Lectures on Gravitational Waves

Geoffrey Compère

Solvay Doctoral School
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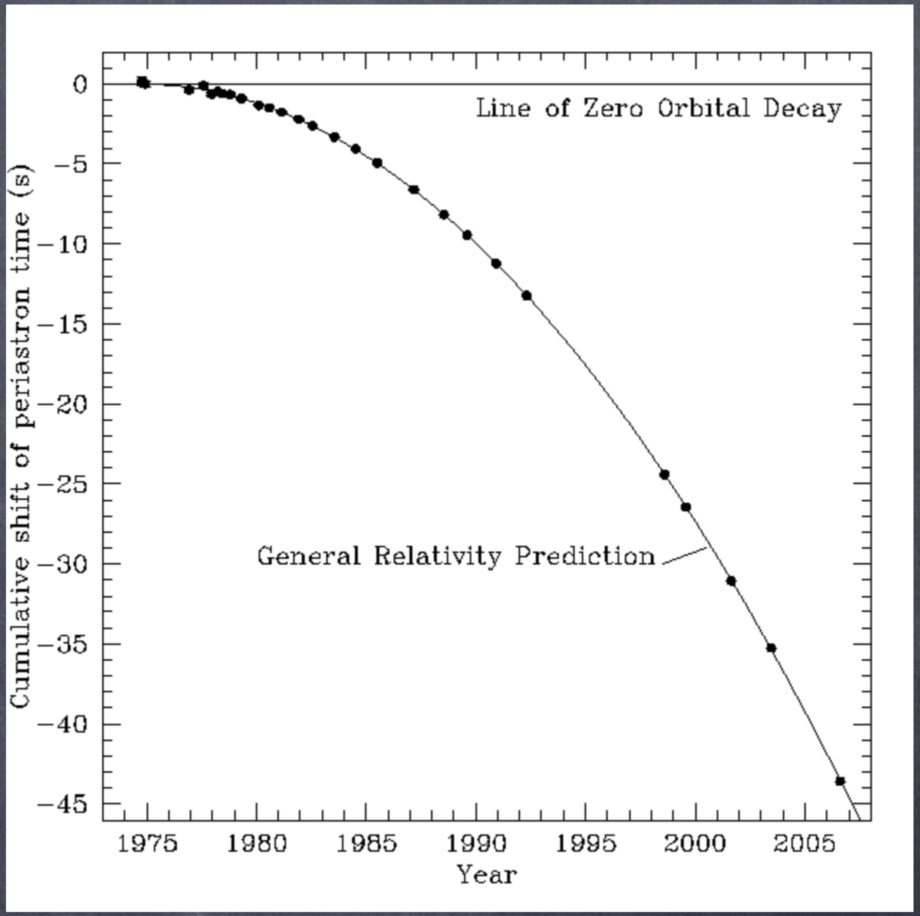
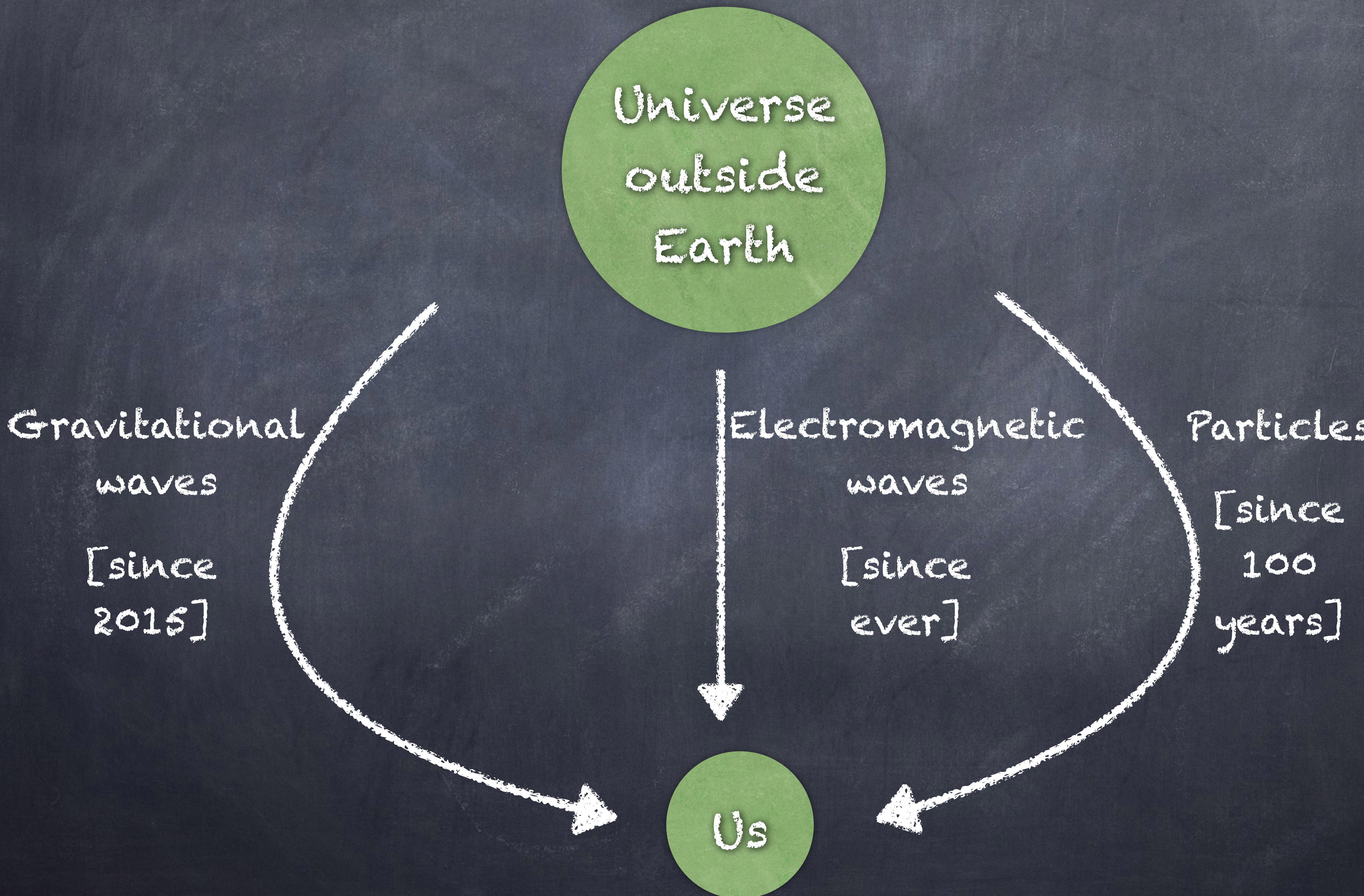
Exciting times...

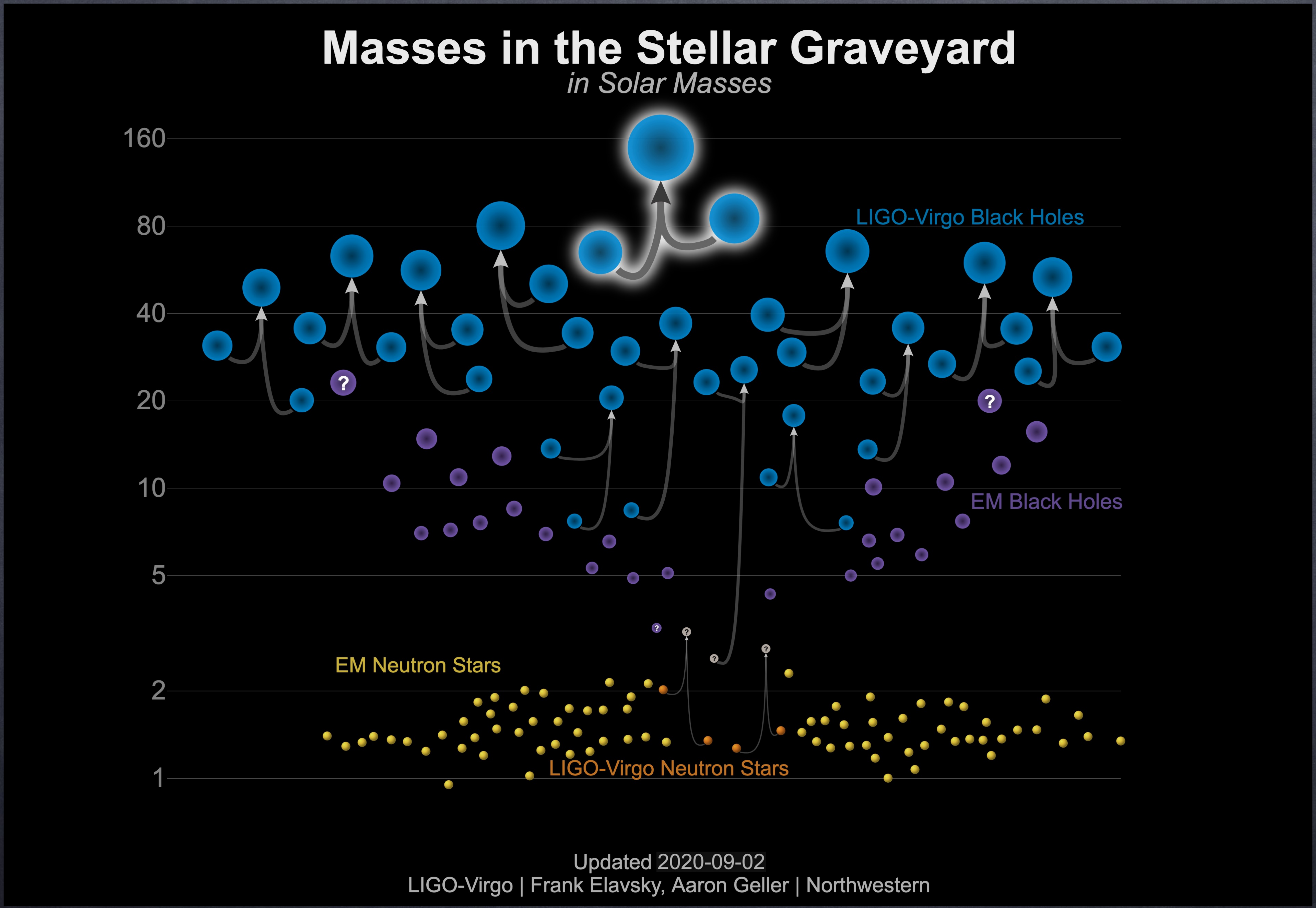
- 1915 Birth of General Relativity and in 1916: prediction of Gravitational Waves
- 1974 Hulse-Taylor pulsar : First indirect detection of GW



