -

For instance, we want to compute

$$[\{Q_\alpha, Q_\beta\}, \varphi] = \dots \qquad (1)$$

$$\psi_\gamma = \dots$$

We use the graded Jacobi identity

Given 3 generic X, Y, Z, we assign a grading

$$X \rightsquigarrow |X| = \begin{cases} 0 & \text{if } X \text{ is bosonic} \\ 1 & \text{if } X \text{ is fermionic} \end{cases}$$

$\Downarrow$

$$(-1)^{|x||z|}\,[x,[y,z]_\pm]_\pm + (-1)^{|x||y|}\,[y,[z,x]]_\pm$$

$$+\,(-1)^{|y||z|}\,[z,[x,y]]_\pm = 0$$

where $[x,y]_+ \equiv \{x,y\}$

$[x,y]_- \equiv [x,y]$

therefore (1) can be rewritten as

$$[\{Q_\alpha, Q_\beta\},\varphi] \sim \pm\{\underbrace{[\varphi, Q_\alpha]}_{\substack{-i\psi_\alpha \\ \| \\ 0}}, Q_\beta\} \pm \{\underbrace{[Q_\beta,\varphi]}_{\substack{i\psi_\beta \\ \| \\ 0}}, Q_\alpha\}$$

$$[\{Q_\alpha, Q_\beta\}, \psi_\gamma] = 0$$ (still applying Jacobi)

$$\Rightarrow \quad \boxed{\{Q_\alpha, Q_\beta\} = 0 \qquad \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0}$$

$$Q^3 \equiv \left(\frac{1}{2} Q^\alpha Q_\alpha\right) Q = 0$$

$$\{Q_\alpha, Q_\beta\} = 0 \underset{\alpha=\beta=1}{\Longrightarrow} \{Q_1, Q_1\} = 0 = 2 Q_1^2$$

$$\Downarrow$$

$$Q_1^2 = 0 \ , \quad Q_2^2 = 0$$

$$Q_1 Q_2 = - Q_2 Q_1$$

$$Q^2 = \frac{1}{2} Q^\alpha Q_\alpha = \frac{1}{2} \epsilon^{\alpha\beta} Q_\beta Q_\alpha \sim Q_1 Q_2 \neq 0$$

Now consider

- $$[\{Q_\alpha, \bar{Q}_{\dot\alpha}\}, \varphi] \underset{\underline{\text{DO IT}}}{=} 2i\, \partial_{\alpha\dot\alpha} \varphi$$

- $$[\{Q_\alpha, \bar{Q}_{\dot\alpha}\}, \psi_\gamma] \underset{\underline{\text{DO IT}}}{=} 2i\, \partial_{\gamma\dot\alpha} \psi_\alpha$$

$$= \underbrace{2i\, \partial_{\alpha\dot\alpha} \psi_\gamma} \underbrace{- 2i\, \partial_{\alpha\dot\alpha} \psi_\gamma + 2i\, \partial_{\gamma\dot\alpha} \psi_\alpha}_{2i\left(\underline{\partial_{\gamma\dot\alpha}} \psi_\alpha - \partial_{\alpha\dot\alpha} \psi_\gamma\right) = 2i\, \epsilon_{\gamma\alpha} \partial^\beta{}_{\dot\alpha} \psi_\beta}$$

From $\mathcal{L} \to -\frac{i}{2}\bar{\psi}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\psi^{\alpha}$

$\frac{\delta\mathcal{L}}{\delta\bar{\psi}^{\dot{\alpha}}} = 0 \rightsquigarrow \partial_{\alpha\dot{\alpha}}\psi^{\alpha} = 0$ EOM for Weyl spinor ψ^{α}

__On-shell__

$[\{Q_{\alpha},\bar{Q}_{\dot{\alpha}}\},\psi_{\gamma}] = 2i\partial_{\alpha\dot{\alpha}}\psi_{\gamma}$

$\Rightarrow \boxed{\{Q_{\alpha},\bar{Q}_{\dot{\alpha}}\} = 2i\partial_{\alpha\dot{\alpha}} \quad \text{on-shell}}$

$i\partial_{\alpha\dot{\alpha}} = \pm P_{\alpha\dot{\alpha}}$

__Question__: how do we go off-shell?

We introduce another pair of scalar fields $F, \bar{F}$

- $$\mathcal{L} \to \tilde{\mathcal{L}} = \frac{1}{2}\partial_{\alpha\dot\alpha}\varphi\,\partial^{\alpha\dot\alpha}\bar{\varphi} - \frac{i}{2}\bar{\psi}^{\dot\alpha}\partial_{\alpha\dot\alpha}\psi^{\alpha} + \underbrace{\frac{1}{4}F\bar{F}}_{\text{no time derivative}}$$

$\Downarrow$

$F, \bar{F}$ do NOT propagate

$F, \bar{F}$ EOM $\rightsquigarrow \begin{cases} \bar{F} = 0 \\ F = 0 \end{cases}$

$\tilde{\mathcal{L}}$ is now invariant under modified SUSY transfs (DO IT)

$$\begin{cases} \delta\varphi = -\varepsilon^{\alpha}\psi_{\alpha} \qquad\qquad \delta\bar{\varphi} = -\bar{\psi}_{\dot{\alpha}}\bar{\varepsilon}^{\dot{\alpha}} \\ \delta\psi_{\alpha} = -2i\,\partial_{\alpha\dot{\alpha}}\varphi\,\bar{\varepsilon}^{\dot{\alpha}} - \varepsilon_{\alpha}F \\ \delta\bar{\psi}_{\dot{\alpha}} = 2i\,\varepsilon^{\alpha}\partial_{\alpha\dot{\alpha}}\bar{\varphi} - \bar{\varepsilon}_{\dot{\alpha}}\bar{F} \\ \delta F = -2i\,\bar{\varepsilon}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\psi^{\alpha} \\ \delta\bar{F} = -2i\,\varepsilon^{\alpha}\partial_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} \end{cases}$$

- $\left[\,\{Q_\alpha, Q_\beta\},\ \begin{matrix}\varphi\\ \psi_\beta\\ F\end{matrix}\,\right] \quad \underset{\text{Jacobi}}{\rightsquigarrow} \quad \{Q_\alpha, Q_\beta\} = 0$

- $\left[\,\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\},\ \begin{matrix}\varphi\\ \psi_\gamma\\ F\end{matrix}\,\right] \quad \underset{\text{Jacobi}}{\rightsquigarrow} \quad \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}}$

(DO IT) <u>off-shell</u>

$F, \bar{F}$ are auxiliary fields and they are needed to close the algebra off-shell.

Now put together $Q_\alpha, \bar{Q}_{\dot\alpha}$ with P_μ, $M_{\nu\rho}$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2 P_{\alpha\dot\alpha} \qquad P_{\alpha\dot\alpha} = (\sigma^\mu)_{\alpha\dot\alpha} P_\mu$$

$$[P_{\alpha\dot\alpha}, Q_\beta] = [P_{\alpha\dot\alpha}, \bar{Q}_{\dot\beta}] = 0$$

$$[M_{\mu\nu}, Q_\alpha] = -\frac{1}{2} (\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta$$

$$[M_{\mu\nu}, \bar{Q}^{\dot\alpha}] = -1/2 \, (\bar{\sigma}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \bar{Q}^{\dot\beta}$$

$$\boxed{\begin{aligned} &[P_\mu, P_\nu] = 0 \\ &[M_{\mu\nu}, P_\rho] = \cdots \\ &[M_{\mu\nu}, M_{\rho\sigma}] = \cdots \end{aligned} \qquad (3)}$$

N=1 SUPERSYMMETRY ALGEBRA in 4D

here $$(\sigma_{\mu\nu})_\alpha{}^\beta = \frac{i}{2}\left(\sigma_{\mu\,\alpha\dot\alpha}\,\bar\sigma_\nu{}^{\dot\alpha\beta} - \mu\leftrightarrow\nu\right)$$

2x2 matrices

$$(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \frac{i}{2}\left(\bar\sigma_\mu{}^{\dot\alpha\alpha}\,\sigma_{\nu\,\alpha\dot\beta} - \mu\leftrightarrow\nu\right)$$

$\rightsquigarrow$ $\gamma_{\mu\nu}$ generators $= \frac{i}{2}[\gamma_\mu, \gamma_\nu]$ (4×4 matrices)

Stress that this algebra closes off shell on $(\varphi, \psi_\alpha, F)$

N=1 susy algebra $\equiv$ $\overline{osp(4|1)}$ superalgebra

Two immediate consequences

1) $\delta\varphi = -\varepsilon^{\alpha}\psi_{\alpha} \sim [Q_{\alpha}, \varphi]$

Start with a bosonic state $\varphi(0)|0\rangle \equiv |b\rangle$

We assume that $Q_{\alpha}|0\rangle = \bar{Q}_{\dot{\alpha}}|0\rangle = 0$, then

$$Q\underbrace{|b\rangle}_{\text{bosonic state}} = [Q,\varphi(0)]|0\rangle \sim \psi(0)|0\rangle = \underbrace{|f\rangle}_{\text{fermionic state}}$$

Q (and similarly $\bar{Q}$) maps

$$|b\rangle \longrightarrow |f\rangle$$
$$|f\rangle \longrightarrow |b\rangle$$

Q (and $\bar{Q}$) changes the spin of the particle

If susy is not broken at a given energy scale

Quark $\rightsquigarrow$ S-Quark

Electron $\rightsquigarrow$ s-electron

fermion $\longrightarrow$ boson

Gauge field $\rightsquigarrow$ Gaugino

boson $\rightsquigarrow$ fermion

Graviton $\rightsquigarrow$ Gravitino

boson $\rightsquigarrow$ fermion

2) From the algebra we still have

$$[P^2, Q_\alpha] = [P^2, \bar{Q}_{\dot\alpha}] = 0 \qquad (2)$$

$$\Downarrow$$

P^2 is still a Casimir op.

$$\Downarrow$$

m is still a good quantum #

From (2) it follows that all particles obtained from a given one by acting with Q or $\bar{Q}$ have the <u>same mass</u>

However, $[W^2, Q_\alpha]$, $[W^2, \bar{Q}_{\dot{\alpha}}] \neq 0$ $\Rightarrow$ W^2 is no longer a Casimir for the whole algebra

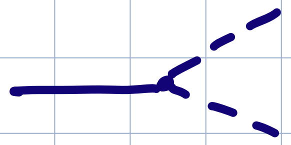
$\Rightarrow$ particles obtained from a given one by acting with $Q, \bar{Q}$ will have different spin.

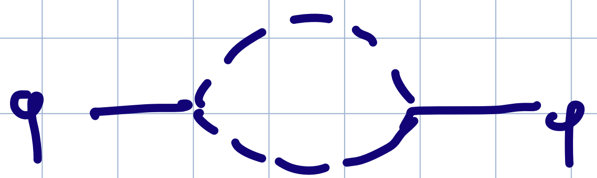
Why do we need SUSY?

1) SUSY was introduced to cure the hierarchy problem in the SM (see Tuning book)

Consider 1L corrections to the Higgs mass.

$$\mathcal{L}_{SM} \rightsquigarrow y_t \underset{\substack{\uparrow \\ \text{Higgs}}}{\varphi} \underbrace{\bar{\psi}_L \psi_R}_{\text{top quark}}$$





$$\rightarrow \quad \delta m_H^2 \sim |y_t|^2 N_c \Lambda^2$$

Λ = UV cutoff

measure m_H, y_t → for a large range of energies $\delta m_H^2 \gg m_H^2$

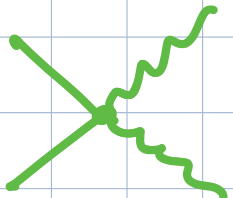
NO GOOD

Fine tuning: $\delta m_H \sim m_H \Rightarrow \Lambda \sim 1$ TeV

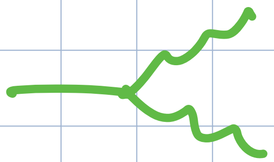
Beyond 1 TeV there must be new physics = new particles that compensate for Λ^2 correction

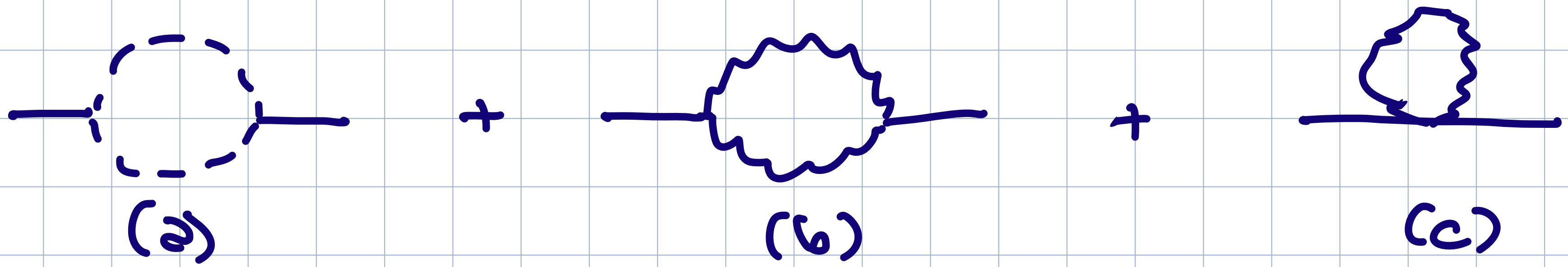
We try by introducing a new pair of scalars φ_L, φ_R with the following couplings to Higgs

$$\mathcal{L}'_{SM} = \mathcal{L}_{SM} - \frac{\boxed{\lambda}}{2}\varphi^2\left(|\varphi_L|^2 + \underset{\sim}{|\varphi_R|^2}\right)$$



$$-\varphi\left(\boxed{\mu_L}\,|\varphi_L|^2+\boxed{\mu_R}\,|\varphi_R|^2\right)$$





(a) $\sim \Lambda^2$

(b) $\sim \log\Lambda$

(c) $\sim \Lambda^2$

What happens is:

$$\begin{cases} \text{if } \lambda=|y_t|^2 \Longrightarrow \Lambda^2\text{-terms cancel} \\ \text{if } \mu_L=\mu_R=m_t \Longrightarrow \log\Lambda \text{ terms cancel} \end{cases}$$

But these are the values for the coupling which make $\mathcal{L}'_{SM}$ SUSY invariant

2) Quantum Gravity → seen as a gauge theory

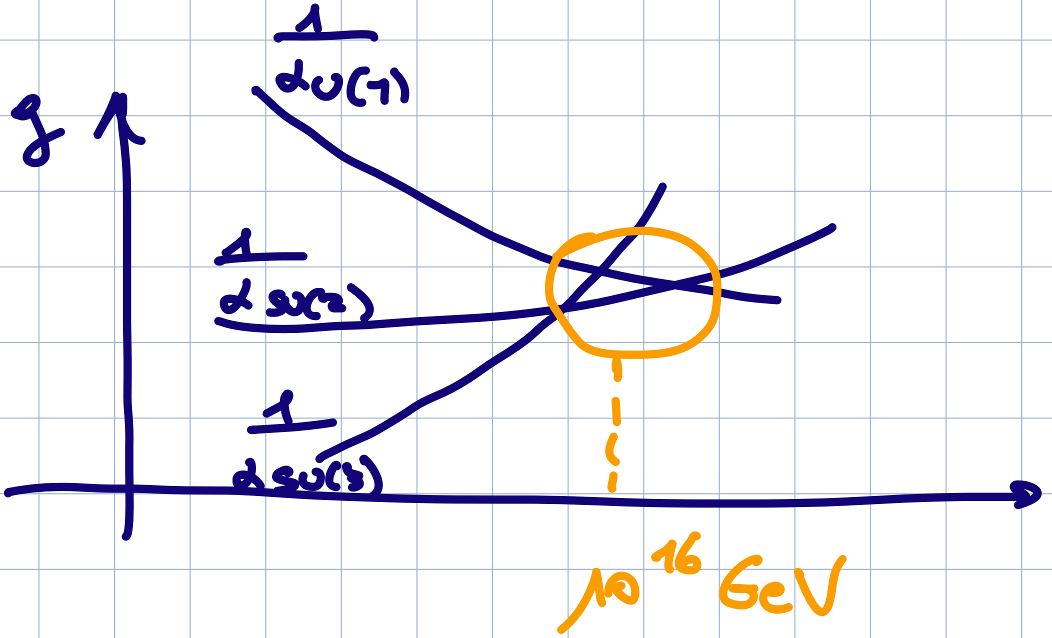
the theory is not renormalizable

$+$ SUSY $\rightsquigarrow$ SUPERGRAVITY

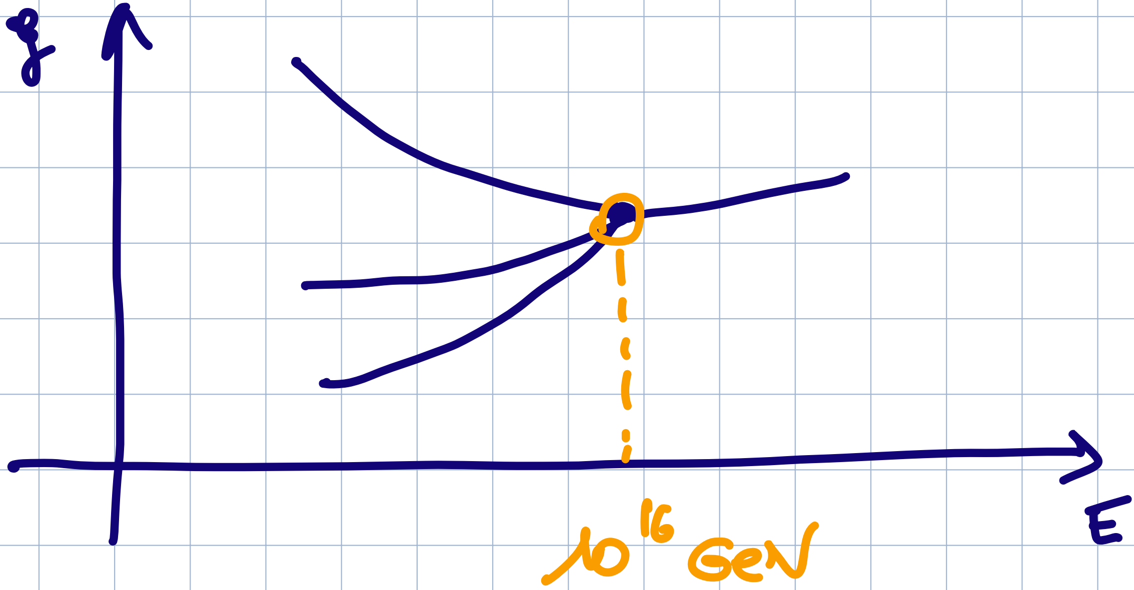
$N = 1, 2, 4$ } we still have milder UV divergences not renormalisable yet

$N = 8$? It might be finite OPEN QUESTION

3)



without susy



Add susy

Modern motivations

4) String theory requires SUSY

5) AdS/CFT correspondence contains SUSY

6) FT with enough SUSY can be solved exactly using SUSY LOCALIZATION

Single SUSY Q and an action s.t. $[Q,S]=0$

$$W = \int [\mathcal{D}\phi]\, e^{-S} \qquad\qquad Q^2 = 0$$

$$\updownarrow$$

$$W(t) = \int [\mathcal{D}\phi]\, e^{-S + t\int \{Q,V\}}$$

Witten

$$W = W(0)$$

$$\frac{dW(t)}{dt} = \int [\mathcal{D}\phi] \int \{Q, V\} \, e^{-S + t \int \{Q,V\}}$$

$$= \int [\mathcal{D}\phi] \left\{ Q, \int V \cdot e^{-S + t \int \{Q,V\}} \right\}$$

$$= \langle 0 | Q (\ldots) | 0 \rangle = 0 \qquad Q|0\rangle = 0$$

$$\Downarrow$$

$$W(t) = W(0) = W(t = +\infty)$$

$\uparrow$

this can be evaluated applying saddle pt prescription which now is <u>not</u> an approximation

Saddle pt $\{Q, \delta V\} = 0$

$$W = e^{-S_{\text{clean}}} \int [\text{Solutions}] \quad \underbrace{Z_{1\text{-loop}}}_{\text{quadratic fluctuations}}$$

[Pestun → loc. for $N=4$ SYM
ABJM ...]