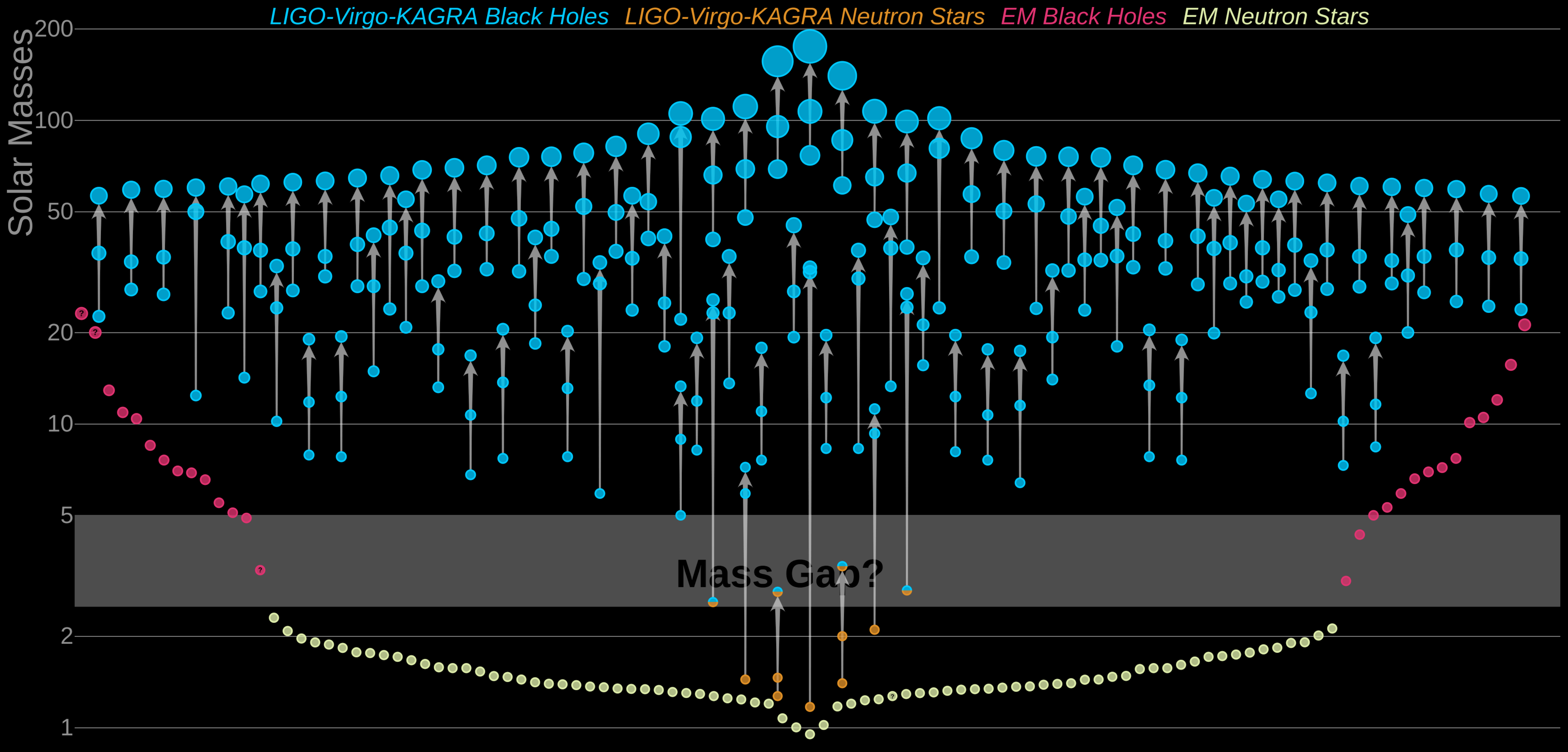
- 2015 First direct detection of binary black hole merger: event GW150914
- 2017 First direct multi-messenger detection of binary neutron star merger: event GW170817
- March 2023 Run 04 of adLigo-adVirgo-Kagra

The three messengers



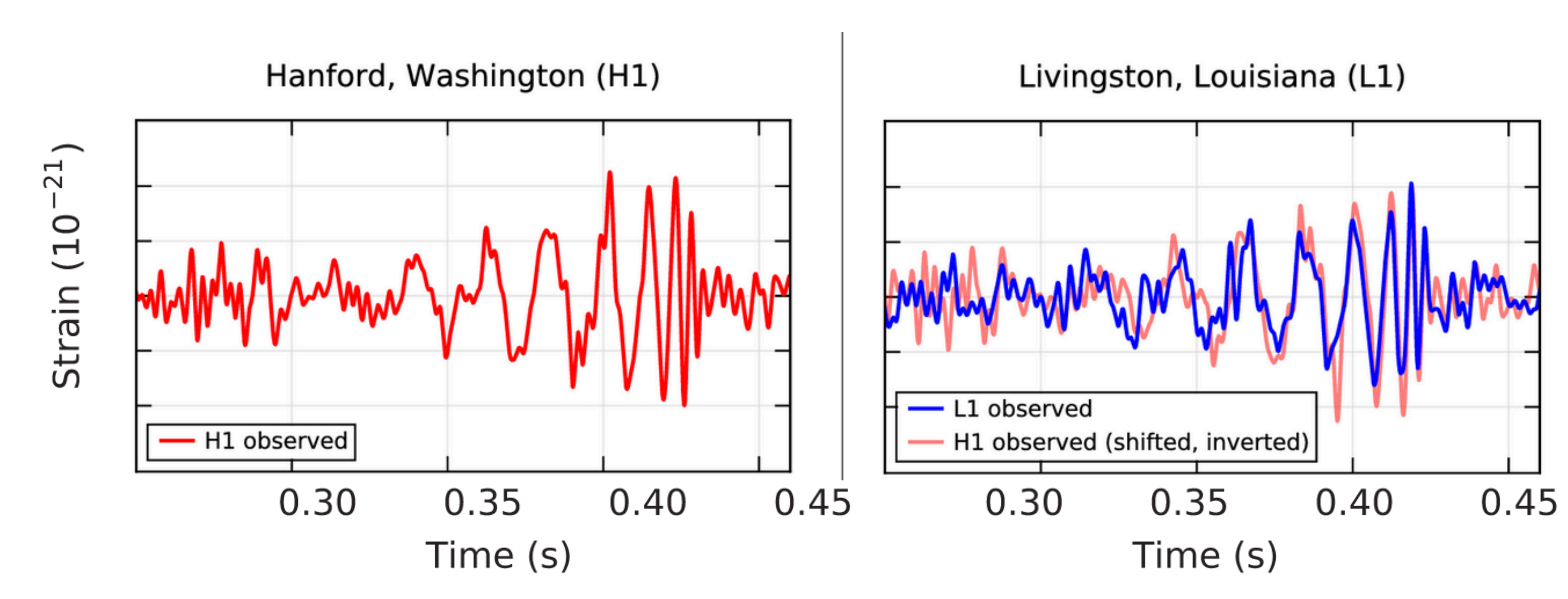

Masses in the Stellar Graveyard
in Solar Masses
160
80
40
20
10
5
2
1
LIGO-Virgo Black Holes
EM Black Holes
EM Neutron Stars
LIGO-Virgo Neutron Stars
Updated 2020-09-02
LIGO-Virgo | Frank Elavsky, Aaron Geller | Northwestern

Masses in the Stellar Graveyard



LIGO-Virgo-KAGRA | Aaron Geller | Northwestern

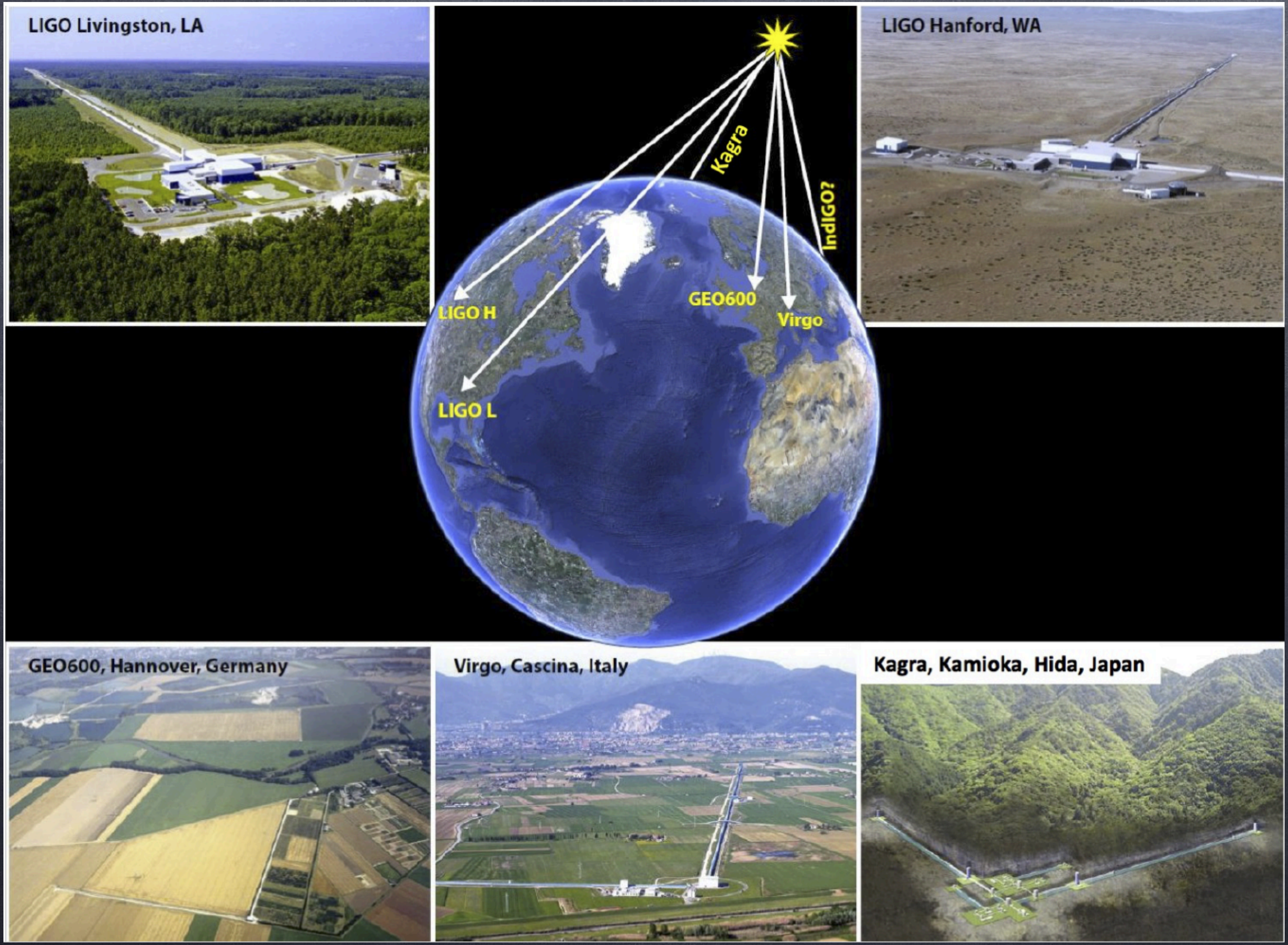
GW150914



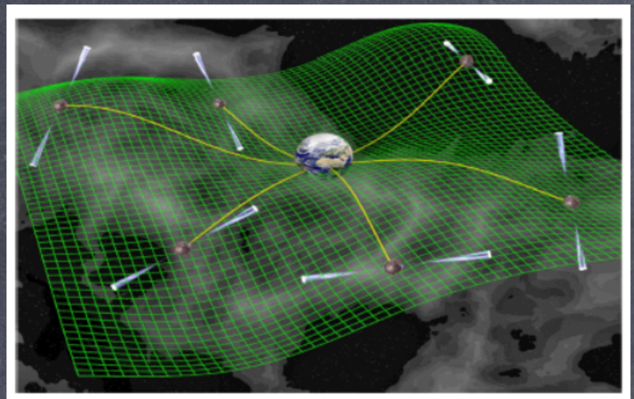
Estimated source parameters

Quantity	Value	Upper/Lower error estimate	Unit
Primary black hole mass	**36.2**	+5.2 -3.8	M sun
Secondary black hole mass	**29.1**	+3.7 -4.4	M sun
Final black hole mass	**62.3**	+3.7 -3.1	M sun
Final black hole spin	**0.68**	+0.05 -0.06	
Luminosity distance	**420**	+150 -180	Mpc
Source redshift, *z*	**0.09**	+0.03 -0.04	
Energy radiated	**3.0**	+0.5 -0.5	M sun