Go back to N=1 susy algebra (3)

the algebra is invariant under U(1) global transfs of $Q, \bar{Q}$

$$\begin{cases} Q_\alpha \longrightarrow Q'_\alpha = e^{i\eta} Q_\alpha \\ \bar{Q}_{\dot{\alpha}} \longrightarrow \bar{Q}'_{\dot{\alpha}} = e^{-i\eta} \bar{Q}_{\dot{\alpha}} \end{cases} \qquad \eta \in \mathbb{R} \text{ (constant)}$$

If we introduce the U(1) generator R

$$\begin{cases} [R, Q_\alpha] = Q_\alpha \\ [R, \bar{Q}_{\dot{\alpha}}] = -\bar{Q}_{\dot{\alpha}} \end{cases} \qquad \begin{aligned} [R, P_\mu] &= 0 \\ [R, M_{\mu\nu}] &= 0 \end{aligned}$$

R- SYMMETRY

$$\begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix} \to \text{Majorana spinor}$$

Further physical consequences of the susy algebra

$$\underbrace{|b\rangle\,,\ Q_\alpha|b\rangle\,,\ Q^2|b\rangle \,\Big|\, \dots\ \bar{Q}\,Q^2|b\rangle\,,\ \bar{Q}^2Q^2|b\rangle}_{\text{finite number of states}}$$

$|f\rangle$

$$\downarrow$$

$$\boxed{\text{SUSY MULTIPLET}}$$

1) How many bosonic and fermionic states does a multiplet contain?

we introduce the operator $(-1)^F$ with

$$F = \begin{cases} 0 & \text{if } (-1)^F \text{ is applied to a bosonic state} \\ 1 & \text{if } (-1)^F \text{ is applied to a fermionic state} \end{cases}$$

$(-1)^F |b\rangle = |b\rangle$ $\qquad$ $(-1)^F |f\rangle = - |f\rangle$

You can check that $\{ (-1)^F , Q_\alpha \} = 0$

$$\{ (-1)^F , \bar{Q}_{\dot{\alpha}} \} = 0$$

$$\{ Q_\alpha , \bar{Q}_{\dot{\alpha}} \} = 2 P_{\alpha\dot{\alpha}} = 2 (\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu$$

we can invert this identity

$$P_\mu = \frac{1}{4} (\bar{\sigma}_\mu)^{\dot{\alpha}\alpha} \{ Q_\alpha , \bar{Q}_{\dot{\alpha}} \}$$

$$P_0 = E = \frac{1}{4} \underbrace{(\bar{\sigma}_0)}_{1}{}^{\dot{\alpha}\alpha} \{ Q_\alpha , \bar{Q}_{\dot{\alpha}} \}$$

$$= \frac{1}{4} \left[\{ Q_1 , \bar{Q}_{\dot{1}} \} + \{ Q_2 , \bar{Q}_{\dot{2}} \} \right]$$

Consider a multiplet made by states $\{ |i\rangle \}_{i=1,\dots}$

$$\mathrm{Tr} \left((-1)^F P_0 \right) = \sum_i \langle i | (-1)^F \underbrace{P_0 | i \rangle} = E | i \rangle$$

$$= E \sum_i \langle i | (-1)^F | i \rangle$$

$$= E\,(\#\,\text{bosons} - \#\,\text{fermions})$$

$$\frac{1}{4}\,\mathrm{Tr}\left((-1)^F\left[\{Q_1,\bar{Q}_{\dot{1}}\}+\{Q_2,\bar{Q}_{\dot{2}}\}\right]\right)$$

$$= \frac{1}{4}\,\mathrm{Tr}\left((-1)^F Q_1\bar{Q}_{\dot{1}} + (-1)^F\,\bar{Q}_{\dot{1}}\,Q_1 + 2\text{-part}\right)$$

cyclicity of Tr

$$= \frac{1}{4}\,\mathrm{Tr}\left((-1)^F Q_1\bar{Q}_{\dot{1}} + \underbrace{Q_1\,(-1)^F}_{\text{they anticommute}}\,\bar{Q}_{\dot{1}} + 2\text{part}\right)$$

$$= \frac{1}{4}\,\mathrm{Tr}\left((-1)^F Q_1\bar{Q}_{\dot{1}} - (-1)^F Q_1\bar{Q}_{\dot{1}} + 2\text{-part}\right)$$

$$= 0$$

$$\# \; \boxed{\#\,\text{bosons} = \#\,\text{fermions}} \quad \text{in a multiplet}$$

2) $$P_0 = \frac{1}{4}\left(\{Q_1, \bar{Q}_{\dot{1}}\} + \{Q_2, \bar{Q}_{\dot{2}}\} \right)$$

Take a generic state $|A\rangle$ (bosonic or fermionic)

$$\langle A | P_0 | A \rangle = E_A$$

$$= \frac{1}{4}\Big(\langle A | Q_1 \underbrace{\bar{Q}_{\dot{1}} | A \rangle} + \langle A | \bar{Q}_{\dot{1}} \underbrace{Q_1 | A \rangle}$$

$$\langle A | Q_2 \underbrace{\bar{Q}_{\dot{2}} | A \rangle} + \langle A | \bar{Q}_{\dot{2}} \underbrace{Q_2 | A \rangle} \Big)$$

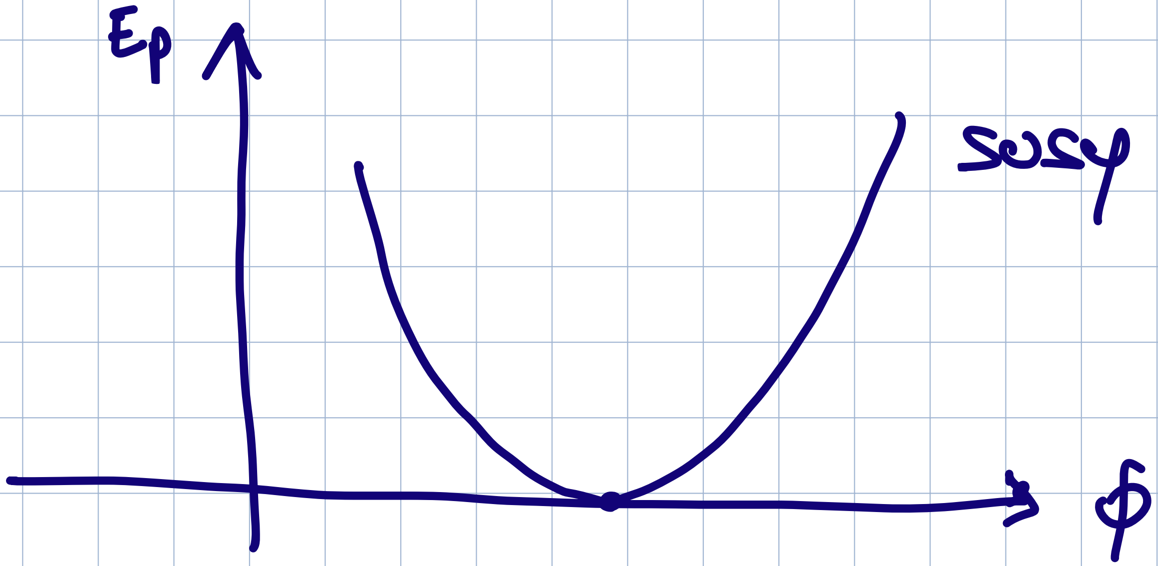
$$= \frac{1}{4} \Big(\| \bar{Q}_{\dot{1}} |A\rangle \|^2 + \| Q_1 |A\rangle \|^2 + \| \bar{Q}_{\dot{2}} |A\rangle \|^2$$

$$+ \| Q_2 |A\rangle \|^2 \Big) \geq 0$$

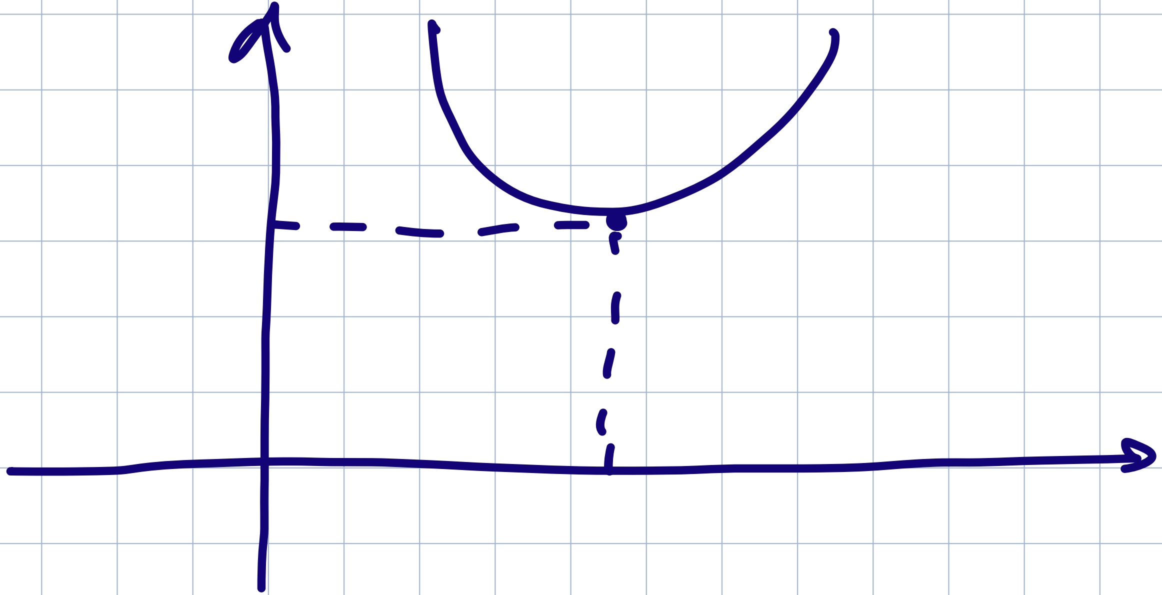
$$\Rightarrow \forall\ |A\rangle \qquad \boxed{E_A \geq 0}$$

We choose $E_0 \equiv \langle 0 | P_0 | 0 \rangle = 0$

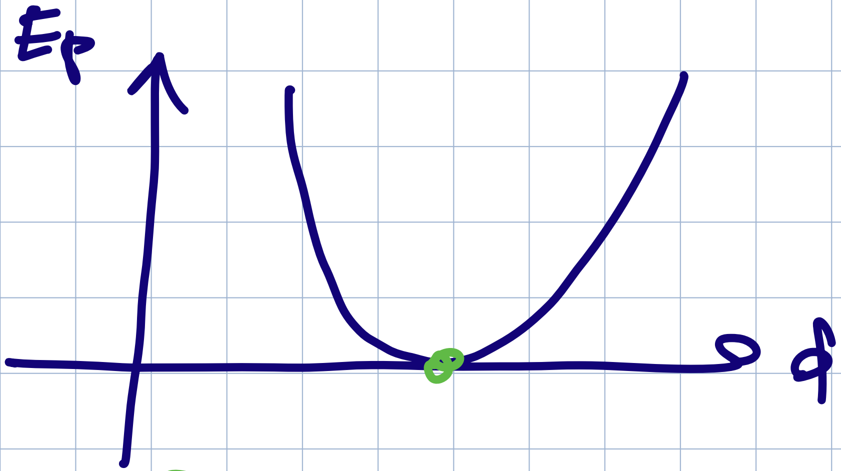
$$\begin{cases} E = E_{kin} + E_{pot} \\ E_A \geq 0 \end{cases} \longmapsto \quad E_{pot} \geq 0$$ for any susy theory



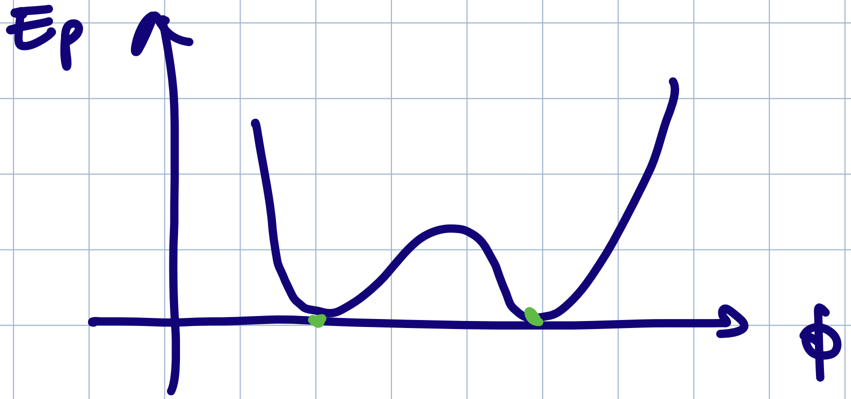
susy is broken when



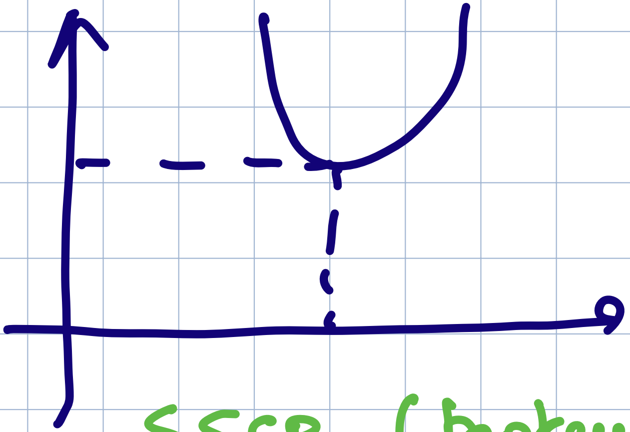
Most general situation



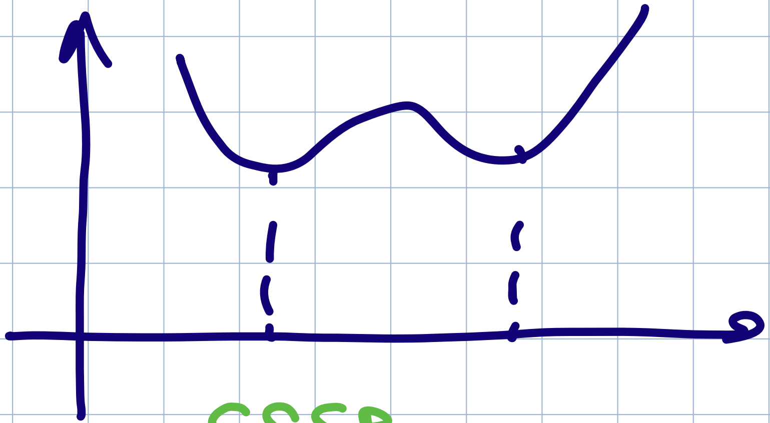
SUSY +
internal symmetry



SUSY
SSB (internal sym)



SSSB (broken susy)
Internal symmetry



SSSB
SSB

lecture 3

4/11/25

N=1 SUSY REPRESENTATIONS

(Sohnius, Terning)

① Massive case

Full spectrum of states $|m, s, s_3\rangle$

for any s fixed $\rightsquigarrow$ $(2s+1)$ states

$s_3 = -s, -s+1, \dots s$

Go to the rest frame $P_\mu = (m, 0, 0, 0)$

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu = 2m\underbrace{(\sigma^0)}_{1}{}_{\alpha\dot{\alpha}}$$

$$\Rightarrow \quad \{Q_1, \bar{Q}_{\dot 1}\} = 2m \qquad\qquad q_1 = \frac{1}{\sqrt{2m}} Q_1$$

$$\{Q_2, \bar{Q}_{\dot 2}\} = 2m \qquad\qquad \bar{q}_{\dot 1} = \frac{1}{\sqrt{2m}} \bar{Q}_{\dot 1}$$

$$\{Q_1, \bar{Q}_{\dot 2}\} = 0 \ \ldots \qquad\qquad \vdots$$

$$\Rightarrow \quad \{q_1, \bar{q}_{\dot 1}\} = 1 \qquad\qquad \{q_2, \bar{q}_{\dot 2}\} = 1$$

$$q_1^2 = 0 \qquad\qquad q_2^2 = 0$$

$$\bar{q}_{\dot 1}^2 = 0 \qquad\qquad \bar{q}_{\dot 2}^2 = 0$$

Two disjoint Clifford algebras

- We define a Clifford vacuum $|m, s, s_3\rangle \equiv |s\rangle$

$$q_1 |s\rangle = 0 \qquad\qquad q_2 |s\rangle = 0$$

STATE	SPIN	N. COMPONENTS
$\vert s\rangle$	s	$(2s+1)$
$\left.\begin{array}{l}\bar{Q}_{\dot{1}}\vert s\rangle \\ \bar{Q}_{\dot{2}}\vert s\rangle\end{array}\right\} = \bar{Q}_{\dot{\alpha}}\vert s\rangle$	$s+\frac{1}{2}$ $s-1/2$	$\left.\begin{array}{l}2(s+1/2)+1 \\ 2(s-1/2)+1\end{array}\right\} 2(2s+1)$
$\underbrace{\bar{Q}_{\dot{\alpha}}\bar{Q}_{\dot{\beta}}}\vert s\rangle$ $\epsilon_{\dot{\beta}\dot{\alpha}}\bar{Q}_{\dot{\gamma}}\bar{Q}^{\dot{\gamma}}\vert s\rangle$ $\wr$ $\bar{Q}^2\vert s\rangle$	s	$(2s+1)$

$$\text{Multiplet} \equiv \{\, |s\rangle,\ \underbrace{\bar{Q}_{\dot{\alpha}}|s\rangle},\ \bar{Q}^2|s\rangle \,\}$$

$$\downarrow$$

$$2(2s+1)$$

$$2(2s+1)$$

Tot # of components $= 4(2s+1) \equiv$ dimension of the multiplet

Relevant examples

	STATE	SPIN	# of COMPONENTS	
1)	$\vert 0\rangle$	$s=0$	1	scalar
	$\bar{Q}_i \vert 0\rangle$	$\frac{1}{2}$	2	Weyl fermion
	$\bar{Q}^2 \vert 0\rangle$	0	1	scalar

massive chiral multiplet

2)	$\vert 1/2\rangle$	$s = 1/2$	2	Weyl fermion
	$\bar{Q}_i \vert 1/2\rangle$	1	3	massive vector
		0	1	scalar
	$\bar{Q}^2 \vert 1/2\rangle$	$1/2$	2	Weyl fermion

massive vector multiplet

3)	$\vert 3/2\rangle$	$3/2$	4	Rarita Schwinger state
	$\bar{Q}_i \vert 3/2\rangle$	2	5	Graviton
		1	3	massive vector
	$\bar{Q}^2 \vert 3/2\rangle$	$3/2$	4	RS fermion

massive Weyl multiplet

② Massless case ($P^2=0$)

Spectrum of states $\rightarrow$ $|E,\lambda\rangle$

E = energy

λ = helicity

Go to the standard frame $P_\mu = (E,0,0,E)$

$$\{Q_\alpha,\bar{Q}_{\dot\alpha}\} = 2(\sigma^\mu)_{\alpha\dot\alpha}P_\mu = 2E\underbrace{(\sigma^0+\sigma^3)}_{\begin{pmatrix}2&0\\0&0\end{pmatrix}}{}_{\alpha\dot\alpha}$$

$$\Rightarrow \quad \{Q_1,\bar{Q}_{\dot 1}\} = 4E \quad \longrightarrow \quad q_1 = \frac{1}{2\sqrt{E}} \quad \{q_1,\bar{q}_{\dot 1}\} = 1$$

$$\{Q_2,\bar{Q}_{\dot 2}\} = 0$$

$\{Q_1, \bar{Q}_{\dot{i}}\} = 0$

Now we have only one Clifford algebra

$$W^0 = \frac{1}{2}\varepsilon^{0ijk} P_i M_{jk} = \frac{1}{2}\cdot E\, \varepsilon^{0312} M_{12} = \frac{E}{2} J_3$$

Helicity op $\Gamma = \frac{1}{E}\vec{J}\cdot\vec{P} \sim J_3$

$$\boxed{W^0 = E\,\Gamma}$$

$$\Gamma|E,\lambda\rangle = \lambda|E,\lambda\rangle$$

Using the algebra we can prove that (DO IT)

$$[W^0, Q_\alpha] = -\frac{E}{2}(\sigma_3)_\alpha{}^\beta Q_\beta \qquad (1)$$

$$[W^0, \bar{Q}_{\dot\alpha}] = \frac{E}{2}\bar{Q}_{\dot\beta}(\sigma_3)^{\dot\beta}{}_{\dot\alpha} \qquad (2)$$

Start with a particular state $|E,\lambda\rangle$ _ It is also an eigenvector of W^0

$$W^0|E,\lambda\rangle = E\lambda\,|E,\lambda\rangle$$

$$W^0\left(Q_1|E,\lambda\rangle\right) = \left([W^0, Q_1] + Q_1 W^0\right)|E,\lambda\rangle$$

$$= \left(-\frac{E}{2}\,(\sigma_3)_1{}^1 + E\lambda\right) Q_1|E,\lambda\rangle$$

$$\Rightarrow\quad W^0\left(Q_1|E,\lambda\rangle\right) = E\left(\lambda - 1/2\right)\left(Q_1|E,\lambda\rangle\right)$$

$$\boxed{Q_1|E,\lambda\rangle = |E,\lambda - 1/2\rangle}$$

Similarly $\boxed{\bar{Q}_{\dot{1}} |E, \lambda\rangle = |E, \lambda + 1/2\rangle}$

Q_1 = lowering operator

$\bar{Q}_{\dot{1}}$ = raising op

$Q_2 |E, \lambda\rangle = 0$

$\bar{Q}_{\dot{2}} |E, \lambda\rangle = 0$

As before, we define a Clifford vacuum $|E, \lambda_0\rangle$ s.t.

$$Q_1 |E, \lambda_0\rangle = 0$$

STATE	HELICITY	# COMPONENTS
$\|E, \lambda_0\rangle$	λ_0	1
$\bar{Q}_{\dot{1}} \|E, \lambda_0\rangle$	$\lambda_0 + 1/2$	1

Multiplet $= \{ |E, \lambda_0\rangle, \bar{Q}_{\dot{1}} |E, \lambda_0\rangle \}$

Relevant examples

	STATE	HELICITY	# COMPONENTS
1)	$|E, 0\rangle$	$\lambda_0 = 0$	1
	$\bar{Q}_{\dot{1}} |E, 0\rangle$	$\frac{1}{2}$	1

massless chiral multiplet $= [0, \frac{1}{2}]$

	STATE	HELICITY	# COMPONENTS
2)	$|E, \frac{1}{2}\rangle$	$\lambda_0 = 1/2$	1
	$\bar{Q}_{\dot{1}} |E, 1/2\rangle$	1	1

3) $\vert E, 1\rangle$	$\lambda_0 = 1$	1
$\bar{Q}_i \vert E, 1\rangle$	$3/2$	1

gravitino multiplet $\equiv [1, 3/2]$

4) $\vert E, 3/2\rangle$	$\lambda_0 = 3/2$	1
$\bar{Q}_i \vert E, 3/2\rangle$	2	1

graviton multiplet

PCT non-invariance : $[0, 1/2]$ is not invariant under PCT.

PCT would produce $[0, -1/2]$

In order to preserve PCT invariance we consider

$$[0,\tfrac{1}{2}] \oplus [-\tfrac{1}{2},0] \longrightarrow \begin{array}{ll} \lambda = -1/2 & 1 \\ \quad\;\; 0 & 2 \\ \quad\;\; 1/2 & 1 \end{array}$$

N-EXTENDED SUSY

$N=1$ susy supercharges $\begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix} \to$ 1 Majorana spinor

$N>1$ " " $\begin{pmatrix} Q^{i}_\alpha \\ \bar{Q}^{\dot{\alpha}}_{i} \end{pmatrix}$ $i = 1, \dots, N$

i = index of fund. repn. of SU(N)

Poincaré ,

$$[Q_\alpha^i, P_\mu] = [\bar{Q}_{\dot\alpha}^i, P_\mu] = 0 \qquad \forall i$$

$$[M_{\mu\nu}, Q_\alpha^i] = -\frac{1}{2} (\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta^i \qquad \forall i$$

$$\bar{Q}_{\dot\alpha}^i$$

$$\{Q_\alpha^i, \bar{Q}_{j\dot\alpha}\} = 2\delta^i_j P_{\alpha\dot\alpha}$$

$$\{Q_\alpha^i, Q_\beta^j\} = \varepsilon_{\alpha\beta} \underbrace{Z^{ij}}_{\text{central charge}} \qquad Z^{ij} = -Z^{ji}$$

$$\{\bar{Q}_{i\dot\alpha}, \bar{Q}_{j\dot\beta}\} = \varepsilon_{\dot\alpha\dot\beta} \bar{Z}_{ij}$$

$\underline{\text{R-symmetry}} \equiv U(1) \times SU(N) = \underbrace{U(N)}_{T_i{}^j \text{ in fund repr.}}$

$$[T_i{}^j \, Q_\alpha^k] = -i \, \delta_i^k \, Q_\alpha^j$$

We study representations of N-extended algebra setting $Z^{ij} = \bar{Z}_{ij} = 0$

① **Massive case** $\rightarrow$ $P_\mu = (m, 0, 0, 0)$

$$\{ Q_\alpha^i , \bar{Q}_{j\dot{\alpha}} \} = 2 \delta^i_j (\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu = 2m (\sigma^0)_{\alpha\dot{\alpha}}$$

$\Rightarrow$ 2 × N Clifford algebras

- Clifford vacuum $\quad Q^{i}_{\alpha}|0\rangle = 0 \quad \forall \alpha = 1,2$
  $\forall i = 1, \dots N$
- Act with $\bar{Q}_{i\dot{1}}$, $\bar{Q}_{i\dot{2}}$ to construct the whole multiplet

<u>STATE</u>	<u>SPIN</u>	<u># COMPONENTS</u>
$\|s\rangle$	s	$\binom{2N}{0}(2s+1) = 1\,(2s+1)$
$\bar{Q}_{i\dot{\alpha}}\|s\rangle$	$\left\{\begin{array}{l} s+1/2 \\ s-1/2 \end{array}\right.$	$\binom{2N}{1}(2s+1) = 2(2s+1)\times N$
$\bar{Q}_{j\dot{\beta}}\,\bar{Q}_{i\dot{\alpha}}\|s\rangle$	$\left\{\begin{array}{l} s+1 \\ s \\ s-1 \end{array}\right.$	$\binom{2N}{2}(2s+1) = \frac{2N(2N-1)}{2}(2s+1)$

$$\vdots$$

$$\bar{Q}_{1\dot{1}}\,\bar{Q}_{2\dot{1}}\cdots\bar{Q}_{N\dot{1}}\,|s\rangle \quad \begin{cases} s+N/2 \\ s+N/2-1/2 \\ \vdots \\ s-N/2 \end{cases} \qquad \binom{2N}{N}(2s+1)$$

$$\vdots$$

$$\bar{Q}_1^{2}\,\bar{Q}_2^{2}\cdots\bar{Q}_N^{2}\,|s\rangle \qquad s \qquad \binom{2N}{2N}(2s+1) = (2s+1)$$

$$\underline{\text{Dim}} = (2s+1)\sum_{k=0}^{2N}\binom{2N}{k} = 2^{2N}(2s+1)$$

$$0 = (1-1)^{2N} = \sum_{k=0}^{2N} (-1)^k \binom{2N}{k}$$

$$= \underbrace{\sum_{k=0}^{N} \binom{2N}{2k}} - \underbrace{\sum_{k=0}^{N} \binom{2N}{2k+1}}$$

\# states at even levels ≡ \# states at odd

② massless case

$$\{Q^{i}_{1}, \bar{Q}_{j1}\} = 4E\,\delta^{i}_{j}$$

N Clifford algebras

- Vacuum state $Q^i_1 |E,\lambda_0\rangle = 0 \quad \forall\, i=1,\dots N$

STATE	HELICITY	# COMP's
$\|E,\lambda_0\rangle$	λ_0	$1 = \binom{N}{0}$
$\bar{Q}_{i1} \|E,\lambda_0\rangle$	$\lambda_0 + 1/2$	$N = \binom{N}{1}$
$\bar{Q}_{j1}\bar{Q}_{i1} \|E,\lambda_0\rangle$ $j \neq i$	$\lambda_0 + 1$	$\frac{N(N-1)}{2} = \binom{N}{2}$
$\vdots$		
$\bar{Q}_{1i}\bar{Q}_{2i}\dots\bar{Q}_{Ni} \|E,\lambda_0\rangle$	$\boxed{\lambda_0 + N/2}$	$1 \quad \binom{N}{N}$

$$\text{Multiplet} \equiv [\lambda_0, \lambda_0 + 1/2, \dots \lambda_0 + N/2] \qquad (4)$$

$$\underline{\text{Dim}} = \sum_{k=0}^{N} \binom{N}{k} = 2^N$$

$$0 = (1-1)^N = \sum_{k=0}^{N} \binom{N}{k} (-1)^k$$

$$= \sum_{k=0}^{[N/2]} \binom{N}{2k} - \sum_{k=0}^{[N/2]} \binom{N}{2k+1}$$

states at even levels $\equiv$ # states at odd level

In general multiplet (4) breaks PT invariance and we have to consider

$$[\lambda_0, \lambda_0 + 1/2, \dots \underbrace{\lambda_0 + \tfrac{N}{2}}_{}] \oplus [-\lambda_0 - \tfrac{N}{2}, \dots -\lambda_0 - \tfrac{1}{2}, \underbrace{-\lambda_0}_{}]$$

We can have a PCT self-conjugate multiplet <u>if</u>

$$\lambda_0 + \frac{N}{2} = -\lambda_0 \iff \boxed{\lambda_0 = -N/4}$$

Two important examples

1) $N=4 \leadsto \lambda_0 = -1 \to [-1, -1/2, 0, \tfrac{1}{2}, 1]$

N=4 vector multiplet

2) $N=8 \leadsto \lambda_0 = -2 \to [-2, -3/2, \dots\ \tfrac{3}{2}, 2]$

N=8 supergravity multiplet

N=4 is the maximal susy in flat spacetime

N=8 " " " " in curved "

Go back to N=1 → Construct multiplets in terms of fields (rather than states)

$$\mathcal{L} = \frac{1}{2}\partial_{\alpha\dot\alpha}\varphi\,\partial^{\alpha\dot\alpha}\bar\varphi - \frac{i}{2}\bar\psi^{\dot\alpha}\partial_{\alpha\dot\alpha}\psi^{\alpha} + \frac{1}{4}F\bar F$$

$$\Phi = (\varphi, \psi, F) \qquad \delta\Phi = i\left[\epsilon^{\alpha}Q_{\alpha} + \bar\epsilon_{\dot\alpha}\bar Q^{\dot\alpha}, \Phi\right]$$

$$[Q_\alpha, \varphi] = i\psi_\alpha$$

$$\underbrace{[\bar{Q}_{\dot\alpha}, \varphi] = 0}$$

CONSTRAINTS

$$[\bar{Q}_{\dot\alpha}, \bar\varphi] = i\bar\psi_{\dot\alpha}$$

$$\underbrace{[Q_\alpha, \bar\varphi] = 0}$$

$$\vdots$$

Antichiral multiplet

STATE	SPIN	# COMPS
$\vert\bar\varphi\rangle \equiv \bar\varphi(0)\vert 0\rangle$	0	1
$\bar{Q}_{\dot\alpha}\vert\bar\varphi\rangle =$ $[\bar{Q}_{\dot\alpha}, \bar\varphi(0)]\vert 0\rangle$ $= i\bar\psi_{\dot\alpha}\vert 0\rangle \equiv \vert\bar\psi\rangle$	1/2	2

$\bar{Q}^2 \lvert\bar{\varphi}\rangle = [\bar{Q}^{\dot{\alpha}}, \bar{\psi}_{\dot{\alpha}}]\lvert 0\rangle$	0	1

$$\simeq \bar{F}(0)\lvert 0\rangle \equiv \lvert\bar{F}\rangle$$

Multiplet $= \{ \bar{\varphi}(x), \bar{\psi}_{\dot{\alpha}}(x), \bar{F}(x) \}$ Antichiral multiplet

Chiral Multiplet

STATE	SPIN	# COMP'S
$\lvert\varphi\rangle = \varphi(0)\lvert 0\rangle$	0	1
$[Q_\alpha, \varphi(0)]\lvert 0\rangle$ $\sim \psi_\alpha$	1/2	2

$[Q^2, \varphi(0)]|0\rangle$ $\qquad\qquad$ 0 $\qquad\qquad$ 1

$\sim F|0\rangle$

Chiral multiplet $\equiv \{\varphi(x), \psi_\alpha(x), F(x)\}$

$[\bar{Q}_{\dot\alpha}, \varphi] = 0 \rightarrow$ chirality constraint

$[Q_\alpha, \bar{\varphi}] = 0 \rightarrow$ antichirality "

Chiral $\oplus$ Antichiral

$\{\varphi, \psi_\alpha, F\} \oplus \{\bar{\varphi}, \bar{\psi}_{\dot\alpha}, \bar{F}\}$

IRR. REPR's OF SUSY ALGEBRA

$$\left\{ \begin{array}{l} \text{1 complex scalar} \\ \text{1 Majorana fermion} \\ \text{1 complex pseudoscalar} \end{array} \right.$$

If we relax any constraint, the most general $N=1$ multiplet (scalar multiplet) is

$$V = \{ \overbrace{C, \underbrace{\chi, \underset{\substack{\uparrow \\ \text{real} \\ \text{scalar}}}{M}, N, \underset{\substack{\uparrow \\ \text{vector} \\ \text{field}}}{A_\mu}, \lambda}_{\text{Majorana spinor}}, D}^{\text{pseudoscalars}} \}$$

Susy transf rules:

$$\varepsilon = \begin{pmatrix} \varepsilon_\alpha \\ \bar{\varepsilon}^{\dot\alpha} \end{pmatrix} \qquad \bar{\varepsilon} = \varepsilon^\dagger \gamma^0$$

$$\begin{cases}
\delta C = \bar{\varepsilon}\gamma^5 \chi \equiv i[\varepsilon^\alpha Q_\alpha + \bar{\varepsilon}_{\dot\alpha}\bar{Q}^{\dot\alpha}, C] \\
\delta \chi = (M + \gamma_5 N)\varepsilon - i\gamma^\mu (A_\mu + \gamma^5 \partial_\mu C)\varepsilon \\
\delta M = \bar{\varepsilon}(\bar{\lambda} - i\not{\partial}\chi) \\
\delta N = \bar{\varepsilon}\gamma^5(\lambda - i\not{\partial}\chi) \\
\delta A_\mu = i\bar{\varepsilon}\gamma_\mu \lambda + \varepsilon\partial_\mu \chi \\
\delta \lambda = -i\gamma^{\mu\nu}\varepsilon\partial_\mu A_\nu - \gamma_5 \varepsilon D \\
\delta D = -i\bar{\varepsilon}\not{\partial}\gamma^5 \lambda
\end{cases}$$

REDUCIBLE REPR.