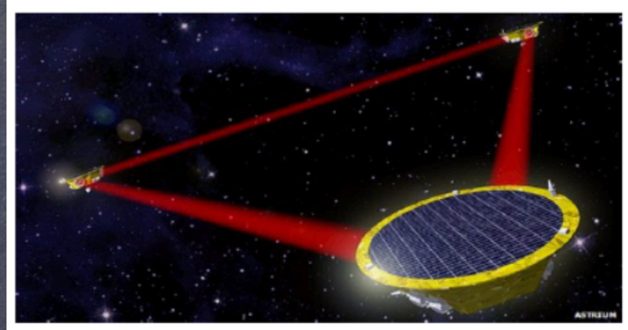
Current GW detectors



Upcoming detectors

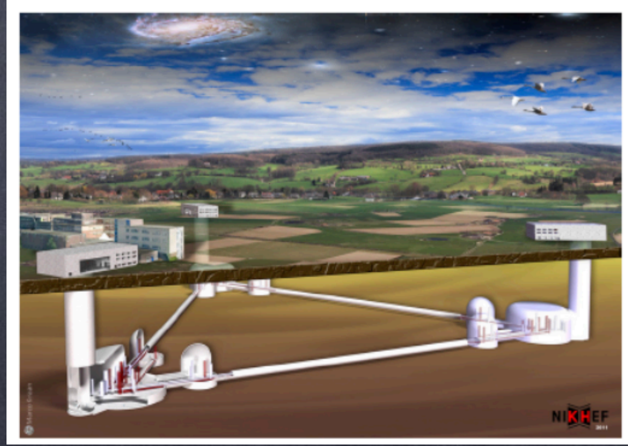


Pulsar Timing Arrays



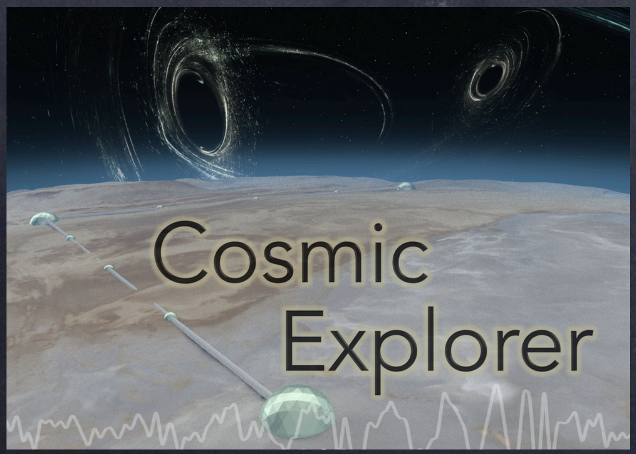
LISA

May 2022: Mission feasibility passed
2037?: Science



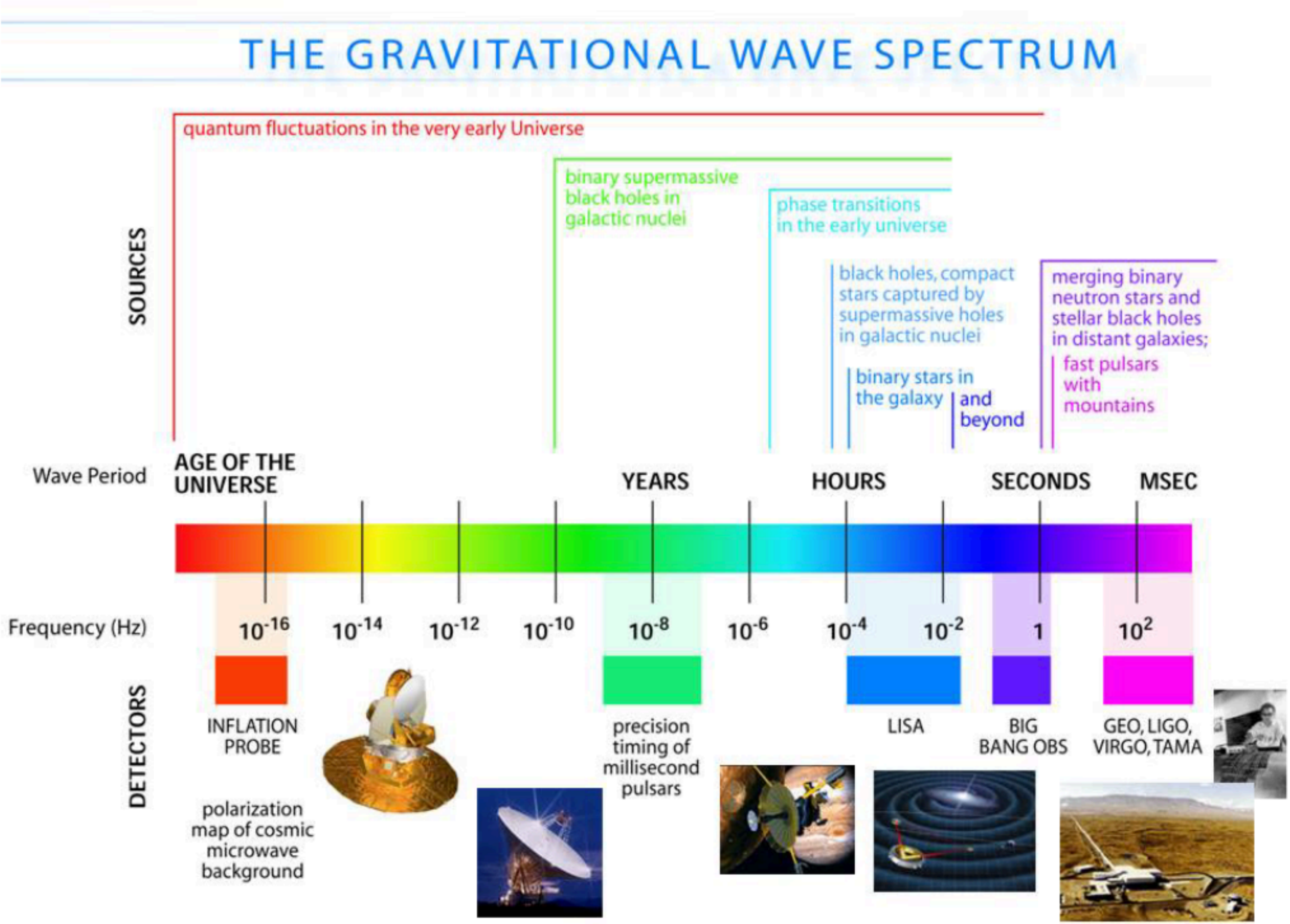
Einstein Telescope

2024: Choice of site
2035?: Science

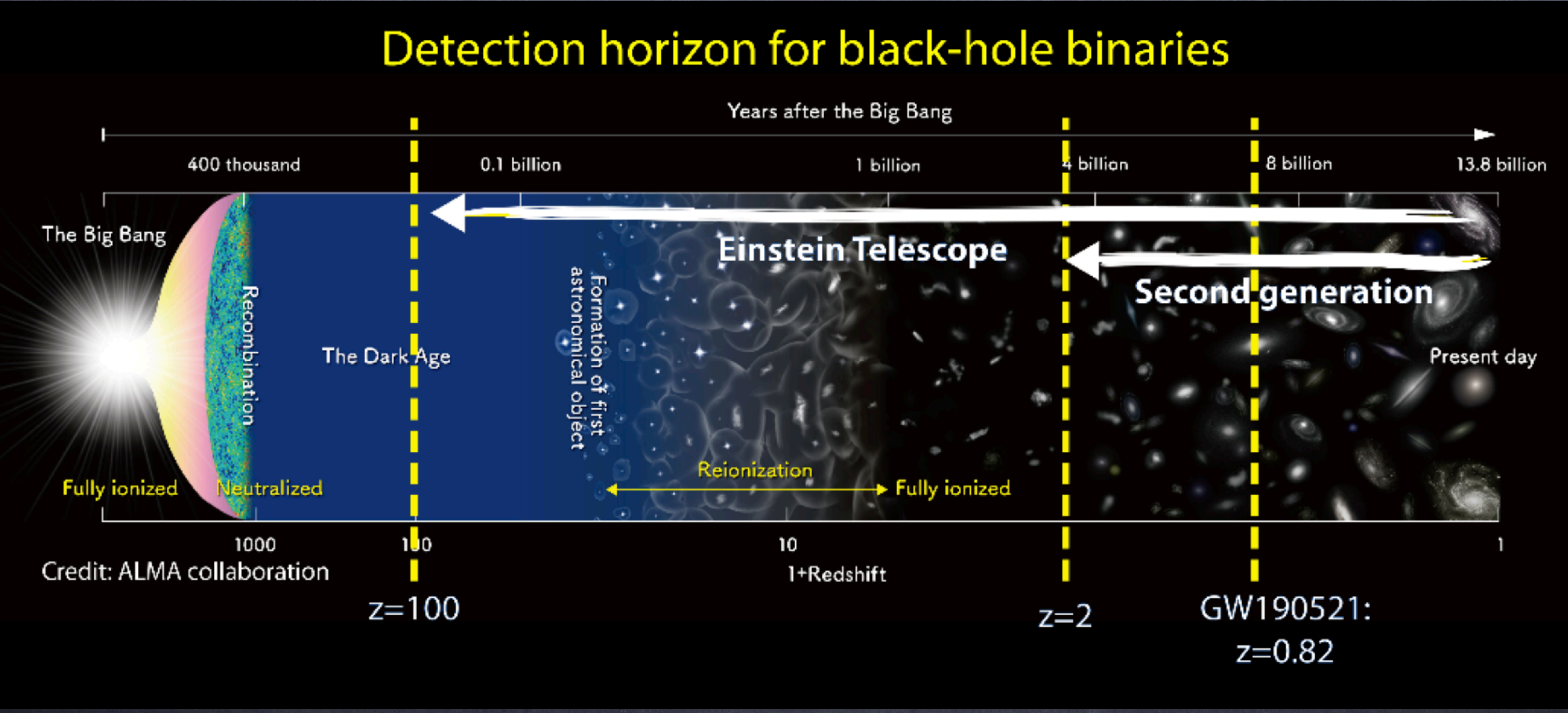


Cosmic Explorer

2035?: Science


THE GRAVITATIONAL WAVE SPECTRUM
SOURCES
quantum fluctuations in the very early Universe
binary supermassive black holes in galactic nuclei
phase transitions in the early universe
black holes, compact stars captured by supermassive holes in galactic nuclei
binary stars in the galaxy
and beyond
merging binary neutron stars and stellar black holes in distant galaxies;
fast pulsars with mountains
Wave Period
AGE OF THE UNIVERSE
YEARS
HOURS
SECONDS
MSEC
Frequency (Hz)
10^{-16}
10^{-14}
10^{-12}
10^{-10}
10^{-8}
10^{-6}
10^{-4}
10^{-2}
1
10^{2}
DETECTORS
INFLATION PROBE
polarization map of cosmic microwave background
precision timing of millisecond pulsars
LISA
BIG BANG OBS
GEO, LIGO, VIRGO, TAMA

3G: Deep in the dark age



LISA Science objectives

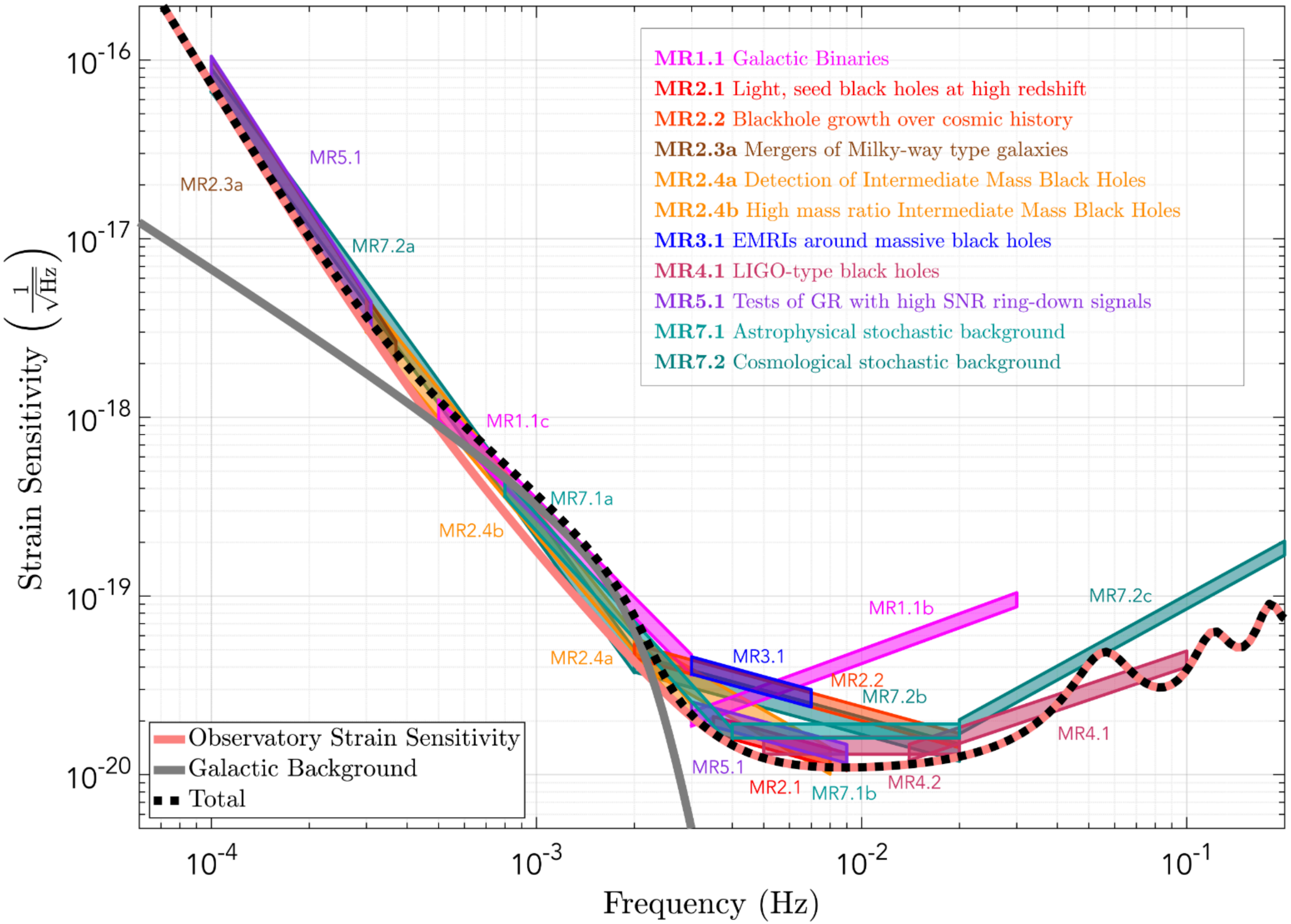


Figure 2: Mission constraints on the sky-averaged strain sensitivity of the observatory for a 2-arm configuration (TDI X), $\sqrt{S_h(f)}$, derived from the threshold systems of each observational requirement.