- $\{\lambda, \partial_{[\mu}A_{\nu]}, D\}$ they transform within this submultiplet DO IT
- $\{M, N, \lambda - i\not{\partial}\chi, \partial^\mu A_\mu, D + \Box C\}$ they are a submult

Moving towards the construction of <u>N=1 SUPERSPACE</u>

+

Review of Minkowski spacetime as a <u>COSET</u>

$$\text{Coset} \equiv \text{Poincaré}/\text{Lorentz} \equiv ISO(1,3)/SO(1,3)$$

= set of equivalence classes where the equivalence law is the following

g_1, g_2 are equivalent $g_1 \sim g_2$ if

$g_1 = g_2 \circ h \qquad h \in SO(1,3)$

We choose as representative of a given equivalence class

$$L(x) = e^{\overset{\uparrow}{i x^\mu P_\mu}}$$

1-1 correspondence between x^μ parameters and pts in Minkowski

Coset description of Minkowski

Every element of the class-x will be of the form

$$g(x) = L(x) \circ h \qquad h \in SO(1,3)$$

Group composition law

$$x' = x'(x,g)$$

$$g \circ L(x) = L(x') \circ h \qquad h = h(x,g)$$

Two relevant cases

1) Choose $g \equiv L(\xi)$

$$L(\xi) \circ L(x) \overset{?}{=} L(x') \circ h$$

we want to find x', h explicitly

$$L(\xi) \circ L(x) = e^{i\xi^\mu P_\mu} \circ e^{ix^\nu P_\nu} = e^{i(\xi^\mu + x^\mu) P_\mu}$$

$$\equiv L(x') \circ h$$

$$\Rightarrow \begin{cases} h = \mathbb{1} \\ x'^\mu = x^\mu + \xi^\mu \end{cases}$$

Translations in Minkowski

2) We choose $g = e^{\frac{i}{2}\lambda^{\mu\nu}M_{\mu\nu}} \in SO(1,3)$

$$g \cdot L(x) = e^{\frac{i}{2}\lambda^{\mu\nu}M_{\mu\nu}} \, e^{ix^{\rho}P_{\rho}} = \, ?$$

$$\underset{\substack{\text{BCH} \\ \text{identity}}}{=} e^{ix'^{\mu}P_{\mu}} \cdot e^{\frac{i}{2}\lambda^{\mu\nu}M_{\mu\nu}} \qquad (5)$$

where

$x'^{\mu} = x^{\nu}\lambda_{\nu}{}^{\mu}$ rotations in Minkowski

How translations and Lorentz transfs are realised on local fields?

1) Translations $\quad x'^{\mu} = x^{\mu} + \xi^{\mu}$

locality $\Longleftrightarrow$ $\phi(x) \equiv \phi'(x') = \phi'(x+\xi)$

$$\phi'(x) = \phi(x-\xi)$$

$$\Big\downarrow$$

$$\delta_0 \phi \equiv \phi'(x) - \phi(x) = \phi(x-\xi) - \phi(x)$$

$$= -\xi^{\mu}\partial_{\mu}\phi + \cdots$$

$$\phi'(x) \equiv e^{-i\xi\cdot P}\phi(x)\,e^{i\xi\cdot P} \underset{|\xi|\ll 1}{\simeq} -i\xi^{\mu}[P_{\mu},\phi]$$

$$\Rightarrow \boxed{[P_{\mu},\phi] = -i\partial_{\mu}\phi}$$

2) Rotations

$$\phi \to \phi' \qquad \phi'(x) = e^{+\frac{i}{2}\lambda^{\mu\nu}M_{\mu\nu}}\, \phi(x)\, e^{-\frac{i}{2}\lambda^{\mu\nu}M_{\mu\nu}}$$

$$= \underbrace{e^{\frac{i}{2}\lambda^{\mu\nu}M_{\mu\nu}}\, e^{-ix\cdot P}}\, \phi(0)\, \underbrace{e^{ix\cdot P}\, e^{-\frac{i}{2}\lambda^{\mu\nu}M_{\mu\nu}}}$$

$$\underset{(5)}{=} e^{ix'\cdot P}\, \underbrace{e^{\frac{i}{2}\lambda\cdot M}\, \phi(0)\, e^{-\frac{i}{2}\lambda\cdot M}}_{\text{internal rotation}}\, e^{-ix'\cdot P}$$

↳ $\phi_{rot}(0)$

$$= \phi_{rot}(x')$$

Lecture 4

5/11/25

SUPERSPACE

Minkowski spacetime $\longrightarrow$ coset $= \text{Poincaré}/\text{Lorentz}$

$\wr$

Supercoset $\equiv \text{Superpoincaré}/\text{Lorentz} = \overline{Osp(4|1)}/SO(1,3)$

let's take a class representative

$$L(x,\theta,\bar{\theta}) = e^{i\left(x\cdot P + \theta^{\alpha} Q_{\alpha} + \bar{\theta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}\right)}$$

$\begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{pmatrix}$ = Majorana spinor $\Longleftrightarrow$ $\begin{pmatrix} \theta_\alpha \\ \bar{\theta}^{\dot\alpha} \end{pmatrix}$ = Majorana spinor

$$\bar{\theta}_{\dot\alpha} = -(\theta_\alpha)^*$$

1-1 correspondence between $(x,\theta,\bar\theta)$ parametrizing the supercoset and a SUPERMANIFOLD whose pts are coordinated by $(x,\theta,\bar\theta)$

↓

N=1 SUPERSPACE

Group composition law

$$g(\xi,\epsilon,\bar\epsilon)\cdot L(x,\theta,\bar\theta) = g(x',\theta',\bar\theta')\cdot h \qquad (1)$$

$h \in SO(1,3)$

Two relevant examples

1) Take $g(\xi, \epsilon, \bar{\epsilon}) \equiv L(\xi, \epsilon, \bar{\epsilon})$

$$L(\xi,\epsilon,\bar{\epsilon}) \circ L(x,\theta,\bar{\theta}) = \; ?$$

$$\parallel$$

$$e^{\underbrace{i\left(\xi\cdot P + \epsilon^{\alpha} Q_{\alpha} + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}\right)}_{A}} \circ e^{\underbrace{i\left(x\cdot P + \theta^{\beta} Q_{\beta} + \bar{\theta}_{\dot{\beta}} \bar{Q}^{\dot{\beta}}\right)}_{B}}$$

We can use BCH identity for A, B s.t. $[A,B]_{\pm} \neq 0$

but
$$\begin{array}{llll} [& A, & [A,B]] & =0 \\ & B & & =0 \end{array}$$

$\Downarrow$

$$e^{A} \cdot e^{B} = e^{A+B+\frac{1}{2}[A,B]_{\pm}}$$

Compute

$$-[\xi\cdot P + \epsilon^{\alpha} Q_{\alpha} + \bar{\epsilon}_{\dot\alpha}\bar{Q}^{\dot\alpha},\ a\cdot P + \theta^{\beta} Q_{\beta} + \bar{\theta}_{\dot\beta}\bar{Q}^{\dot\beta}]$$

$$= -[\epsilon^{\alpha} Q_{\alpha},\ \bar{\theta}_{\dot\beta}\bar{Q}^{\dot\beta}] - [\bar{\epsilon}_{\dot\alpha}\bar{Q}^{\dot\alpha},\ \theta^{\beta} Q_{\beta}]$$

$$= +\,\epsilon^{\alpha}\bar{\theta}_{\dot\beta}^{\;\uparrow}\{Q_{\alpha},\ \bar{Q}^{\dot\beta}_{\downarrow}\} + \overset{\uparrow}{\bar{\epsilon}_{\dot\alpha}}\,\theta^{\beta}\{\bar{Q}^{\dot\alpha}_{\downarrow},\ Q_{\beta}\}$$

$$= -\epsilon^{\alpha}\bar{\theta}^{\dot\beta}\,2P_{\alpha\dot\beta} - \bar{\epsilon}^{\dot\alpha}\theta^{\beta}\,2P_{\beta\dot\alpha}$$

$\Downarrow$

$$L(\xi,\varepsilon,\bar\varepsilon)\circ L(x,\theta,\bar\theta) =$$

$$e^{i\left[(x^{\alpha\dot\alpha}+\xi^{\alpha\dot\alpha}+i\varepsilon^{\alpha}\bar\theta^{\dot\alpha}+i\bar\varepsilon^{\dot\alpha}\theta^{\alpha})P_{\alpha\dot\alpha}+(\theta^{\alpha}+\varepsilon^{\alpha})Q_{\alpha}+(\bar\theta_{\dot\alpha}+\bar\varepsilon_{\dot\alpha})\bar Q^{\dot\alpha}\right]}$$

$$= L\left(x+\xi+i\varepsilon\bar\theta+i\bar\varepsilon\theta,\ \theta+\varepsilon,\ \bar\theta+\bar\varepsilon\right)$$

$$\Downarrow$$

In superspace this corresponds to

$$\begin{cases} x'^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} + \overbrace{\xi^{\alpha\dot\alpha} + i\bar\varepsilon^{\dot\alpha}\theta^{\alpha} + i\varepsilon^{\alpha}\bar\theta^{\dot\alpha}}^{\delta x^{\alpha\dot\alpha}} \\ \theta'^{\alpha} = \theta^{\alpha} + \varepsilon^{\alpha} \\ \bar\theta'^{\dot\alpha} = \bar\theta^{\dot\alpha} + \bar\varepsilon^{\dot\alpha} \end{cases} \qquad (2)$$

SUPERTRANSLATION

in superspace

2) Rotations

Consider composition law (1) with

$$g = e^{\frac{i}{2}\lambda\cdot M}$$

$$e^{\frac{i}{2}\lambda\cdot M} \circ L(x,\theta,\bar{\theta}) = L(x',\theta',\bar{\theta}') \circ e^{\frac{i}{2}\lambda\cdot M}$$

where (infinitesimal λ)

$$\left\{ x'^{\mu} = x^{\mu} - x^{\nu}\lambda_{\nu}{}^{\mu} \right.$$

$$\left\{\begin{aligned} \theta'^{\alpha} &= \theta^{\beta}\left(e^{-\frac{i}{4}\lambda^{\mu\nu}M_{\mu\nu}}\right)_{\beta}{}^{\alpha} \\ \bar{\theta}'^{\dot{\alpha}} &= \left(e^{\frac{i}{4}\lambda^{\mu\nu}M_{\mu\nu}}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\,\bar{\theta}^{\dot{\beta}} \end{aligned}\right.$$

Minkowski → Define smooth functions $f(x)$

Superspace → Define smooth functions $\Phi(x,\theta,\bar{\theta})$

SUPERFIELD

We will only consider <u>LOCAL SUPERFIELDS</u>

$(x,\theta,\bar\theta) \to (x',\theta',\bar\theta')$ the superfield
superteanslations transforms as
$\Phi \to \Phi'$ s.t.

$$\boxed{\Phi'(x',\theta',\bar\theta') = \Phi(x,\theta,\bar\theta)}$$

$x' = x + \delta x$ (from (2))

$\theta' = \theta + \varepsilon$

$\bar\theta' = \bar\theta + \bar\varepsilon$

$$\Phi'(x,\theta,\bar\theta) = \Phi(x-\delta x, \theta-\varepsilon, \bar\theta-\bar\varepsilon)$$

$$\Downarrow$$

Functional variation

$$\delta_0 \Phi \equiv \Phi'(x,\theta,\bar\theta) - \Phi(x,\theta,\bar\theta)$$

$$= \Phi(x-\delta x, \theta-\varepsilon, \bar\theta-\bar\varepsilon) - \Phi(x,\theta,\bar\theta)$$

$$\underset{\text{infinitesimal}}{=} -\delta x^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}\Phi - \varepsilon^{\alpha}\frac{\partial\Phi}{\underbrace{\partial\theta^{\alpha}}} - \bar\varepsilon^{\dot\alpha}\frac{\partial\Phi}{\underbrace{\partial\bar\theta^{\dot\alpha}}}$$

$$\frac{\partial}{\partial\theta^{\alpha}} \equiv \partial_{\alpha}$$

$$\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \equiv \bar{\partial}_{\dot{\alpha}}$$

Convention : $\begin{cases} \partial_\alpha \theta^\beta = - \delta_\alpha{}^\beta \\ \left[\bar{\partial}_{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}{}^{\dot{\beta}} \right. \end{cases}$ $\qquad \theta^\beta \overleftarrow{\partial}_\alpha = \delta_\alpha{}^\beta$

$$\delta_0 \Phi = - \delta x^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \Phi - \varepsilon^\alpha \partial_\alpha \Phi - \bar{\varepsilon}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} \Phi \qquad (3)$$

$$\Downarrow$$

$$L(\xi, \varepsilon, \bar{\varepsilon}) \, \Phi(x, \theta, \bar{\theta}) \, L^{-1}(\xi, \varepsilon, \bar{\varepsilon})$$

$$\underset{\text{infinit.}}{=} -i\xi^{\alpha\dot\alpha}[P_{\alpha\dot\alpha},\Phi] - i\epsilon^{\alpha}[Q_\alpha,\Phi] - i\bar\epsilon_{\dot\alpha}[\bar Q^{\dot\alpha},\Phi] \qquad (4)$$

Comparing eqs (3) and (4) we obtain

$$\left\{\begin{aligned} [P_{\alpha\dot\alpha},\Phi] &= -i\partial_{\alpha\dot\alpha}\Phi \\ [Q_\alpha,\Phi] &= -i\left(\partial_\alpha + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\right)\Phi \\ [\bar Q_{\dot\alpha},\Phi] &= -i\left(\bar\partial_{\dot\alpha} - i\theta^{\alpha}\partial_{\alpha\dot\alpha}\right)\Phi \end{aligned}\right. \qquad \forall\,\Phi$$

$$\Rightarrow \quad \left\{\begin{aligned} P_{\alpha\dot\alpha} &= -i\partial_{\alpha\dot\alpha} \\ Q_\alpha &= -i\left(\partial_\alpha + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}\right) \\ \bar Q_{\dot\alpha} &= -i\left(\bar\partial_{\dot\alpha} - i\theta^{\alpha}\partial_{\alpha\dot\alpha}\right) \end{aligned}\right.$$

they satisfy the correct algebra

Differential calculus

Do we like $\partial_{\alpha\dot\alpha}, \partial_\alpha, \bar\partial_{\dot\alpha}$?