GW and string theory

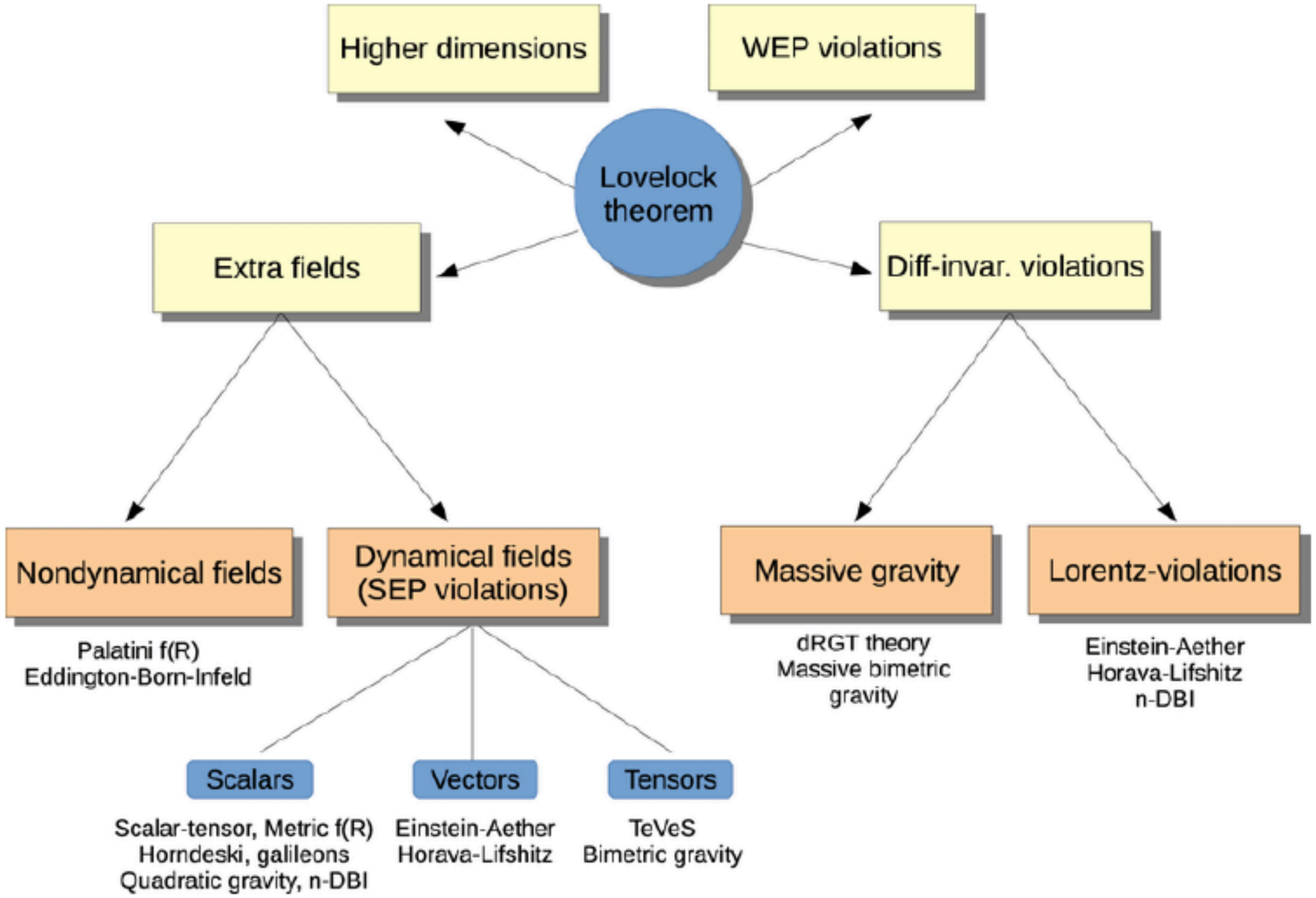
- (Huge) Gap between Planck scale and observations
- Effective field theory approach: GR+suppressed modifications
- Current and future observations: Strong field GR (no quantum, yet?)

Modifications of GR

- Lovelock's theorem:

  "In four spacetime dimensions the only divergence-free symmetric rank-2 tensor constructed solely from the metric $g_{\mu\nu}$ and its derivatives up to second differential order, and preserving diffeomorphism invariance, is the Einstein tensor plus a cosmological term."

- Relaxing one or more of the assumptions allows for a plethora of alternative theories:



Berti et al., CQG **32**, 243001 (2015)

Diffeomorphic invariant higher curvature corrections start at O(R^3)

Leading order in the curvature expansion:
the Einstein-Hilbert action

$$S_\Lambda + S_{\rm EH} = \Lambda \int d^4x \sqrt{|g|} + \frac{1}{16\pi G} \int d^4x \sqrt{|g|} R \,.$$

$$\tilde{R}^{\mu\nu\rho\sigma} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} R_{\alpha\beta}{}^{\rho\sigma} \,.$$

Four terms at O(R^2): (one is parity-odd)

$$S_{(2)} = \frac{\ell^2}{16\pi G} \int d^4x \sqrt{|g|} \left(b_1 R^2 + b_2 R_{\mu\nu} R^{\mu\nu} + b_3 R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} \right) + b_4 \tilde{\mathcal{X}}_4$$

But two combinations are topological (Gauss-Bonnet and Pontryagin):

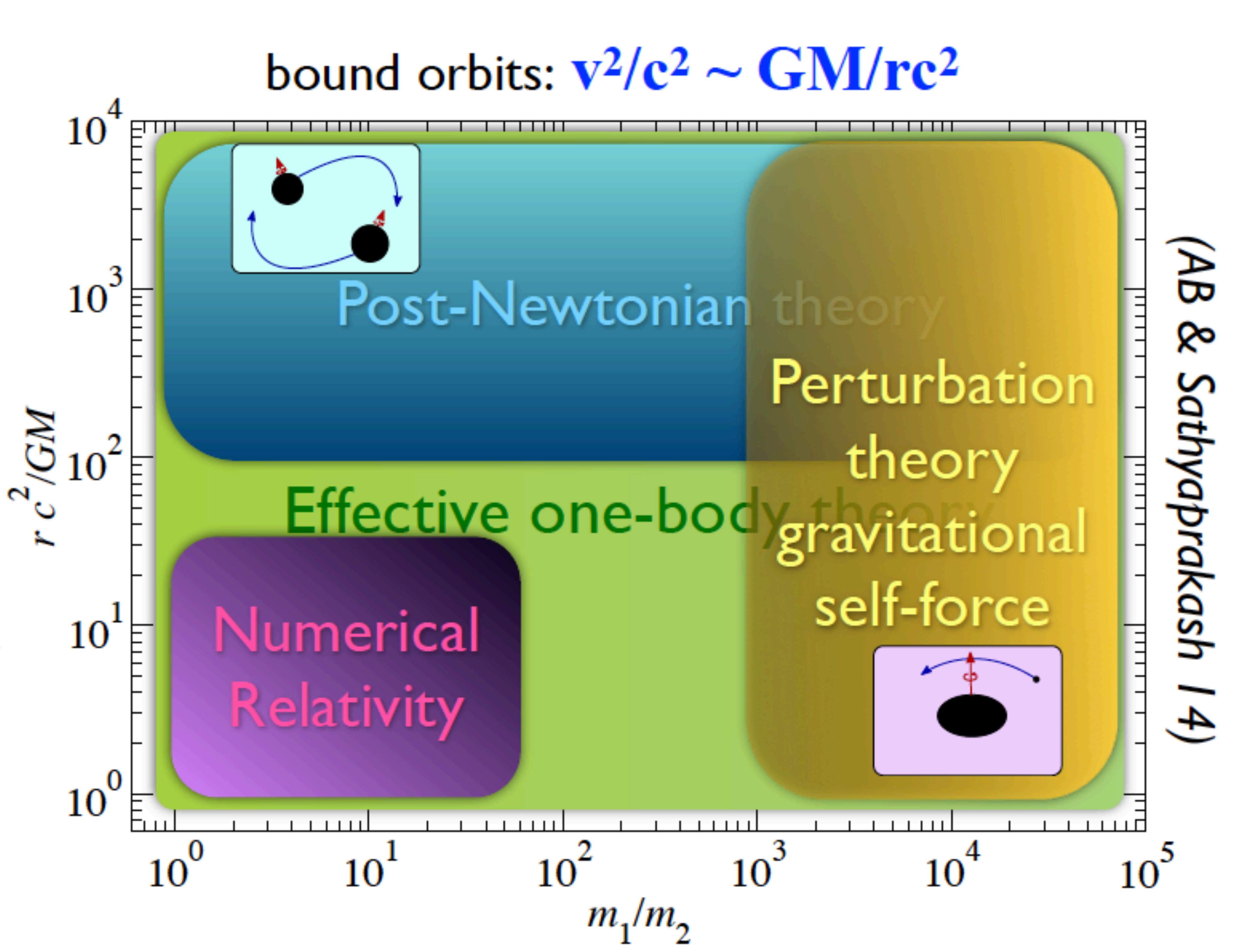
$$\mathcal{X}_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \,, \qquad \tilde{\mathcal{X}}_4 = R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma} \,.$$