We would like these derivatives to be covariant respect to supertranslations

- $\partial_{\alpha\dot\alpha}\Phi$
  $$\delta_\varepsilon(\partial_{\alpha\dot\alpha}\Phi) \equiv i[\varepsilon^\beta Q_\beta, \partial_{\alpha\dot\alpha}\Phi]$$
  $$\overset{?}{=} \underbrace{\partial_{\alpha\dot\alpha}}_{P_{\alpha\dot\alpha}}(i[\varepsilon^\beta Q_\beta, \Phi) = \partial_{\alpha\dot\alpha}(\delta_\varepsilon \Phi)$$
  TRUE

- $\partial_\alpha\Phi$
  $$\delta_\varepsilon(\partial_\alpha\Phi) \equiv i[\varepsilon^\beta Q_\beta, \partial_\alpha \Phi]$$

$$\overset{?}{=} i\,\partial_\alpha\,[\epsilon^\beta Q_\beta, \Phi] = \partial_\alpha(\delta_\epsilon \Phi)$$

$$\{\partial_\alpha, Q_\beta\} = 0 \quad ?$$

$$\{\partial_\alpha, -i(\partial_\beta + i\bar\theta^{\dot\beta}\partial_{\beta\dot\beta})\} = 0$$

$$\delta_{\bar\epsilon}(\partial_\alpha\Phi) = i[\bar\epsilon_{\dot\beta}\bar Q^{\dot\beta}, \partial_\alpha\Phi]$$

$$= \dots$$

$$\{\partial_\alpha, \bar Q_{\dot\beta}\} = 0 \quad ?$$

$$\downarrow$$

$$\{\partial_\alpha, -i(\bar\partial_{\dot\beta} - i\theta^\beta\partial_{\beta\dot\beta})\} = \partial_{\alpha\dot\beta} \neq 0 \quad !!$$

We need covariantize $\partial_\alpha, \bar{\partial}_{\dot\alpha}$

$$\partial_\alpha \to D_\alpha$$

$$\bar{\partial}_{\dot\alpha} \to \bar{D}_{\dot\alpha}$$

s.t. $\{D_\alpha, Q_\beta\} =$

$\{D_\alpha, \bar{Q}_{\dot\beta}\} = 0$

$$\{\bar{D}_{\dot\alpha}, Q_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

$$\begin{cases} D_\alpha = \partial_\alpha - i\bar{\theta}^{\dot\alpha}\partial_{\alpha\dot\alpha} \\ \bar{D}_{\dot\alpha} = \bar{\partial}_{\dot\alpha} + i\theta^\alpha \partial_{\alpha\dot\alpha} \end{cases}$$

check that they anticommute with $Q, \bar{Q}$

$$\{D_\alpha, \bar{D}_{\dot\alpha}\} = 2P_{\alpha\dot\alpha}$$

$$\{D_\alpha, D_\beta\} = 0 \qquad (D^3 = 0)$$

$$\{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0 \qquad (\bar{D}^3 = 0)$$

$$\begin{cases} D_\alpha = i Q_\alpha - 2i\bar{\theta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \\ \bar{D}_{\dot{\alpha}} = i \bar{Q}_{\dot{\alpha}} + 2i\theta^{\alpha} \partial_{\alpha\dot{\alpha}} \end{cases}$$

they differ by a total $\partial_{\alpha\dot{\alpha}}$-derivative

Conclusion: $D_A = \left(\partial_{\alpha\dot{\alpha}}, D_\alpha, \bar{D}_{\dot{\alpha}} \right)$

$A = (\alpha\dot{\alpha}), \alpha, \dot{\alpha}$

In bosonic manifolds

$$[D_\mu, D_\nu] = \underbrace{T_{\mu\nu}}_{\text{torsion}}{}^{\rho} D_\rho + \underbrace{R_{\mu\nu}}_{\text{curvature}}$$

$$\Big\downarrow$$

$$[D_A, D_B]_{\pm} = T_{AB}{}^{C} D_C + R_{AB}$$

Comparing with the D-algebra we find that

$R_{AB} = 0 \implies$ N=1 superspace is FLAT

$$T_{\alpha\dot{\alpha}}{}^{(\beta\dot{\beta})} = -2i\, \delta_\alpha{}^\beta\, \delta_{\dot{\alpha}}{}^{\dot{\beta}}$$ with TORSION

Expansion in components

$\{\theta_\alpha, \theta_\beta\} = 0 \ldots$

$$\Phi(x,\theta,\bar\theta) = c(x) + \theta^\alpha \chi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha}(x)$$
$$+ \theta^2 M(x) + \bar\theta^2 \bar{M}(x) + \theta^\alpha\bar\theta^{\dot\alpha} A_{\alpha\dot\alpha}(x) \qquad (6)$$
$$+ \bar\theta^2\theta^\alpha\lambda_\alpha(x) + \theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x)$$
$$+ \theta^2\bar\theta^2 D(x)$$

$$\theta^2 = \frac{1}{2}\theta^\alpha\theta_\alpha$$
$$\bar\theta^2 = \frac{1}{2}\bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}$$

$$\{ c, \chi_\alpha, \bar{\chi}_{\dot\alpha}, M, \bar{M}, A_{\alpha\dot\alpha}, \lambda_\alpha, \bar{\lambda}_{\dot\alpha}, D \} \equiv \text{components of } \Phi$$

$$c = \Phi \big|_{\theta = \bar\theta = 0}$$

$$\chi_\alpha = -D_\alpha \Phi \big|$$

Check it

$$\bar\chi_{\dot\alpha} = -\bar{D}_{\dot\alpha} \Phi \big|$$

$$M = D^2 \Phi \big|$$

$$D^2 \equiv \frac{1}{2} D^\alpha D_\alpha$$

$$\bar{M} = \bar{D}^2 \Phi \big|$$

$$\bar{D}^2 = \frac{1}{2} \bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha}$$

$$A_{\alpha\dot\alpha} = D_\alpha \bar{D}_{\dot\alpha} \Phi \big|$$

$$\lambda_\alpha = \bar{D}^2 D_\alpha \Phi \Big|$$

$$\bar{\lambda}_{\dot\alpha} = D^2 \bar{D}_{\dot\alpha} \Phi \Big|$$

$$D = D^2 \bar{D}^2 \Phi \Big|$$

Claim: the set of components realises a representation of the susy algebra

In fact, apply a susy transf to Φ

$$\delta\Phi = i\left[\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot\alpha}\bar{Q}^{\dot\alpha}, \Phi\right]$$

$$\|$$

$$\delta C + \theta^\alpha \delta\chi_\alpha + \bar{\theta}_{\dot\alpha}\delta\bar{\chi}^{\dot\alpha} + ----$$

$$\Rightarrow \begin{cases} \delta C = \\ \delta \chi_\alpha = \\ \delta \bar{\chi}_{\dot\alpha} = \\ \vdots \end{cases}$$

Check that the algebra closes on this set of components

Φ in eq. (6) is an alternative realization of the real vector multiplet

CONSTRAINED SUPERFIELDS

Def: chiral superfield : $\bar{D}_{\dot\alpha} \Phi = 0$

antichiral " : $D_\alpha \bar{\Phi} = 0$

Obs : $\bar{D}_{\dot\alpha}\Phi = 0$ chiral superfield

$$\wr$$

$$0 = \bar{D}_{\dot\alpha}\Phi\Big|_{\theta=\bar\theta=0}\uparrow = \left(i\bar{Q}_{\dot\alpha}\Phi + \cancel{2i\theta^{\alpha}}\partial_{\alpha\dot\alpha}\Phi\right)\Big|$$

$$\bar{D}_{\dot\alpha} = i\bar{Q}_{\dot\alpha} + 2i\theta^{\alpha}\partial_{\alpha\dot\alpha}$$

Chirality cond $\Longleftrightarrow$ $[\bar{Q}_{\dot\alpha},\Phi]\Big| = 0 \Rightarrow [\bar{Q}_{\dot\alpha},\varphi]=0$

$\uparrow$

$\Phi\big|$ = its lowest component $= \varphi$

same condition already introduced for chiral multiplets let's solve the constraint $\bar{D}_{\dot\alpha}\Phi = 0$

we introduce a new set of bosonic coords

$$x_L^{\alpha\dot\alpha} \equiv x^{\alpha\dot\alpha} + i\theta^\alpha\bar\theta^{\dot\alpha}$$

$\Rightarrow$ $\bar{D}_{\dot\alpha}\, x_L^{\alpha\dot\alpha} = 0$. Moreover, $\bar{D}_{\dot\alpha}\,\theta_\alpha = 0$

$$\boxed{\bar\partial_{\dot\alpha}} + \theta^\beta\partial_{\beta\dot\alpha}$$

therefore, the general expression for a <u>chiral superfield</u>

$$\Phi(x,\theta,\bar\theta) = \varphi(x_L) + \theta^\alpha\psi_\alpha(x_L) + \theta^2 F(x_L)$$

Components of a chiral superfield $= \{ \varphi , \psi_\alpha , F \} \equiv$ chiral multiplet

Exercise – Find $\delta \varphi = --$

$\delta \psi_\alpha = ---$ under susy transfs

$\delta F = ---$

and check that they are exactly the transfs of a chiral multiplet

Antichiral $\quad D_\alpha \bar{\Phi} = 0$

$$x_R^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} - i\,\theta^\alpha \bar\theta^{\dot\alpha} \quad \Rightarrow \quad D_\alpha x_R^{\alpha\dot\alpha} = 0 \;,\; D_\alpha \bar\theta^{\dot\beta} = 0$$

$$\rightarrow \quad \bar\Phi(x,\theta,\bar\theta) = \bar\varphi(x_R) + \bar\theta_{\dot\alpha}\bar\Psi^{\dot\alpha}(x_R) + \bar\theta^2 \bar F(x_R)$$

Integrazione in superspazio

$$x^\mu \rightsquigarrow \int d^4x$$

$$\begin{matrix}\theta^\alpha \\ \bar\theta^{\dot\alpha}\end{matrix} \rightsquigarrow \begin{matrix}\int d\theta^\alpha & ? \\ \int d\bar\theta^{\dot\alpha} & ?\end{matrix}$$

Toy model: 1D superspazio (x,θ)

$$\phi(x,\theta) = a(x) + \theta\, b(x)$$

$$\int d\theta \, \phi(x,\theta) = \quad ?$$

Requirements

1) $\int d\theta \, \phi(x,\theta)$ is a superfield $\Longrightarrow$

$$\int d\theta \, (a + \theta b) = A + \theta B$$

2) Linearity

$$\int d\theta \; \alpha (a + \theta b) = \alpha \int d\theta \, a + \alpha \int d\theta \, \theta b$$

3) Integral has to be translational invariant

- $d\theta = d(\theta + \varepsilon) \equiv d\theta'$ $\varepsilon =$ constant

$$\int d\theta\, \Phi(x,\theta) \equiv \int d\theta'\, \Phi(x,\theta')$$

let's work out this cond

$$\int d\theta\,(a+\theta b) \equiv \int d\theta'\,(a+\theta' b) = \int d\theta\,\big(a+(\theta+\varepsilon)b\big)$$

$$\int d\theta\, a + \int d\theta\, \theta b = \int d\theta\,(a+\varepsilon b) + \int d\theta\, \theta b$$

$$\Rightarrow \int d\theta\, \varepsilon b = \varepsilon b \int d\theta \equiv 0$$

$$\rightarrow \quad \left[\begin{array}{l} \int d\theta = 0 \\ \int d\theta\, \theta = 1 = \theta \frac{\partial}{\partial\theta} \end{array}\right.$$

$\delta(\theta-\theta')$?

$$\int d\theta \; \delta(\theta-\theta') \underbrace{\Phi(x,\theta)}_{(a+\theta b)} = \underbrace{\Phi(x,\theta')}_{(a+\theta' b)}$$

$$\Rightarrow \quad \delta(\theta-\theta') = (\theta-\theta')$$

Generalisation to 4D

$\theta_\alpha \, , \, \bar{\theta}_{\dot{\alpha}}$

$$\int d^2\theta \ldots \equiv \int \frac{1}{2}\, d\theta^\alpha \, d\theta_\alpha \; \ldots$$

$$\int d^2\bar{\theta} \ldots \equiv \int \frac{1}{2}\, d\bar{\theta}_{\dot{\alpha}} \, d\bar{\theta}^{\dot{\alpha}} \; \ldots$$

$$\int d^2\theta = 0 \qquad \int d^2\bar{\theta} = 0$$

$$\int d^2\theta \; \theta^2 = -1 \equiv D^2 \theta^2 \Big|_{\theta=\bar{\theta}=0}$$

$$\int d^2\bar{\theta} \; \bar{\theta}^2 = -1 \equiv \bar{D}^2 \bar{\theta}^2 \Big|$$

In general

$$\int d^2\theta \; \phi(x,\theta,\bar{\theta}) = D^2 \phi(x,\theta,\bar{\theta}) \Big|_{\theta=\bar{\theta}=0}$$

$$\int d^2\bar{\theta} \; \phi(x,\theta,\bar{\theta}) = \bar{D}^2 \phi(x,\theta,\bar{\theta}) \Big|_{\theta=\bar{\theta}=0}$$

$$\int d^2\theta \, d^2\bar{\theta} \; \phi(x,\theta,\bar{\theta}) = D^2 \bar{D}^2 \Phi(x,\theta,\bar{\theta}) \Big|$$

Witten → Supermanifolds

$$\delta^{(2)}(\theta-\theta') = (\theta-\theta')^2$$

$$\delta^{(2)}(\bar\theta-\bar\theta') = (\bar\theta-\bar\theta')^2$$

$$\delta^{(4)}(\theta-\theta') = \delta^{(2)}(\theta-\theta')\,\delta^{(2)}(\bar\theta-\bar\theta')$$

$$= (\theta-\theta')^2(\bar\theta-\bar\theta')^2$$

Integration by parts

In 1D

$$\int d\theta\, f(x,\theta)\frac{\partial}{\partial\theta}g(x,\theta) = -\int d\theta\, \frac{\partial f}{\partial\theta}(x,\theta)\cdot g(x,\theta)$$

FIELD THEORY IN SUPERSPACE

Lagrangian FT $\rightsquigarrow$ $\mathcal{L}$? $S = \int d^4x\, d^2\theta\, d^2\bar\theta\, \mathcal{L}$

If ϕ is a superfield $\Rightarrow$ ϕ^n still a superfield

$D_\alpha \phi$ $\Big\}$ " "

$\bar D_{\dot\alpha} \phi$

$\partial_{\alpha\dot\alpha}\phi$ " "

$\Rightarrow$ $\mathcal{L}(\phi^n, D_\alpha\phi, \bar D_{\dot\alpha}\phi, \partial_{\alpha\dot\alpha}\phi$, powers of derivatives ...)

$\equiv$ superfield

Action : $S = \int \overbrace{d^4x\, d^2\theta\, d^2\bar\theta}^{d^8z}\, \mathcal{L} = \int d^4x\, D^2 \bar D^2 \mathcal{L}\Big|_{\theta=\bar\theta=0}$

we go to components

Caveat: For chiral superfield $\phi \to \phi^n$ still chiral

$$\bar{D}_{\dot\alpha}\phi^{n} = n\,\phi^{m-1}\,\bar{D}_{\dot\alpha}\phi = 0$$

$$\Rightarrow \quad \int d^4x\, d^2\theta\, \phi^n$$

$$\int d^4x\, d^2\bar\theta\, \bar\Phi^n$$

Claim: S is manifestly susy invariant ⇒ susy FT

$$\delta_{susy} S = \int d^8z\, \delta_{susy}\mathcal{L} = \int d^8z\; i\left[\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar{Q}^{\dot\alpha}, \Phi\right]$$

$$= i\int d^4x \; D^2\bar{D}^2 \Big(\varepsilon^\alpha (Q_\alpha \mathcal{L}) + \bar{\varepsilon}_{\dot\alpha} (\bar{Q}^{\dot\alpha}\mathcal{L}) \Big)\Big|$$

$$= i\int d^4x \Big[\varepsilon^\alpha Q_\alpha (D^2\bar{D}^2\mathcal{L}) + \bar{\varepsilon}_{\dot\alpha}\bar{Q}^{\dot\alpha}(D^2\bar{D}^2\mathcal{L}) \Big]\Big|$$

$$\simeq i\int d^4x \Big[\underset{\overset{\shortparallel}{0}}{\cancel{\varepsilon^\alpha D_\alpha (D^2\bar{D}^2\mathcal{L})}}\Big| + \bar{\varepsilon}_{\dot\alpha}\, \underbrace{\bar{D}^{\dot\alpha}(D^2\bar{D}^2\mathcal{L})}\Big| \quad \Big]$$

$$\downarrow$$

$$\bar{D}^{\dot\alpha} D^2\bar{D}^2\mathcal{L} = [\bar{D}^{\dot\alpha}, D^2]\,\bar{D}^2\mathcal{L}$$

$$= 2i\,\partial^{\alpha\dot\alpha}(D_\alpha\bar{D}^2\mathcal{L})$$

$$\simeq \int d^4x \; \bar{\varepsilon}_{\dot\alpha}\; \partial^{\alpha\dot\alpha}(D_\alpha\bar{D}^2\mathcal{L})\Big| = 0$$

(Total derivative)

Lecture 5

6/11/25

FT in superspace (continue...)

$$S = \int d^4x \, d^2\theta \, d^2\bar\theta \; \mathcal{L}\left(\Phi, D_\alpha \Phi, \bar D_{\dot\alpha} \Phi, \ldots\right)$$

$$= \int d^4x \; D^2 \bar D^2 \mathcal{L} \Big|_{\theta=\bar\theta=0}$$

- S is manifestly susy invariant
- Dimensional analysis

$$\{ \underset{\frac{1}{2}}{Q_\alpha}, \underset{\frac{1}{2}}{\bar Q_{\dot\alpha}} \} = 2 \underset{1}{P_{\alpha\dot\alpha}}$$

Group element $e^{i(\underset{-\frac{1}{2}}{\theta^\alpha}\underset{\frac{1}{2}}{Q_\alpha} + \underset{-\frac{1}{2}}{\bar\theta_{\dot\alpha}}\underset{\frac{1}{2}}{\bar Q^{\dot\alpha}} + \cdots)}$ $[D_\alpha = \partial_\alpha - \cdots] = \frac{1}{2}$

$$[\theta_\alpha] = -\frac{1}{2} \quad \sim \quad [d\theta_\alpha] = \frac{1}{2} \qquad \int d\theta\, \theta = 1$$

$$[\bar\theta_{\dot\alpha}] = -\frac{1}{2} \quad \sim \quad [d\bar\theta_{\dot\alpha}] = \frac{1}{2}$$

$$\Rightarrow \quad S = \int \underbrace{d^4x\, d^2\theta\, d^2\bar\theta}_{-4 + 2\cdot\frac{1}{2} + 2\cdot\frac{1}{2}} \mathcal{L} \quad \Rightarrow \quad \boxed{[\mathcal{L}] = 2}$$

- (Anti) chiral integrals

If $\mathcal{L}_c$ is chiral $(\bar D_{\dot\alpha}\mathcal{L}_c = 0) \Rightarrow \underbrace{\int d^4x\, d^2\theta\, \mathcal{L}_c}_{\text{chiral integral}}$

If $\bar{\mathcal{L}}_c$ is antichiral $(D_\alpha \bar{\mathcal{L}} = 0) \Rightarrow \underbrace{\int d^4x\, d^2\bar\theta\, \bar{\mathcal{L}}_c}$

antichiral wit.

Simplest example

Consider a pair $\Phi, \bar{\Phi}$ of chiral + antichiral superfields

$$\Phi = \varphi + \theta^{\alpha} \psi_{\alpha} + \theta^{2} F \qquad (1)$$
$$\bar{\Phi} = \bar{\varphi} + \bar{\theta}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} + \bar{\theta}^{2} \bar{F}$$

Usual kinetic term for $\varphi \sim \int d^4x \, \partial_{\alpha\dot{\alpha}} \underset{1}{\varphi} \, \partial^{\alpha\dot{\alpha}} \underset{1}{\bar{\varphi}}$

$$\underset{1}{\Phi} = \underset{1}{\varphi} + \underset{-\frac{1}{2}}{\theta^{\alpha}} \underset{\frac{3}{2}}{\psi_{\alpha}} + \underset{-1}{\theta^{2}} \underset{2}{F}$$

$\rightarrow$ $[\mathcal{L}] = 2$

$\rightarrow$ $\mathcal{L}(\phi, \bar{\Phi})$ s.t. it is <u>real</u>

$\rightarrow$ $[\phi] = [\bar{\Phi}] = 1$

$$\Downarrow$$

$$S = \int d^4x \, d^2\theta \, d^2\bar{\theta} \quad \phi\bar{\Phi}$$

We go to components

$$S = \int d^4x \; D^2 \bar{D}^2 (\phi\bar{\Phi}) \Big|$$

$$D^2 \equiv \frac{1}{2} D^\alpha D_\alpha$$

$$= \int d^4x \, D^2 \left(\phi \bar{D}^2 \bar{\Phi}\right) \Big|$$

$$= \int d^4x \left[D^2\phi\, \bar D^2\bar\Phi \Big| + \underbrace{\phi\, D^2 \bar D^2 \bar\Phi \Big|}_{} + \underbrace{D^\alpha\phi\, D_\alpha \bar D^2\bar\Phi}_{} \Big| \right]$$

$$\rightarrow \quad \phi\, D^2\bar D^2\bar\Phi = \phi\,\{D^2,\bar D^2\}\bar\Phi = -\Phi\,\Box\,\bar\Phi$$

$$\{D^2,\bar D^2\} \underset{\text{using } N=1 \text{ superalgebra}}{=} -\Box + \bar D^{\dot\alpha} D^2 \bar D_{\dot\alpha} = \overbrace{-\Box + D^\alpha \bar D^2 D_\alpha}$$

$$\rightarrow \quad D^\alpha\phi\, D_\alpha\bar D^2\bar\Phi = D^\alpha\phi\,[D_\alpha,\bar D^2]\,\bar\Phi = -2i\, D^\alpha\phi\,\partial_{\alpha\dot\alpha}\bar D^{\dot\alpha}\bar\Phi$$

$$[D_\alpha,\bar D^2] = \overbrace{-2i\,\partial_{\alpha\dot\alpha}\bar D^{\dot\alpha}}$$

$$\downarrow$$

$$S = \int d^4x \left[D^2\Phi \bar{D}^2\bar{\Phi}\Big| - \Phi \Box \bar{\Phi}\Big| - 2i D^\alpha \Phi \, \partial_{\alpha\dot\alpha} \bar{D}^{\dot\alpha}\bar{\Phi}\Big| \right]$$

From expressions (1) we obtain the $\Phi, \bar\Phi$ components

$\Phi| = \varphi \qquad D_\alpha \Phi| = -\psi_\alpha \qquad D^2\Phi| = -F$

$\bar\Phi| = \bar\varphi \qquad \bar{D}_{\dot\alpha}\bar\Phi| = -\bar\psi_{\dot\alpha} \qquad \bar{D}^2\bar\Phi| = -\bar{F}$

$$S_{kin} = \int d^4x \left\{ F\bar{F} - \varphi \Box \bar\varphi - 2i \psi^\alpha \partial_{\alpha\dot\alpha} \bar\psi^{\dot\alpha} \right\}$$

Interactions ?

Chiral interactions $\rightarrow g\int d^4x \, d^2\theta \, \Phi^n + \bar{g}\int d^4x \, d^2\bar\theta \, \bar\Phi^n$

the most general form of an interacting action is

$$S = \int d^4x\, d^2\theta\, d^2\bar\theta\, \underbrace{K(\Phi,\bar\Phi)}_{\text{Kahler term}} + \int d^4x\, d^2\theta\, \underbrace{W(\Phi)}_{\text{superpotential}} + \int d^4x\, d^2\bar\theta\, \bar{W}(\bar\Phi)$$

non-linear σ-model in superspace

ACTION PRINCIPLE → EOM

Ordinary bosonic case $\quad S[\varphi]$

$$\delta S = \int d^4x' \, \frac{\delta S}{\delta\varphi(x')}\, \delta\varphi(x') \qquad S = \int d^4x\, \mathcal{L}(\varphi(x))$$

$$= \int d^4x'\, d^4x \, \frac{\delta \mathcal{L}(\varphi(x))}{\delta\varphi(x')} \cdot \delta\varphi(x') \equiv 0 \quad \forall\, \delta\varphi$$

$\downarrow$

$$\frac{\delta\varphi(x)}{\delta\varphi(x')} = \delta^{(4)}(x-x')$$

Generalization to superspace

$$\frac{\delta\Phi(x,\theta,\bar\theta)}{\delta\Phi(x',\theta',\bar\theta')} = \delta^{(4)}(x-x')\,\underbrace{\delta^{(2)}(\theta-\theta')\,\delta^{(2)}(\bar\theta-\bar\theta')}_{(\theta-\theta')^2\,(\bar\theta-\bar\theta')^2}$$

<u>Caveat</u> : if ϕ is chiral ($\bar D_{\dot\alpha}\phi=0$) the rule is

$$\frac{\delta\phi(x,\theta,\bar\theta)}{\delta\phi[x',\theta',\bar\theta')} = \boxed{\bar D^2}\,\delta^{(4)}(x-x')\,\delta^{(4)}(\theta-\theta')$$

$$\frac{\delta\bar\phi(x,\theta,\bar\theta)}{\delta\bar\phi(x',\theta',\bar\theta')} = \boxed{D^2}\,\delta^{(4)}(x-x')\,\delta^{(4)}(\theta-\theta')$$

Apply to $S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\; \Phi \bar{\Phi}$

$$\delta S = \int d^8 z' \; \frac{\delta S(\Phi, \bar{\Phi})}{\delta \bar{\Phi}(x', \theta', \bar{\theta}')} \; \delta \bar{\Phi}(x', \theta', \bar{\theta}')$$

$$= \int d^8 z' \int d^8 z \; \Phi(x, \theta, \bar{\theta}) \; \underbrace{\frac{\delta \bar{\Phi}(x, \theta, \bar{\theta})}{\delta \bar{\Phi}(x', \theta', \bar{\theta}')}}_{D^2 \, \delta^{(4)}(x-x') \, \delta^{(4)}(\theta-\theta')} \; \delta \bar{\Phi}(x', \theta', \bar{\theta}')$$

by parts

$$= \int d^8 z' \int d^8 z \; D^2 \Phi(x, \theta, \bar{\theta}) \; \delta^{(4)}(x - x') \, \delta^{(4)}(\theta - \theta') \; \delta \bar{\Phi}(x', \theta', \bar{\theta}')$$

$$= \int d^8 z \; D^2 \Phi(z) \; \delta \bar{\Phi}(z) \equiv 0 \qquad \forall \, \delta \bar{\Phi}$$

$$\Rightarrow \boxed{D^2 \Phi = 0} \qquad \boxed{\bar{D}^2 \bar{\Phi} = 0}$$

EOM

Exercise : Write these eqs in components

- $D^2\Phi \big| = 0 \quad \rightsquigarrow \quad F = 0$
- $\bar{D}_{\dot\alpha}(D^2\Phi) \big| = 0 \quad \rightsquigarrow \quad \partial_{\alpha\dot\alpha}\psi^{\alpha} = 0$
- $\bar{D}^2(D^2\Phi) \big| = 0 \quad \rightsquigarrow \quad \Box \varphi = 0$

R-symmetry invariance of S

$$[R, Q_\alpha] = -Q_\alpha \qquad [R, \bar{Q}_{\dot\alpha}] = \bar{Q}_{\dot\alpha}$$

Group element $e^{i\left(\theta^{\alpha}Q_{\alpha}+\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}+\dots\right)}$

$$\Downarrow$$

$$\begin{cases} R(\theta_\alpha) = 1 \\ R(\bar{\theta}_{\dot{\alpha}}) = -1 \end{cases}$$

In general $\Phi, \bar{\Phi}$ can have non-trivial R-charge

$$R(\Phi) = \omega \qquad R(\bar{\Phi}) = -\omega$$

&

$$S_{kin} = \int \underbrace{d^4x\, d^2\theta\, d^2\bar{\theta}\ \Phi\bar{\Phi}}_{R=0}$$

Chiral interactions

$$\underbrace{\int d^4x \, d^2\theta \, \phi^n}_{R = -2 + n\omega}$$

R-symmetry is preserved by the interaction only if we choose

$$\boxed{\omega = \frac{2}{n}}$$

Generalization to interactions

$$\int d^4x \, d^2\theta \, \underbrace{W(\phi)}$$ generic superpotential

R-symmetry is preserved only if $\boxed{R(W) = 2}$

Superfield components have different R-charge

$$\Phi = \varphi + \theta^\alpha \underset{\substack{\uparrow \\ \omega-1}}{\psi_\alpha} + \theta^2 \underset{\substack{\uparrow \\ \omega-2}}{F}$$

$R: \quad \omega \qquad \omega$

WESS-ZUMINO MODEL

$$S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\; \Phi \bar{\Phi}$$

$$+ \int d^4x\, d^2\theta \left(\frac{m}{2} \Phi^2 + \frac{\lambda}{3!} \Phi^3 \right) + \int d^4x\, d^2\bar{\theta} \left(\frac{\bar{m}}{2} \bar{\Phi}^2 + \frac{\bar{\lambda}^3}{3!} \bar{\Phi}^3 \right)$$

For simplicity let's take $m = \bar{m}$, $\lambda = \bar{\lambda}$

In components (Do IT)

$$S = \int d^4x \, \Big\{ -\varphi \Box \bar{\varphi} - 2i \psi^{\alpha} \partial_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} + \underbrace{F\bar{F}}$$

$$- \underbrace{m \varphi F} + \frac{m}{2} \psi^{\alpha} \psi_{\alpha} - \underbrace{m \bar{\varphi} \bar{F}} + \frac{m}{2} \bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}$$

$$- \underbrace{\frac{\lambda}{2} \varphi^2 F} + \lambda \varphi \psi^{\alpha} \psi_{\alpha} - \underbrace{\frac{\lambda}{2} \bar{\varphi}^2 \bar{F}} + \lambda \bar{\varphi} \bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} \Big\}$$

Go on-shell for $F, \bar{F}$

EOM

$$\frac{\delta S}{\delta F} = 0 \rightarrow \bar{F} = m\varphi + \frac{\lambda}{2} \varphi^2$$

$$\frac{\delta S}{\delta \bar{F}} = 0 \rightarrow F = m\bar{\varphi} + \frac{\lambda}{2} \bar{\varphi}^2$$

algebraic Equations

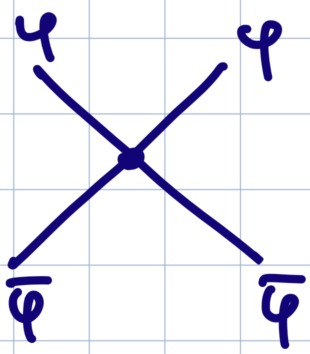
Replace inside the action (DO IT)

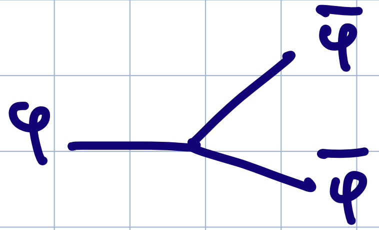
$$S = \int d^4x \Big\{ -\varphi \Box \bar\varphi - 2i\, \psi^\alpha \partial_{\alpha\dot\alpha} \bar\psi^{\dot\alpha}$$

$$- 4m^2 \varphi\bar\varphi + \frac{m}{2} \psi^\alpha \psi_\alpha + \frac{m}{2} \bar\psi_{\dot\alpha} \bar\psi^{\dot\alpha}$$

$$- 2m\lambda \left(\varphi \bar\varphi^2 + \bar\varphi \varphi^2 \right) - \lambda^2 \underbrace{\varphi^2 \bar\varphi^2}_{\text{quartic interaction}}$$

$$+ \underbrace{\lambda \varphi\, \psi^\alpha \psi_\alpha + \lambda \bar\varphi\, \bar\psi_{\dot\alpha} \bar\psi^{\dot\alpha}}_{\text{yukawa couplings}} \Big\}$$





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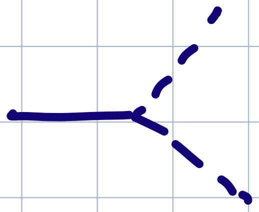


+ h.c.

susy transfs : $\delta\psi_\alpha = -2i\bar{\epsilon}^{\dot\alpha}\partial_{\alpha\dot\alpha}\varphi - \epsilon_\alpha F$

$$\xrightarrow[\text{on shell}]{} -2i\bar{\epsilon}^{\dot\alpha}\partial_{\alpha\dot\alpha}\varphi - \epsilon_\alpha\underbrace{\left(m\bar\varphi + \frac{\lambda}{2}\bar\varphi^2\right)}_{\uparrow}$$

On-shell susy transfs are <u>no-longer</u> <u>linear</u>

Quantisation in superspace

Define the generating functional

$$W[J,\bar J] = \int[\mathcal{D}\Phi\mathcal{D}\bar\Phi]\, e^{iS[\Phi,\bar\Phi]+i\int d^6z\, J\Phi + i\int d^6\bar z\, \bar J\bar\Phi}$$

$$\bar D_{\dot\alpha}J=0\ ,\quad D_\alpha\bar J=0$$

1) $S[\phi,\bar\phi] = \int \phi\bar\phi \qquad (m=0)$

$\wr$

$$W_0[J,\bar J] = e^{-\int d^8z\; J\,\frac{D^2\bar D^2}{\Box}\,\bar J} \qquad \Leftarrow \qquad (3)$$

Suggestion: $\int d^6z\; J\phi = -\int d^8z\; J\,\frac{D^2}{\Box}\,\phi$

$\Uparrow$

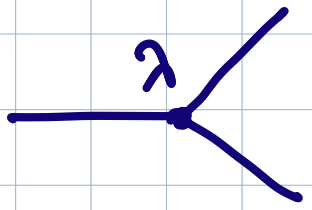
$$\int d^8z\; J\,\frac{D^2}{\Box}\,\phi = \int d^6z\; \bar D^2\left(J\,\frac{D^2}{\Box}\,\phi\right)$$

$$= \int d^6z\; J\,\frac{\overbrace{\{\bar D^2, D^2\}}^{-\Box}}{\Box}\,\phi$$

From (3) we can read the propagator

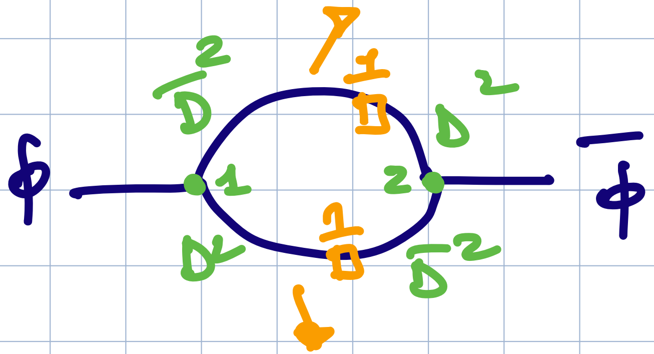
$$\langle \phi \bar\phi \rangle = \frac{\bar D^2 D^2}{\Box} \rightarrow \langle \phi(z), \bar\phi(z') \rangle = \frac{D^2 \bar D^2}{\Box}\,\delta^{(8)}(z-z')$$

Interaction vertex



$$\frac{\lambda}{3!}\phi^3$$

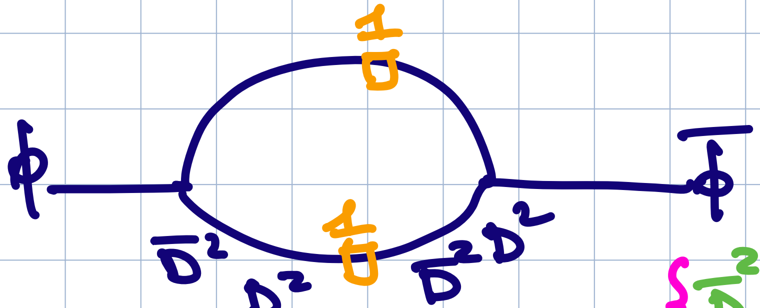
$$\delta^{(4)}(\theta_1-\theta_2) = (\theta_1-\theta_2)^2(\bar\theta_1-\bar\theta_2)^2$$



SUPERGRAPH

$$\delta^{(4)}(\theta_1-\theta_2) \quad \delta^{(4)}(x_1-x_2)$$

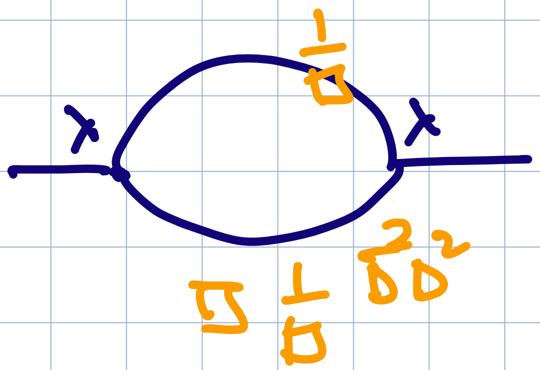
$=$



$$\underbrace{\qquad\qquad}_{} = \{\bar D^2_1 D^2\}\bar D^2 D^2\,\delta^{(8)}(z_1-z_2)$$

$-\Box$

$=$



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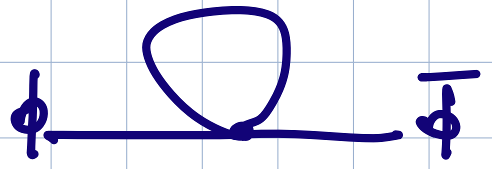


ordinary Feynman integral

Perturbatively, one can show that whatever is the supergraph we start with, UV divergences are always proportional to a complete (non-chiral) integral –

$$\text{UV diverg.} \sim \int \underline{d^8 z}\, f(\Phi, \bar{\Phi}) \times \begin{pmatrix}\text{div.}\\ \text{Feynman}\\ \text{integral}\end{pmatrix}$$

↑
This is due to the way D-algebra works

$\Longrightarrow$ they only correct the Kahler part $\int \Phi \bar{\Phi}$ while the superpotential part is left untouched.