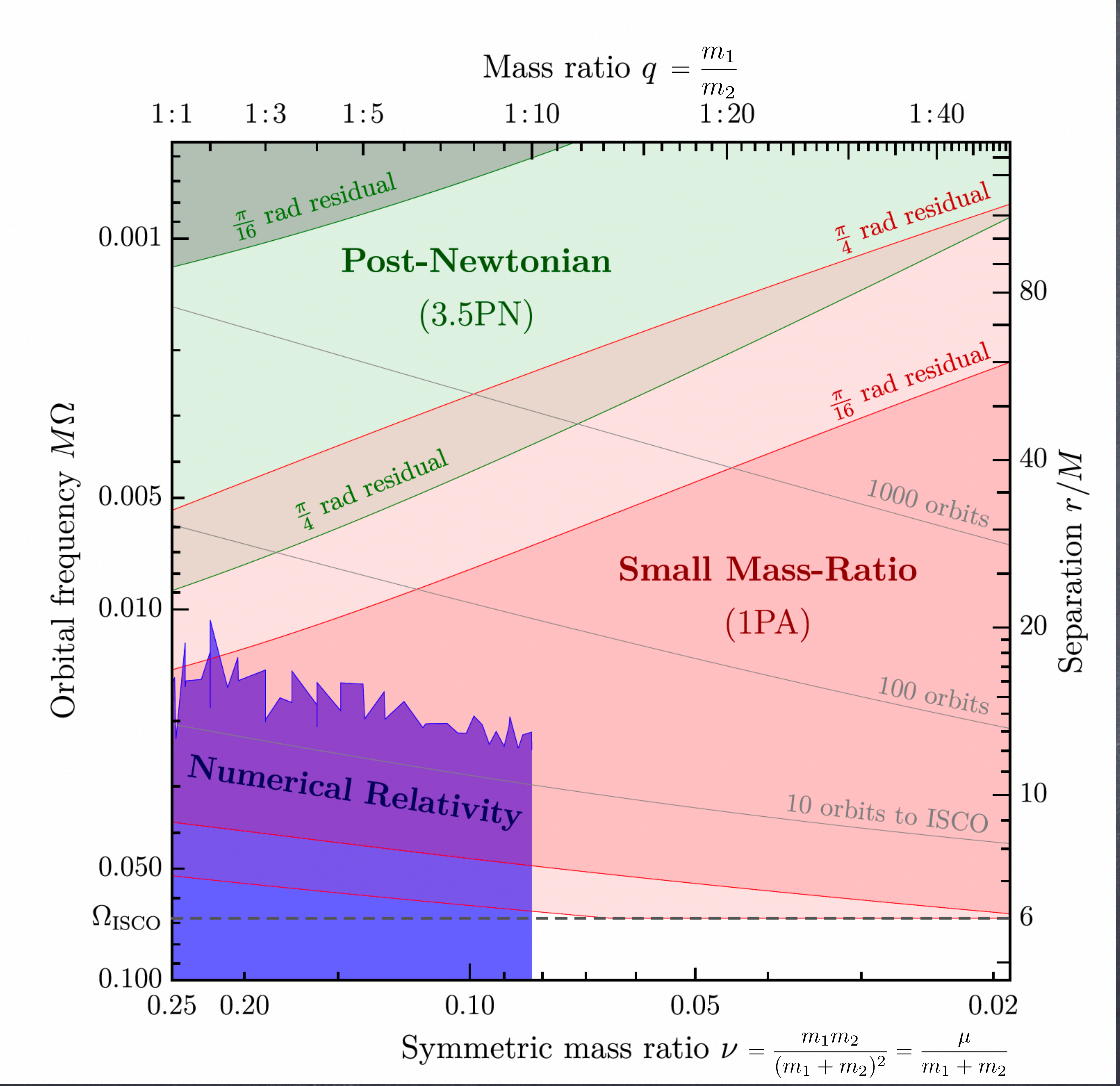
The other two parameters can be absorbed by a field-redefinition of the metric:

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + Q_{\mu\nu} \,, \qquad \mathcal{E}^{(0)}_{\mu\nu} = -\frac{1}{\sqrt{|g|}} \frac{\delta S^{(0)}}{\delta g^{\mu\nu}} \,,$$

$$\sqrt{|g|} \mathcal{L}^{(0)}[g] = \sqrt{|\tilde{g}|} (\mathcal{L}^{(0)}[\tilde{g}] + \mathcal{E}^{(0)}_{\mu\nu} Q^{\mu\nu}) + \mathcal{O}(Q^2) \,.$$

These lectures:
Two-body problem in GR





[van de Meent, Pfeiffer, 2020]

Outline of these lectures

1. Post-Newtonian/Post-Minkowksian theory
    1.1. Propagation, linear GR theory
    1.2. Interaction with test masses, freely falling frame, TT and detector frames, energy of GW
    1.3. Generation of GW
    1.4. Quasi-circular inspiral of compact binaries
    1.5. Post-Newtonian corrections, the 1PN Einstein-Infeld-Hoffmann equations
    1.6. Effective one-body resummation
2. Black hole Perturbation theory
    2.1. Regge-Wheeler and Zerilli equations
    2.2. Quasi-normal modes of Schwarzschild - Black Hole Spectroscopy
    2.3. Newman-Penrose formalism, Petrov's classification, Teukolsky equation
    2.4. Quasi-normal modes of Kerr
    2.5. Mathisson-Papapetrou-Dixon theory

References

1. M. Maggiore, Gravitational Waves, Volume 1. Sections 1, 3 and 4.
2. L. Blanchet, Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries, 1310.1528
3. N. Straumann, General Relativity
4. N. Deruelle, Mini-course on the 2-body problem in general relativity and scalar-tensor theories of gravity, ULB 2020.
5. L. Barack and A. Pound, Self-force and radiation reaction in general relativity, 1805.10385
6. A. Harte, Motion in classical field theories and the foundations of the self-force problem, 1405.5077
7. E. Berti, V. Cardoso, A. Starinets, 0905.2975
8. G. Compère, A. Fiorucci, 1801.07064

1.1. Propagation, linear GR theory

General Relativity

$$S = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}R[g] + S_M$$

↑

Minimal coupling

$$\eta_{\mu\nu} \to g_{\mu\nu}$$
$$\partial_\mu \to D_\mu$$

$$T^{\mu\nu} = \frac{2c}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}}$$

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

Diffeomorphism invariance

$$x^\mu \to x'^\mu(x)$$

$$g'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}(x(x'))$$

Perturbation theory

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad\qquad |h_{\mu\nu}| \ll 1$$

Residual gauge transformations:

$$x^\mu \mapsto x'^\mu = x^\mu + \xi^\mu(x) \qquad\qquad |\partial_\mu \xi_\nu| \ll 1$$

$$h_{\mu\nu}(x) \mapsto h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

Symmetries:

$$\xi_\mu = b_\mu$$

$$\xi_\mu = a_{[\mu\nu]} x^\nu$$

→ $h_{\mu\nu}$ invariant under Poincaré transformations

Linearized Riemann

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho})$$

Exercise: Prove it!

We define

$$h \equiv \eta^{\mu\nu} h_{\mu\nu}$$

Trace-reversed perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

Note

$$\bar{h} \equiv \eta^{\mu\nu} \bar{h}_{\mu\nu} = h - 2h = -h$$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} \bar{h}$$

Exercise

Prove

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\nu\rho} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho})$$

Remember

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma})$$

$$R^\alpha_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\beta\delta} + \Gamma^\alpha_{\epsilon\gamma}\Gamma^\epsilon_{\beta\delta} - (\gamma \leftrightarrow \delta)$$

board

Linearized Einstein

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - \partial^\rho\partial_\nu\bar{h}_{\mu\rho} - \partial^\rho\partial_\mu\bar{h}_{\nu\rho} = -\frac{16\pi G}{c^4}T_{\mu\nu}$$

We can use the gauge freedom to go to
harmonic / de Donder gauge / Lorenz gauge

$$\partial^\mu\bar{h}_{\mu\nu} = 0$$

Indeed, under $x^\mu \mapsto x'^\mu = x^\mu + \xi^\mu(x)$

$$\bar{h}_{\mu\nu} \mapsto \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho)$$

$$\partial^\nu\bar{h}_{\mu\nu} \mapsto \partial^\nu\bar{h}_{\mu\nu} - \Box\xi_\mu$$

If originally $\partial^\nu\bar{h}_{\mu\nu} = f_\mu(x)$ we need to solve $\Box\xi_\mu = f_\mu(x)$

to reach harmonic gauge. This is always possible.

In harmonic gauge

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Acting with ∂^ν we get by consistency $\partial^\nu T_{\mu\nu} = 0$

- This equation is important both for generation of GW and for propagation

Outside sources,

$$\Box \bar{h}_{\mu\nu} = 0 = (-\frac{1}{c^2}\partial_t^2 + \nabla^2)\bar{h}_{\mu\nu}$$

→ **GW propagate at light speed**

TT gauge

For ξ^μ harmonic $\Box\xi^\mu = 0$ then $\xi_{\mu\nu} \equiv \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho$ is harmonic.

→ Out of the 10 components of $\bar{h}_{\mu\nu}$ only 2 propagate as a wave.

We choose ξ^0 such that $\bar{h} = 0$

Proof: $tr(\bar{h}_{\mu\nu} + \xi_{\mu\nu}) = \bar{h} + 2\partial_\alpha\xi^\alpha - 4\partial_\alpha\xi^\alpha$

Solve the ODE $\partial_0\xi^0 + \partial_i\xi^i = \frac{1}{2}\bar{h}$ for $\xi^0(x^0, x^i)$

Residual gauge parameter: $\xi^0(x^i)$ harmonic

It implies

$$\bar{h}_{\mu\nu} = h_{\mu\nu}$$

TT gauge

For ξ^μ harmonic $\Box\xi^\mu = 0$ then $\xi_{\mu\nu} \equiv \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho$ is harmonic.

→ Out of the 10 components of $\bar{h}_{\mu\nu}$ only 2 propagate as a wave.

We choose ξ^i such that $h^{0i} = 0$

Proof: $$h_{0i} + \xi_{0i} = h_{0i} + \partial_0\xi_i + \partial_i\xi_0$$

Solve the ODE $\partial_0\xi^i = -h_{0i} - \partial_i\xi_0$ for $\xi^i(x^0, x^j)$

Residual gauge parameter: $\xi^i(x^j)$ harmonic

It implies

$$h_{0i} = 0$$

Harmonic gauge reduces to

$$\mu = 0 \rightarrow \partial^0 h_{00} + \partial^i h_{0i} = \partial^0 h_{00} = 0 \quad \longrightarrow \quad h_{00}(\vec{x})$$

$$\mu = i \rightarrow \partial^j h_{ij} = 0$$

Newtonian potential
that is irrelevant for propagation
We set it to zero

$$h_{0\mu} = 0, \qquad h_{ii} = 0, \qquad \partial^i h_{ij} = 0$$

We reached TT gauge. Note that it cannot be reached inside of sources because $\Box \bar{h}^{\mu\nu} \neq 0$

We write a decomposition in plane waves in terms of the polarisation tensor

$$h_{ij}^{TT}(x) = e_{ij}(\vec{k}) e^{i\vec{k}\cdot\vec{x} - i\omega t}$$

$$k^\mu = (\frac{\omega}{c}, \vec{k}), \qquad \frac{\omega}{c} = |\vec{k}|, \qquad \hat{n} = \frac{\vec{k}}{|k|}$$

Polarized GW

Choose $\hat{n}$ along the z axis

$$h_{ij}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - \frac{z}{c})]$$

This gives the linear solution

$$ds^2 = -c^2dt^2 + dz^2 + (1 + h_+ \cos[\omega(t - \frac{z}{c})])dx^2$$
$$+(1 - h_+ \cos[\omega(t - \frac{z}{c})])dy^2 + 2h_\times \cos[\omega(t - \frac{z}{c})]dxdy$$

TT Projector

Given a plane wave $h_{\mu\nu}(x)$ propagating along the direction $\hat{n}$ already in harmonic gauge but not yet in TT gauge, we reach TT gauge as

$$h_{ij}^{TT} = \Lambda_{ij,kl} h_{kl}$$

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \qquad P_{ij} \equiv \delta_{ij} - n_i n_j$$

Exercise

Prove the following properties of

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \qquad P_{ij} \equiv \delta_{ij} - n_i n_j$$

(i) projector $\Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}$

(ii) transverse $n^i \Lambda_{ij,kl} = 0 = n^k \Lambda_{ij,kl}$

(iii) traceless $\Lambda_{ii,kl} = 0 = \Lambda_{ij,kk}$

(iv) harmonic $\Box h_{ij} = 0 = \partial^k \bar{h}_{kl} \Rightarrow \Box(\Lambda_{ij,kl}h_{kl}) = 0$

$$\partial^j(\Lambda_{ij,kl}h_{kl}) = 0$$

board

Exercise

Prove the following properties of

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \qquad P_{ij} \equiv \delta_{ij} - n_i n_j$$

(i) projector $\quad \Lambda_{ij,kl}\Lambda_{kl,mn} = \Lambda_{ij,mn}$

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(iii) traceless $\quad \Lambda_{ii,kl} = 0 = \Lambda_{ij,kk}$

(iv) harmonic $\quad \Box h_{ij} = 0 = \partial^k \bar{h}_{kl} \Rightarrow \Box(\Lambda_{ij,kl}h_{kl}) = 0$

$$\partial^j(\Lambda_{ij,kl}h_{kl}) = 0$$

Hints:

$P_{ii} = 2$

$\Box n_i = -2n_i$

$\partial_a n_i \partial_b n_i = \gamma_{ab} \qquad \partial_i = n_i\partial_r + \partial_i\theta^a\partial_a$

$\partial_i\theta^a\partial_a n_j = P_{ij}$

$n_i P_{ij} = 0$

board

1.2. Interaction with test masses, freely falling frame, TT and detector frames, energy of GW

Interaction of GW with test masses

- Detectors idealized as test masses
- Computations done in a reference frame. Physics invariant under the choice
- GW are simple in TT gauge. It corresponds to a specific reference frame/observer
- Detector is more intuitive in another frame, the detector proper frame
- We need to switch between the two frames
- Physical intuition comes from the geodesic deviation equation

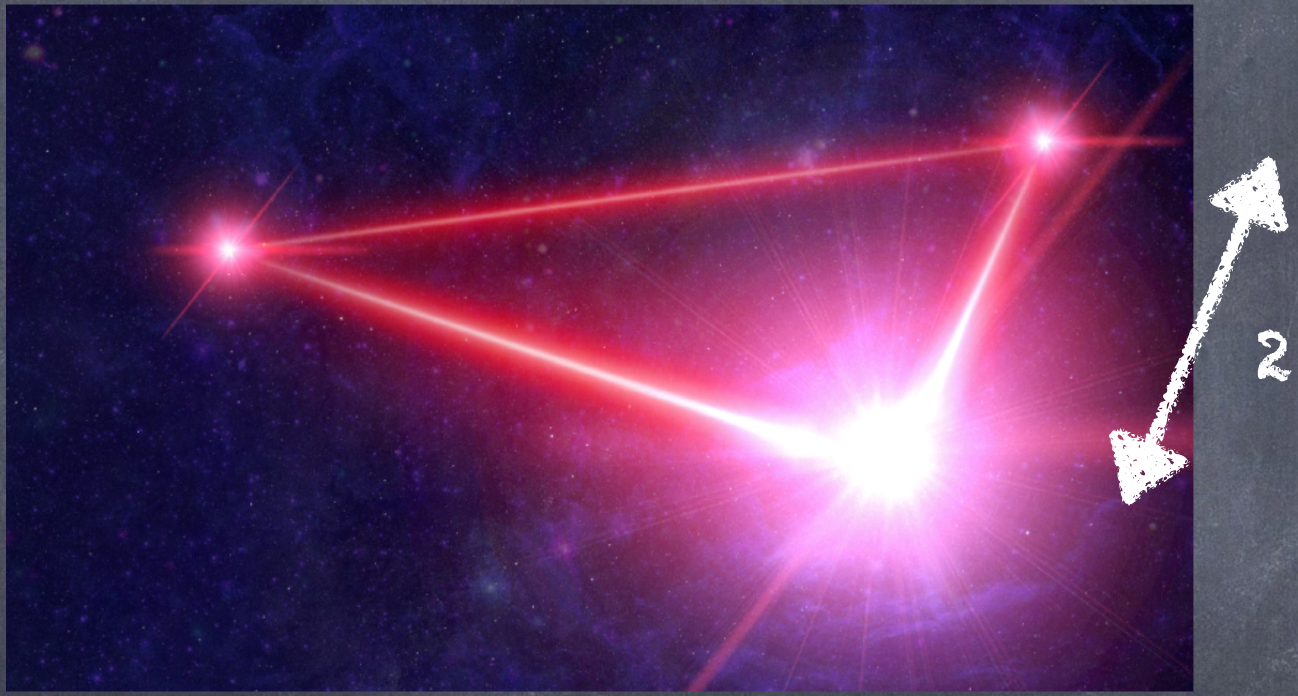
Local inertial frame

- It is always possible to choose coordinates such that the Christoffel symbols vanish at one point
- The resulting coordinate system around that point is called a local inertial frame

Freely falling frame

- It is always possible to choose coordinates such that on an entire timeline geodesic, all Christoffel symbols vanish
- For each point P in this frame, the geodesic equation becomes
$$\frac{d^2 x^\mu}{d\tau^2}|_P = 0.$$
- In this frame, a test mass is freely falling. It gives a realization of the equivalence principle.

Laser Interferometer Space Antenna (LISA)



3 drag-free satellites:
Spacecrafts that adjust their position with thrusters
in order to remain centred about a freely floating mass

To be launched in 2034

Fermi normal coordinates

A freely falling frame naturally define a special coordinate system around the test mass located at (t,0,0,0).
It is called a Fermi normal coordinate system.

Since

$$\Gamma^{\alpha}_{\beta\gamma}|_{\gamma} = 0$$

the metric around the test mass has no linear term in x^i

There are quadratic terms proportional to the Riemann tensor,

$$ds^2 \approx -c^2 dt^2[1 + R_{0i0j}x^i x^j] - 2c dx^i dt(\frac{2}{3}R_{0jik}x^j x^k)$$
$$+dx^i dx^j[\delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l]$$

Geodesic equation

Consider a curve $x^\mu(\lambda)$

The interval ds between 2 points separated by $d\lambda$ is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} d\lambda^2$$

The velocity is $u^\mu = \dfrac{dx^\mu}{d\lambda}$

For a timelike curve the proper time τ is defined as

$$c^2 d\tau^2 = -ds^2 = -g_{\mu\nu} u^\mu u^\nu d\lambda^2$$

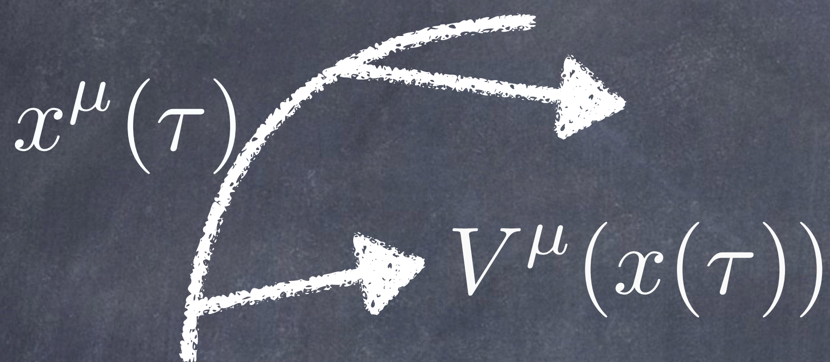
or after using $\lambda = \tau$ $\quad g_{\mu\nu} u^\mu u^\nu = -c^2$

The classical trajectory of a particle test mass m is obtained by extremizing the action

$$S = -m \int_{\tau_i}^{\tau_f} d\tau \longrightarrow \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x) u^\nu u^\rho = 0$$

Parallel transport

Consider a geodesics with proper time τ



We introduce the covariant derivative along the curve $x^\mu(\tau)$

$$\frac{DV^\mu}{D\tau} \equiv \frac{\partial V^\mu}{\partial \tau} + \Gamma^\mu_{\nu\rho} V^\nu \frac{dx^\rho}{d\tau}$$

Property: $\dfrac{DV^\mu}{D\tau}$ transforms as a vector.