$\Longleftrightarrow$ Only the Kahler term undergoes non-trivial renormalization.

Superpotential DOES NOT renormalise

↑

NON-RENORM. THEOREM
FOR THE SUPERPOTENTIAL

D-algebra provides a perturbative proof of this theorem

Seiberg's proof (Weinberg, III volume)

Consider WZ model

$$W(\phi) = \underbrace{\frac{m}{2}}_{}\phi^2 + \underbrace{\frac{\lambda}{3!}}_{}\phi^3 \qquad \overline{W}(\overline{\Phi})$$

promote m, λ to be background <u>chiral</u> superfields

1) **Holomorphy**: $W(\phi, \lambda, m)$

If holomorphy survives at quantum level, W cannot get corrections proportional to $\overline{m}, \overline{\lambda}, \overline{\Phi}$

2) We can assign $U(1)$ charges to ϕ, m, λ

	$U(1)_R$	$U(1)$ (does not act on $\theta, \bar\theta$)
ϕ	1	1
m	0	-2
λ	-1	-3

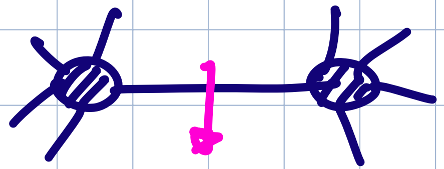
Assumption : these $U(1)$'s are not broken at quantum level

$$W \to W_{eff} = m\phi^2 f\left(\frac{\lambda\phi}{m}\right)$$

$$= m\phi^2 \sum_n a_n \frac{\lambda^n \phi^n}{m^n}$$

$$= \sum_n a_n \lambda^n \phi^{n+2} / m^{n-1} \qquad (6)$$

$\frac{1}{m^2} \to$



$\frac{1}{p^2+m^2} \longrightarrow p^2 \to 0$

From 1PI diagrams we will never obtain UV div. corrections prop. to $\frac{1}{m^p}$ $p>0$

$\Rightarrow$ In (6) $m-1 \leq 0 \rightsquigarrow m = 0$
$\qquad\qquad\qquad\qquad\qquad\qquad = 1$

$$\Rightarrow \quad W_{eff} = \underset{\underset{\frac{1}{2}}{\uparrow}}{a_0}\, m \phi^2 + \underset{\underset{\frac{1}{3!}}{\uparrow}}{a_1}\, \lambda \phi^3$$

the same as the classical superpotential

Conclusion : What is the renormalization pattern of WZ model?

- there are UV div. contributions $\sim \quad \phi \bar{\Phi} \quad \Rightarrow$

ϕ renormalizes nontrivially

$$\boxed{\phi_R = Z_\phi^{1/2} \, \phi}$$

$$W(\phi, m, \lambda) = \underset{W}{W_R}(\phi_R, m_R, \lambda_R)$$

$$\Rightarrow \quad m\phi^2 = m_R \phi_R^2 = m_R Z_\phi \phi^2$$

$\rightsquigarrow$ mass renorm. $\quad \boxed{m_R = Z_\phi^{-1} m}$

$$\lambda \phi^3 \equiv \lambda_R \phi_R^3 = \lambda_R Z_\phi^{3/2} \phi^3$$

↝ coupl. const renorm

$$\boxed{\lambda_R = Z_\phi^{-3/2} \lambda}$$

RUNNING COUPLING CONSTANT

N=1 4D SYM

Given number of chiral superfields Φ^a $a=1,2,\dots\#$ in a repr. of a group $G \to \{T^A\}$

G transf (global) $\Phi^a \to \Phi'^a = \left(e^{i\lambda^A T^A}\right)^a{}_b \Phi^b$

Can we make G-invariance local consistently with susy?

$$[\delta_{susy}, \delta_G(x)] = \begin{cases} \delta'_{susy}(x) \\ 0 \end{cases}$$

In case of δ'_{SUSY} non-trivial $\Rightarrow [\delta'_{SUSY}(x), \delta'_{SUSY}(x)]$

$= \delta_{\text{Diffeomorphisms}}$

$\underbrace{\delta_{SUGRA}}_{\text{no good}}$

$\Downarrow$

Gauge symmetry is consistent with RIGID SUSY only if $[\delta_{SUSY}, \delta_{G(x)}] = 0$

<u>Conclusion</u>: All the components of a given superfield have to belong to the same G-repr-

$\Downarrow$

Consequence: In $N=1$ 4D superspace, matter (fund. rep.) and gauge fields (adj. rep.) Cannot belong to the same multiplet.

If chiral Φ^a describe matter, we have to "invent" another superfield to include gauge fields.

Lecture 6

7/11/25

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① Abelian gauge theory

No matter for the moment

Bosonic case → A_μ with $S = -\frac{1}{4}\int d^4x\, F_{\mu\nu} F^{\mu\nu}$

We look a susy generalization of this action

$$F_{\mu\nu} \rightsquigarrow F_{\alpha\dot\alpha\,\beta\dot\beta} = (\sigma^\mu)_{\alpha\dot\alpha} (\sigma^\nu)_{\beta\dot\beta}\, F_{\mu\nu}$$

$$F_{\alpha\dot\alpha\,\beta\dot\beta} = -F_{\beta\dot\beta\,\alpha\dot\alpha}$$

$$F_{\alpha\dot\alpha\,\beta\dot\beta} = \epsilon_{\dot\alpha\dot\beta}\, f_{\alpha\beta} + \epsilon_{\alpha\beta}\, \bar f_{\dot\alpha\dot\beta} \qquad (1)$$

$$f_{\alpha\beta} = f_{\beta\alpha}$$

$$\bar f_{\dot\alpha\dot\beta} = \bar f_{\dot\beta\dot\alpha}$$

$$\tilde F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = (\sigma_\mu)^{\alpha\dot\alpha}(\sigma_\nu)^{\beta\dot\beta}\, \tilde F_{\alpha\dot\alpha\,\beta\dot\beta}$$

$$\tilde F_{\alpha\dot\alpha\,\beta\dot\beta} = \epsilon_{\dot\alpha\dot\beta}\, f_{\alpha\beta} - \epsilon_{\alpha\beta}\, \bar f_{\dot\alpha\dot\beta} \qquad (2)$$

$$\begin{cases} f_{\alpha\beta} = \text{self dual part of } F_{\mu\nu} \\ \bar f_{\dot\alpha\dot\beta} = \text{anti-self dual part of } F_{\mu\nu} \end{cases}$$

$$S = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu} \underset{\underline{\text{DO IT}}}{=} \int d^4x \left(f^{\alpha\beta} f_{\alpha\beta} + \bar f^{\dot\alpha\dot\beta}\, \bar f_{\dot\alpha\dot\beta} \right)$$

we introduce a new chiral superfield W_α

$\bar{W}_{\dot\alpha}$

satisfying Bianchi's constraint

$$\boxed{D^\alpha W_\alpha - \bar{D}_{\dot\alpha} \bar{W}^{\dot\alpha} = 0}$$

General solution to this constraint –

Remember that there is a general identity

$$D^\alpha \bar{D}^2 D_\alpha = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha}$$

$\Downarrow$

$$\begin{cases} W_\alpha = \bar{D}^2 D_\alpha V \\ \bar{W}_{\dot\alpha} = D^2 \bar{D}_{\dot\alpha} V \end{cases}$$

V = vector multiplet

$$V = c + \theta^\alpha \chi_\alpha + \bar\theta_{\dot\alpha} \bar\chi^{\dot\alpha} - \theta^2 M - \bar\theta^2 \bar M + \theta^\alpha \bar\theta^{\dot\alpha} \overset{\text{vector}}{A_{\alpha\dot\alpha}}$$

$$- \bar\theta^2 \theta^\alpha \lambda_\alpha - \theta^2 \bar\theta_{\dot\alpha} \bar\lambda^{\dot\alpha} + \theta^2 \bar\theta^2 \tilde D \qquad \underline{\text{real}}$$

<u>Main obs</u> : W_α, $\bar W_{\dot\alpha}$ are invariant under

(4) $\quad V \rightarrow V' = V + i(\Lambda - \bar\Lambda)$

$\Lambda, \bar\Lambda \rightarrow$ superfield parameters of $U(1)$ transf

$$\begin{cases} \bar D_{\dot\alpha} \Lambda = 0 \\ D_\alpha \bar\Lambda = 0 \end{cases}$$

$$\Lambda(x, \theta, \bar\theta) = \Lambda_1(x_L) + \theta^\alpha \Lambda_{2\alpha}(x_L) + \theta^2 \Lambda_3(x_L)$$

Λ_1, Λ_3 = complex scalars

$\Lambda_{2\alpha}$ = left Weyl spinor

$$\bar{\Lambda}(x,\theta,\bar{\theta}) = \bar{\Lambda}_1(x_R) + \bar{\theta}_{\dot{\alpha}}\bar{\Lambda}_2^{\dot{\alpha}}(x_R) + \bar{\theta}^2\bar{\Lambda}_3(x_R)$$

$$\begin{cases} \mathrm{Re}\,\Lambda_1 \\ \mathrm{Im}\,\Lambda_1 \\ \mathrm{Re}\,\Lambda_3 \\ \mathrm{Im}\,\Lambda_3 \\ \Lambda_{2\alpha},\ \bar{\Lambda}_{2\dot{\alpha}} \end{cases}$$

too many gauge parameters?

Writing (4) in components (DO IT)

$$\begin{cases} \delta C = i\,\underbrace{(\bar{\Lambda}_1 - \Lambda_1)}_{\mathrm{Im}\,\Lambda_1} \\ \delta\chi_\alpha = -i\,\Lambda_{2\alpha} \\ \delta\bar{\chi}_{\dot{\alpha}} = i\,\bar{\Lambda}_{2\dot{\alpha}} \end{cases}$$

$(\delta C = C' - C)$

SUPERGAUGE TRANSF.

$$\begin{cases} \delta M = -i\Lambda_3 \\ \delta \bar{M} = i\bar{\Lambda}_3 \\ \delta A_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}}\underbrace{(\Lambda_1+\bar{\Lambda}_1)}_{\mathrm{Re}\,\Lambda_1} \\ \delta\lambda_\alpha = 0 \\ \delta\bar{\lambda}_{\dot{\alpha}} = 0 \\ \delta\tilde{D} = 0 \end{cases}$$

WZ GAUGE to set
$$\begin{cases} C=0 \quad \text{using } \mathrm{Im}\,\Lambda_1 & V|=0 \\ \chi_\alpha=\bar{\chi}_{\dot{\alpha}}=0 \quad \text{using } \Lambda_{2\alpha},\ \bar{\Lambda}_{2\dot{\alpha}} & D_\alpha V|=0 \\ & \bar{D}_{\dot{\alpha}} V|=0 \\ M=\bar{M}=0 \quad \text{using } \Lambda_3,\ \bar{\Lambda}_3 & D^2 V|=0 \\ & \bar{D}^2 V|=0 \end{cases}$$

We are left with

$$(A_{\alpha\dot\alpha},\ \lambda_\alpha,\ \bar\lambda_{\dot\alpha},\ \tilde D) \quad + \quad \mathrm{Re}\Lambda_1 \equiv \lambda$$

$$\Big\downarrow$$

$\delta A_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha}\lambda$ usual gauge transf

Caveat: WZ gauge breaks susy

For instance $\delta_{susy}\lambda_\alpha \simeq \bar\epsilon^{\dot\alpha} A_{\alpha\dot\alpha} \neq 0$

$\Rightarrow$ susy transf leads us into something that no longer satisfies WZ gauge conds

therefore, we have to apply a gauge-restoring gauge transf to get back to WZ gauge

In the WZ gauge the action is invariant under modified susy transf

$$\delta^{WZ}_{susy} = \delta_{susy} + \delta^{WZ}_{G-restoring}$$

Still in the WZ gauge → $V \rightsquigarrow (A_{\alpha\dot\alpha}, \lambda_\alpha, \bar\lambda_{\dot\alpha}, \tilde{D})$

therefore we find

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$$

$$W_\alpha \longrightarrow \begin{cases} W_\alpha \big| = \lambda_\alpha \\ D_{(\alpha} W_{\beta)} \big| = \partial_{\alpha\dot\alpha} A^{\dot\alpha}{}_{\beta} + \partial_{\beta\dot\alpha} A^{\dot\alpha}{}_{\alpha} \\ \qquad\qquad \equiv f_{\alpha\beta} \\ D^\alpha W_\alpha \big| = \tilde{D} \end{cases}$$

$$\Big(\quad D^2 W_\alpha \Big) = 2i\,\partial_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}$$

Supergauge action

Dimensions $[A_{\alpha\dot\alpha}] = 1 \implies [V] = 0$

$\implies [W_\alpha] = [\bar W_{\dot\alpha}] = 3/2$

$$\rightsquigarrow \quad S = \int d^4x \underbrace{d^2\theta}_{D^2}\ \underbrace{\tfrac{1}{2} W^\alpha W_\alpha}_{W^2} + \int d^4x \underbrace{d^2\bar\theta}_{\bar D^2}\ \underbrace{\tfrac{1}{2} \bar W_{\dot\alpha} \bar W^{\dot\alpha}}_{\bar W^2}$$

Exercise: write S in components in the WZ gauge

$$S = \int d^4x \Big[\# \overbrace{\Big(f_{\alpha\beta} f^{\alpha\beta} + \bar f_{\dot\alpha\dot\beta} \bar f^{\dot\alpha\dot\beta} \Big)}^{F_{\mu\nu}F^{\mu\nu}}$$

$$+ \; \# \; \underbrace{\lambda^{\alpha} \partial_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}}_{\substack{\text{dynamical}\\ \text{gaugino}}} \; + \; \# \; \underset{\substack{\uparrow\\ \text{auxiliary}\\ \text{real scalar}}}{\tilde{D}^{2}} \; \Big]$$

Obs: If we consider

$$\operatorname{Im} S = \int W^2 - \int \bar{W}^2 \underset{\underline{\text{DO IT}}}{=} \int d^4x \; \partial\text{-total derivative}$$

with trivial b.c. $\Longrightarrow \int \bar{W}^2 = \int W^2$

therefore $S = \int d^4x \, d^2\theta \, W^2$

with nontrivial b.c. $\sim$ introduce a complex coupling $\tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g^2}$

and consider

$$S = \mathrm{Im}\left\{ \tau \int W^2 \right\}$$

$$= \frac{4\pi}{g^2}\,\mathrm{Re}\int W^2 + \underbrace{\frac{\theta}{2\pi}\,\mathrm{Im}\int W^2}_{\text{instantonic action}}$$

Obs: Is $S = \int d^4x\, d^2\theta\; W^2$ a chiral integral?

$$S = \int d^4x\, d^2\theta\; \frac{1}{2} W^\alpha W_\alpha = \frac{1}{2}\int d^4x\, d^2\theta\; \underline{\bar{D}^2}\left(D^\alpha V \,.\, \bar{D}^2 D_\alpha V \right)$$

$$= \frac{1}{2}\int d^4x\, d^2\theta\, d^2\bar{\theta}\; D^\alpha V \,.\, \bar{D}^2 D_\alpha V$$

It is NOT a chiral integral

Coupling to matter

Take a chiral superfield Φ ($\bar{\Phi}$) describing matter charged under U(1)

$$\begin{cases} \Phi \to \Phi' = e^{i\Lambda}\Phi & \bar{D}_{\dot{\alpha}}\Lambda = 0 \\ \bar{\Phi} \to \bar{\Phi}' = \bar{\Phi}\, e^{-i\bar{\Lambda}} & D_{\alpha}\bar{\Lambda} = 0 \end{cases}$$

SUPERGAUGE TRANSF.s $\Lambda = \Lambda(x,\theta,\bar{\theta})$

$\bar{\Lambda} = \bar{\Lambda}(x,\theta,\bar{\theta})$

$$S_{kin} = \int d^8z\, \Phi\bar{\Phi} \to \int d^8z\, \Phi'\bar{\Phi}' = \int e^{i\Lambda}\Phi\bar{\Phi}\, e^{-i\bar{\Lambda}}$$

$$\neq \int \Phi\bar{\Phi}$$

⊕ V (prepotential) s.t. under $U(1)$ supergauge transfs

$$V \to V' = V + i(\bar\Lambda - \Lambda)$$

$$e^V \to e^{V'} = e^{V + i(\bar\Lambda - \Lambda)}$$

$$= e^{i\bar\Lambda} e^V e^{-i\Lambda}$$

$$\boxed{e^V \to e^{V'} = e^{i\bar\Lambda} e^V e^{-i\Lambda}}$$

Consider $\int d^8z \, \underbrace{\bar\Phi e^V \Phi}$ invariant

MINIMAL COUPLING IN SUPERSPACE

$$\bar\Phi e^V \Phi \to \bar\Phi' e^{V'} \Phi' = \bar\Phi \cancel{e^{-i\bar\Lambda}} \cancel{e^{i\bar\Lambda}} e^V \cancel{e^{-i\Lambda}} \cancel{e^{i\Lambda}} \Phi$$

U(1) - supergauge invariant action is

$$S_{tot} = \int d^8 z \; \underbrace{\bar\Phi e^V \Phi}_{} \; + \; \frac{1}{g^2}\int d^6 z \; W^2$$

$$\bar\Phi\Phi + \bar\Phi V \Phi + \frac{1}{2}\bar\Phi V^2 \Phi + \dots\dots$$

$$= \bar\Phi\Phi + \underbrace{\bar\Phi(e^V - 1)\Phi}_{\text{interaction part}}$$

Summarising

① matter charged under G

$$\Phi' = e^{i\Lambda}\Phi \qquad \bar D\Lambda = 0$$

$$\bar\Phi' = \bar\Phi e^{-i\bar\Lambda} \qquad D\bar\Lambda = 0$$

② Introduce a gauge superfield $$e^{V} \to e^{V'} = e^{+i\bar{\Lambda}} e^{V} e^{-i\Lambda}$$

③ Corresponding field strength $$W_\alpha = \bar{D}^2 D_\alpha V = \bar{D}^2 \left(e^{-V} D_\alpha e^{V} \right)$$
$$\bar{W}_{\dot{\alpha}} = D^2 \bar{D}_{\dot{\alpha}} V = -D^2 \left(e^{V} \bar{D}_{\dot{\alpha}} e^{-V} \right)$$

④ $$S_{TOT} = \int d^8 z \, \bar{\Phi} e^{V} \Phi + \frac{1}{2g^2} \int d^6 z \, W^\alpha W_\alpha$$

2) Non-abelian case

We have a set of chiral superfields Φ^a $a = 1, 2 \ldots \#$

We can take $G = SU(N)$ and ϕ^a in fund. rep.

$(a = 1, \dots N)$

① $$\phi^a \to \phi'^a = \left(e^{i\Lambda^A(x,\theta,\bar\theta)T^A}\right)^a{}_b \, \phi^b$$

$$\bar\Phi_a \to \bar\Phi'_a = \bar\Phi_b \left(e^{-i\bar\Lambda^A(x,\theta,\bar\theta)T^A}\right)^b{}_a$$

$$\begin{cases} \bar D \Lambda^A = 0 \\ D \bar\Lambda^A = 0 \end{cases}$$

② Gauge superfields $\to V^A \quad \sim \quad e^{V^A T^A} \equiv e^V$

$$e^V \to e^{V'} = e^{i\bar\Lambda} e^V e^{-i\Lambda}$$

③ Field strengths $\quad W_\alpha = \bar{D}^2 \left(e^{-V} D_\alpha e^{V} \right)$

$$\bar{W}_{\dot\alpha} = D^2 \left(e^{V} \bar{D}_{\dot\alpha} e^{-V} \right)$$

they are covariant under gauge transfs

$$W_\alpha \to W'_\alpha = e^{i\Lambda} \, W_\alpha \, e^{-i\Lambda}$$

$$\bar{W}_{\dot\alpha} \to \bar{W}_{\dot\alpha} = e^{i\bar\Lambda} \, \bar{W}_{\dot\alpha} \, e^{-i\bar\Lambda}$$

(covariant under G-transfs)

④ $$S = \int d^8z \, \underbrace{\bar\Phi \, e^{V} \, \Phi}_{\equiv \; \bar\Phi_a \left(e^{V}\right)^a{}_b \, \Phi^b} + \frac{1}{2g^2} \int d^6z \, \mathrm{Tr}\left(W^\alpha W_\alpha \right)$$

WZ gauge : $V^A\big| = 0 \qquad \forall A = 1, \dots N^2-1$

$$D_\alpha V^A\big| = 0 = \bar{D}_{\dot\alpha} V^A\big|$$

$$D^2 V^A\big| = 0 = \bar{D}^2 V^A\big|$$

Gauge covariant derivatives

$D_A \equiv (D_\alpha, \bar{D}_{\dot\alpha}, \partial_{\alpha\dot\alpha})$ covariant respect to SUSY transfs.

We want to construct derivatives that are covariant respect to G-transfs : ∇_A

$\phi \rightarrow \phi' = e^{i\Lambda}\phi \qquad (\nabla_A\phi)' = e^{i\Lambda}\nabla_A\phi$

<u>Solution</u> : ∇_A should then transform as

$$\nabla_A \to \nabla_A' = e^{i\Lambda} \nabla_A e^{-i\Lambda}$$

$$\begin{cases} \nabla_\alpha = e^{-V} D_\alpha e^{V} \\ \bar\nabla_{\dot\alpha} = \bar D_{\dot\alpha} \\ \nabla_{\alpha\dot\alpha} = \frac{i}{2}\{\nabla_\alpha, \bar\nabla_{\dot\alpha}\} \end{cases}$$

$$\nabla_\alpha \phi \equiv e^{-V} D_\alpha \left(e^{V}\phi\right)$$

$$[\nabla_A, \nabla_B] = T_{AB}{}^{C} \nabla_C - i F_{AB}$$

$$\Bigg\downarrow \quad \underline{\text{DO IT}}$$

Torsion $\quad T_{\alpha\dot\alpha}{}^{(\beta\dot\beta)} = -2i\, \delta_\alpha^{\beta}\, \delta_{\dot\alpha}^{\dot\beta}$

Curvature : $F_{\alpha\beta} = 0 \qquad F_{\dot\alpha\dot\beta} = 0 \qquad F_{\alpha\dot\beta} = 0$

$$F_{\alpha,\beta\dot\beta} = \frac{1}{2}\,\varepsilon_{\alpha\beta}\left(e^{-V}\,\overline{W}_{\dot\beta}\,e^{V}\right)$$

$$F_{\dot\alpha,\beta\dot\beta} = \frac{1}{2}\,\varepsilon_{\dot\alpha\dot\beta}\,W_{\beta}$$

$$F_{\alpha\dot\alpha,\beta\dot\beta} = \frac{1}{4}\left(\varepsilon_{\dot\alpha\dot\beta}\,\underbrace{\nabla_{(\alpha}W_{\beta)}}_{f_{\alpha\beta}} + \varepsilon_{\alpha\beta}\,\underbrace{\overline{\nabla}_{(\dot\alpha}\overline{W}_{\dot\beta)}}_{\bar{f}_{\dot\alpha\dot\beta}}\right)$$

RENORMALIZATION PROPERTIES OF SYM

1) <u>N= 1</u> $\rightsquigarrow$ Chiral + Gauge Vector superfields

the most general action ($V \to gV$)

$$S = \frac{1}{2g^2}\int d^6z\, W^\alpha W_\alpha + \int d^8z\, \bar{\Phi}_a \left(e^{gV}\right)^a{}_b \Phi^b$$

$$+ \int d^6z\, \underbrace{P(\Phi)}_{\text{gauge invariant}} + \text{h.c.}$$

$$\underbrace{a_{a_1 \cdots a_n}}_{G\text{-singlet}}\, \phi^{a_1} \cdots \phi^{a_n}$$

If we have also a U(1) gauge invariance → V term we can add an extra term

<u>Fayet-Iliopoulos term</u>

$$S_{FI} = \xi \int d^8z\, V$$

$\hookrightarrow V \to V' = V + i(\bar{\Lambda} - \Lambda)$

We quantise the system (functional quant.)

Propagators

- scalar $\int \bar{\Phi}\Phi \rightsquigarrow \boxed{\langle \Phi\bar{\Phi}\rangle = \frac{D^2\bar{D}^2}{\Box}}$

- Vector

$$\frac{1}{2g^2}\int d^6z\, W^\alpha W_\alpha = \frac{1}{2g^2}\int d^8z \left(e^{-gV}D^\alpha e^{gV}\right)\bar{D}^2\left(e^{-gV}D_\alpha e^{gV}\right)$$

$$\xrightarrow[\text{quadratic part}]{} \frac{1}{2\cancel{g^2}}\cancel{g^2}\int d^8z\, D^\alpha V\,\bar{D}^2 D_\alpha V$$

$$= -\frac{1}{2}\int d^8z\, V D^\alpha \bar{D}^2 D_\alpha V$$

kinetic term for the vector

But $D^\alpha \bar{D}^2 D_\alpha$ is not invertible

$$D^\alpha \bar{D}^2 D_\alpha \underbrace{\left(D^2 \bar{D}^2 V + \bar{D}^2 D^2 V\right)}_{\text{nontrivial kernel}} = 0$$

We choose the superspace <u>gauge-fixing</u>

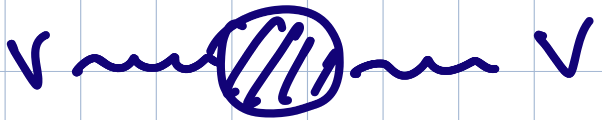
$$D^2 V = 0, \qquad \bar{D}^2 V = 0$$

we add $\quad S_{gf} = 2\int d^8 z \, D^2 V \, \bar{D}^2 V$

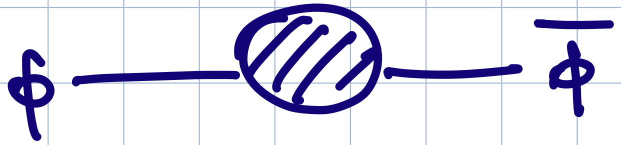
$$\int d^8 z \, V D^\alpha \bar{D}^2 D_\alpha V + \int d^8 z \, V \left(D^2 \bar{D}^2 + \bar{D}^2 D^2\right) V$$

$$= \int d^8z \, V \underbrace{\left(D^\alpha \bar{D}^2 D_\alpha + \{ D^2 , \bar{D}^2 \} \right)}_{\substack{||| \\ -\Box}} V$$

$$= - \int d^8z \, V \Box V \implies \boxed{\langle V V \rangle = \frac{1}{\Box}}$$



$\rightsquigarrow$ UV div's



$\rightsquigarrow$ UV div's

We need renormalize the theory

We have 2 Non-ren. theorems

1) $P(\Phi) \to$ superpotential does not renormalise

2) gV does not renormalise

Parameters : $\phi^a \,,\; V \,,\; \underset{\substack{\uparrow \\ \text{gauge} \\ \text{coupling}}}{g} \,,\; \underbrace{m \,,\; \lambda}_{\text{in } P(\phi)}$

$$\begin{cases} \phi^a_R = Z_\phi^{1/2}\, \phi^a \\ V_R = Z_V^{1/2}\, V \\ g_R = Z_g\, g \end{cases}$$

$$m_R = Z_m\, m$$

$$\lambda_R = Z_\lambda\, \lambda$$

↙ cubic int.

NR-1 $\Rightarrow$ $$Z_\lambda Z_\phi^{3/2} = 1$$ $$Z_m Z_\phi = 1$$ $\Bigg\} \rightarrow$ Z_m, Z_λ determined by Z_ϕ

NR-2 $\Rightarrow$ $$gV = g_R V_R \Leftrightarrow Z_g Z_V^{1/2} = 1 \Rightarrow Z_g$$ not indep.

$\leadsto$ Only 2 independent ren. functions Z_ϕ, Z_V

$\Rightarrow$ we will find non-trivial β_g and β_λ

We can have asymptotic freedom if ϕ^a
$a = 1, \dots p$

$$\underline{\underline{p \leq 2}}$$

2) N=2 SYM

$\rightsquigarrow$ it has SU(2) R-symmetry

Two kinds of multiplets

- Vector multiplet $\rightarrow \left(1, \frac{1}{2}, \frac{1}{2}, 0, 0\right)$
- Hypermultiplets $\rightarrow \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right)$

We do realise these repr's in terms of N=1 superfields

- Vector multiplet = N=1 vector superfield V

$$\oplus$$

N=1 chiral superf. Φ

both in adj repr. of G

$$\begin{array}{llcl} V \longrightarrow & 1 \ldots & \multirow{2}{*}{} & \ldots \underset{\text{SU(2) singlet}}{A_{ai}} \ldots \\ \Phi \longrightarrow & \underset{\text{SU(2) singlet}}{\varphi} & & \ldots \end{array} \qquad \underset{\text{SU(2) doublet}}{\begin{pmatrix} \lambda_\alpha \\ \psi_\alpha \end{pmatrix}}$$

- Hyper = Chiral superfield Q_1

$$\oplus$$

Chiral superfield $Q_2^\dagger$

both in the same G-rep.

$$\begin{array}{ll} Q_1 \longrightarrow \ldots \\ Q_2^\dagger \longrightarrow \ldots \end{array} \underset{\text{SU(2) doublet}}{\begin{pmatrix} q_{1\alpha} \\ q_{2\alpha}^\dagger \end{pmatrix}}$$

$$S_{N=2} = \frac{1}{2g^2}\int d^6z\, W^\alpha W_\alpha + \frac{1}{g^2}\int d^8z\, e^{-gV}\bar{\Phi} e^{gV}\Phi$$

$$+\int d^8z\, Q^i e^{gV}\bar{Q}_i + \boxed{\tilde{g}}\int d^6z\, Q_i \underset{\uparrow}{\Phi^i{}_j}\, Q^j + h.c.$$

$$\Phi = \Phi^A T^A$$

N=2 imposes : 1) $\tilde{g} = g$

2) No superpotential terms

Renormalisation:

Parameters	V	Φ	Q_1	Q_2	g
	Z_V	Z_ϕ	Z_1	Z_2	Z_g

- **NR-2** $\rightarrow$ g^V does not run $\Longrightarrow$ $Z_g \sim Z_V^{-1/2}$

- $N=2$ inv. $\rightarrow$ $\underline{Z_V = Z_\phi}$

- **NR-1** $\rightarrow$ $\underbrace{Z_g}_{Z_V^{-1/2}} Z_\phi^{1/2} \, Z_{Q_1}^{1/2} \, Z_{Q_2}^{1/2} = 1$

$$\left(\cancelto{1}{Z_\phi Z_V^{-1}}\right)^{1/2} Z_{Q_1}^{1/2} \, Z_{Q_2}^{1/2} = 1$$

- $SU(2)$ inv. $\rightarrow$ $Z_{Q_1} = Z_{Q_2}$

Conclusione $Z_{Q_1} = Z_{Q_2} = 1$

We are left with only 1 ren. funct. → Z_V

$\Rightarrow \quad Z_g = Z_V^{-1/2} \quad \rightsquigarrow \quad \beta_g$

<u>But</u>: Pert. calc's in superspace reveal that UV div's occur only at 1 loop ⇒ β_g is 1L exact

3) <u>N=4 SYM</u>

In this case we have <u>ONLY ONE</u> superfield that contains
$$\begin{cases} 1 \text{ vector field} \\ 4 \text{ fermions} \\ 6 \text{ real scalars} \end{cases}$$ → fund. of

$SO(6) \sim SU(4)$

(R-symmetry)

In terms of $N=1$ superfields –

Construct 3 complex scalars

$$\varphi_1 = q_1 + i q_2 \qquad \bar{\varphi}_1 = q_1 - i q_2$$

$$\varphi_2 = q_3 + i q_4$$

$$\varphi_3 = q_5 + i q_6$$

they are in fund repr.

of $SU(3) \subset SU(4)$

(R-symmetry)

$$\text{N=4 vector Mult.} \;\Big|\!\longrightarrow\; \underbrace{\begin{array}{c}1\ \ \text{N=1 vector superfields}\\ \oplus\\ 3\ \ \text{chiral superfields}\end{array}}_{\text{they are all in the adjoint of the gauge group}}$$

$$S = \frac{1}{g^2}\,\mathrm{Tr}\Big\{ \int d^6z\, \frac{1}{2} W^\alpha W_\alpha + \int d^8 z\, e^{-gV}\bar{\Phi}_a\, e^{gV}\Phi^a$$

$$+ \frac{g}{3!}\int d^6z\; i f_{abc}\, \Phi^a\, [\Phi^b, \Phi^c]$$

$$+ \frac{g}{3!}\int d^6\bar{z}\; i f_{abc}\, \bar{\Phi}_a\, [\bar{\Phi}_b, \bar{\Phi}_c] \Big\}$$

Renormalization

Parameters $\phi_1 \quad \phi_2 \quad \phi_3 \quad v \quad g$

$Z_1 \quad Z_2 \quad Z_3 \quad Z_v \quad Z_g$

- **NR-2** $\Longrightarrow \quad Z_g = Z_v^{-1/2}$
- **NR-1** $\Longrightarrow \quad Z_g \, Z_1^{1/2} Z_2^{1/2} \, Z_3^{1/2} = 1$
- N=4 SUSY $\Longrightarrow \quad Z_1 = Z_2 = Z_3 = Z_v$

$$\Rightarrow \quad Z_v = Z_1 = Z_2 = Z_3 = Z_g = 1$$

no renormalization

$\underline{\mathcal{N}=4 \text{ is finite}} \iff \beta_g = 0 \iff$ SUPERCONFORMAL INVARIANCE

Superconf algebra in 4D $= su(2,2|4)$ (superalgebra)

$$su(2,2|4) \supset \underbrace{su(2,4)}_{\text{conf algebra in 4D}}$$

Bosonic generators $\rightarrow$ $\begin{cases} \text{Poincaré} \\ D \quad \text{(dilatation)} \\ K_\mu \quad \text{(special conf. transf)} \end{cases}$

Fermionic generators $\rightarrow$ $\begin{cases} Q^i_\alpha \quad \bar{Q}_i^{\dot{\alpha}} \quad i=1,2,3,4 \\ S^i_\alpha \quad \bar{S}_i^{\dot{\alpha}} \quad \text{special} \end{cases}$

SUSY Trans/3

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"Introductory lectures on Higher-Spin theories"