The vector is parallely transported along the curve.

Geodesic deviation

Now consider 2 geodesics, each one with proper time τ

$$\frac{d^2(x^\mu+\xi^\mu)}{d\tau^2}+\Gamma^\mu_{\nu\rho}(x+\xi)\frac{d(x^\nu+\xi^\nu)}{d\tau}\frac{d(x^\rho+\xi^\rho)}{d\tau}=0$$

If $|\xi^\mu|$ is negligible with respect to the variation of the gravitational field, we can expand at first order in $\xi^\mu(\tau)$

$$\frac{d^2\xi^\mu}{d\tau^2}+2\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau}+\xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}=0$$

Exercise

Prove: $$\frac{D^2\xi^\mu}{D\tau^2}=-\xi^\sigma R^\mu{}_{\nu\sigma\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

Geodesic deviation

$$\frac{D^2\xi^\mu}{D\tau^2} = \frac{D}{D\tau}\left(\frac{\partial\xi^\mu}{\partial\tau} + \Gamma^\mu_{\nu\rho}\xi^\nu\frac{dx^\rho}{d\tau}\right)$$

$$\frac{D^2\xi^\mu}{D\tau^2} = \frac{\partial^2\xi^\mu}{\partial\tau^2} + \frac{dx^\lambda}{d\tau}\partial_\lambda\Gamma^\mu_{\nu\rho}\xi^\nu\frac{dx^\rho}{d\tau} + \Gamma - terms$$

Use $$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau} + \xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

$$\frac{D^2\xi^\mu}{D\tau^2} = -\xi^\sigma\partial_\sigma\Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \frac{dx^\lambda}{d\tau}\partial_\lambda\Gamma^\mu_{\nu\rho}\xi^\nu\frac{dx^\rho}{d\tau} + \Gamma - terms$$

$$\frac{D^2\xi^\mu}{D\tau^2} = \xi^\sigma\left(\partial_\nu\Gamma^\mu_{\sigma\rho} - \partial_\sigma\Gamma^\mu_{\nu\rho}\right)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} + \Gamma - terms$$

$\longrightarrow$ $$\frac{D^2\xi^\mu}{D\tau^2} = -\xi^\sigma R^\mu{}_{\nu\sigma\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

TT frame

This is the coordinate frame in which the metric is in TT gauge

Consider a test mass initially at rest at $\tau = 0$.

The geodesic equation is

$$\frac{d^2x^i}{d\tau^2}|_{\tau=0} = -[\Gamma^i_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}]_{\tau=0}$$

$$= -[\Gamma^i_{00}(\frac{dx^0}{d\tau})^2]$$

because

$$\frac{dx^i}{d\tau}|_{\tau=0} = 0$$

At linear order, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ after taking Cartesian coordinates for the background.

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\sigma}(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho})$$

$$\Gamma^{i}_{00} = \frac{1}{2}(2\partial_0 h_{0i} - \partial_i h_{00}) = 0$$

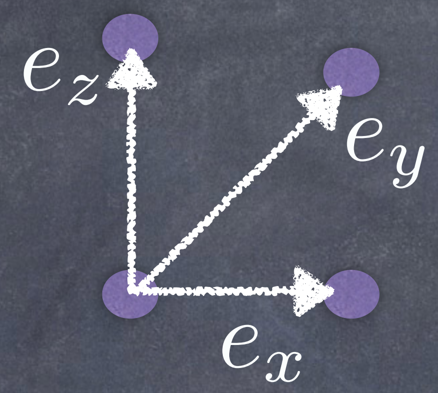
in TT gauge. Therefore,

$$\frac{d^2x^i}{d\tau^2}|_{\tau=0} = 0$$

In TT frame, particles which were at rest before the arrival of the wave remain at rest even after the arrival of the wave!
(non-linear effects are highly subleading)

The coordinates stretch themselves so that the position of the free test masses do not change

We can define the coordinates using freely falling test masses



What about time? In TT gauge, $h_{00} = h_{0i} = 0$

The proper time along a timelike trajectory $x^\mu(\tau)$ is obtained from

$$c^2 d\tau^2 = c^2 dt^2 - (\delta_{ij} + h_{ij}^{TT}) \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} d\tau^2$$

For a test mass initially at rest, $\frac{dx^i}{d\tau} = 0 \qquad \forall \tau$

$\longrightarrow$ $\tau = t$

In TT gauge, the proper time of a free test mass initially at rest is the coordinate time

Of course, proper distances change!

Consider the distance between $(t, x_1, 0, 0)$ and $(t, x_2, 0, 0)$

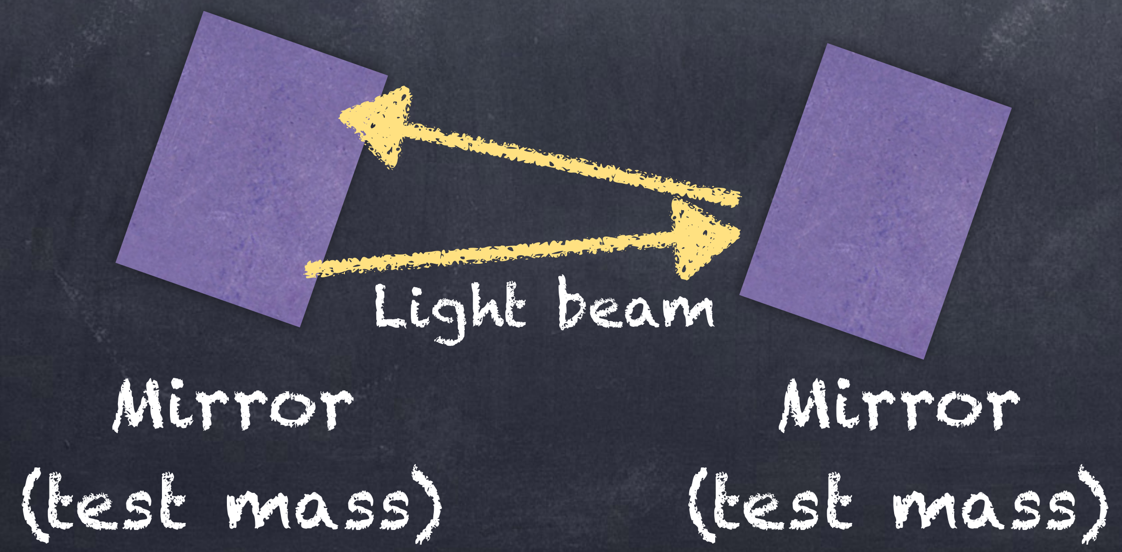
The coordinate distance $L = x_2 - x_1$ is a constant

But the proper distance after the passage of the wave is

$$s = \int_1^2 \sqrt{g_{xx}} dx = \sqrt{g_{xx}}(x_2 - x_1)$$

$$= \sqrt{1 + h_+ \cos(\omega t)} L \approx L(1 + \frac{1}{2} h_+ \cos(\omega t))$$

This is the basis of interferometry



Earth detector frame (LIGO/Virgo)

- Distance computed using rigid rulers
- No free fall with respect to the local inertial frame

$\vec{a} = -\vec{g}$ is the acceleration of the laboratory

- Local rotation with respect to local gyroscopes (e.g. Foucault pendulum)

$\vec{\Omega}$ is the angular velocity of the laboratory with respect to local gyroscopes.

Earth detector frame (LIGO/Virgo)

Result up to O(r^2) is $r = \sqrt{x^i x^i}$

inertial acc. grav. redshift Lorentz time dil.

$$ds^2 \approx -c^2 dt^2 [1 + \frac{2}{c^2}\vec{a}\cdot\vec{x} + \frac{1}{c^4}(\vec{a}\cdot\vec{x})^2 - \frac{1}{c^2}(\vec{\Omega}\times\vec{x})^2 + R_{0i0j}x^i x^j]$$

GW and varying grav.

$$+ 2cdtdx^i[\frac{1}{c}\epsilon_{ijk}\Omega^j x^k - \frac{2}{3}R_{0ijk}x^j x^k]$$

Sagnac effect GW and varying grav.

$$+ dx^i dx^j [\delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l]$$

GW and varying grav.

This is the detector frame used by experimentalists on Earth

We denote by L_B the typical variation scale of the metric

$$\frac{\vec{a}}{c^2} \sim \frac{1}{L_B}, \qquad R_{\mu\nu\rho\sigma} \sim \frac{1}{L_B^2}$$

Earth detector frame (LIGO/Virgo)

$$ds^2 \approx -c^2 dt^2 [1 + \frac{2}{c^2}\vec{a}\cdot\vec{x} + \frac{1}{c^4}(\vec{a}\cdot\vec{x})^2 - \frac{1}{c^2}(\vec{\Omega}\times\vec{x})^2 + R_{0i0j}x^i x^j]$$
$$+ 2c\,dt\,dx^i [\frac{1}{c}\epsilon_{ijk}\Omega^j x^k - \frac{2}{3}R_{0ijk}x^j x^k]$$
$$+ dx^i dx^j [\delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l]$$

We denote by L_B the typical variation scale of the metric

$$\frac{\vec{a}}{c^2} \sim \frac{1}{L_B}, \qquad R_{\mu\nu\rho\sigma} \sim \frac{1}{L_B^2}$$

At order $O(\frac{r}{L_B})^0$ ⟶ Minkowski (in contrast to TT gauge where there is no such expansion!)

$O(\frac{r}{L_B})^1$ ⟶ Newtonian corrections

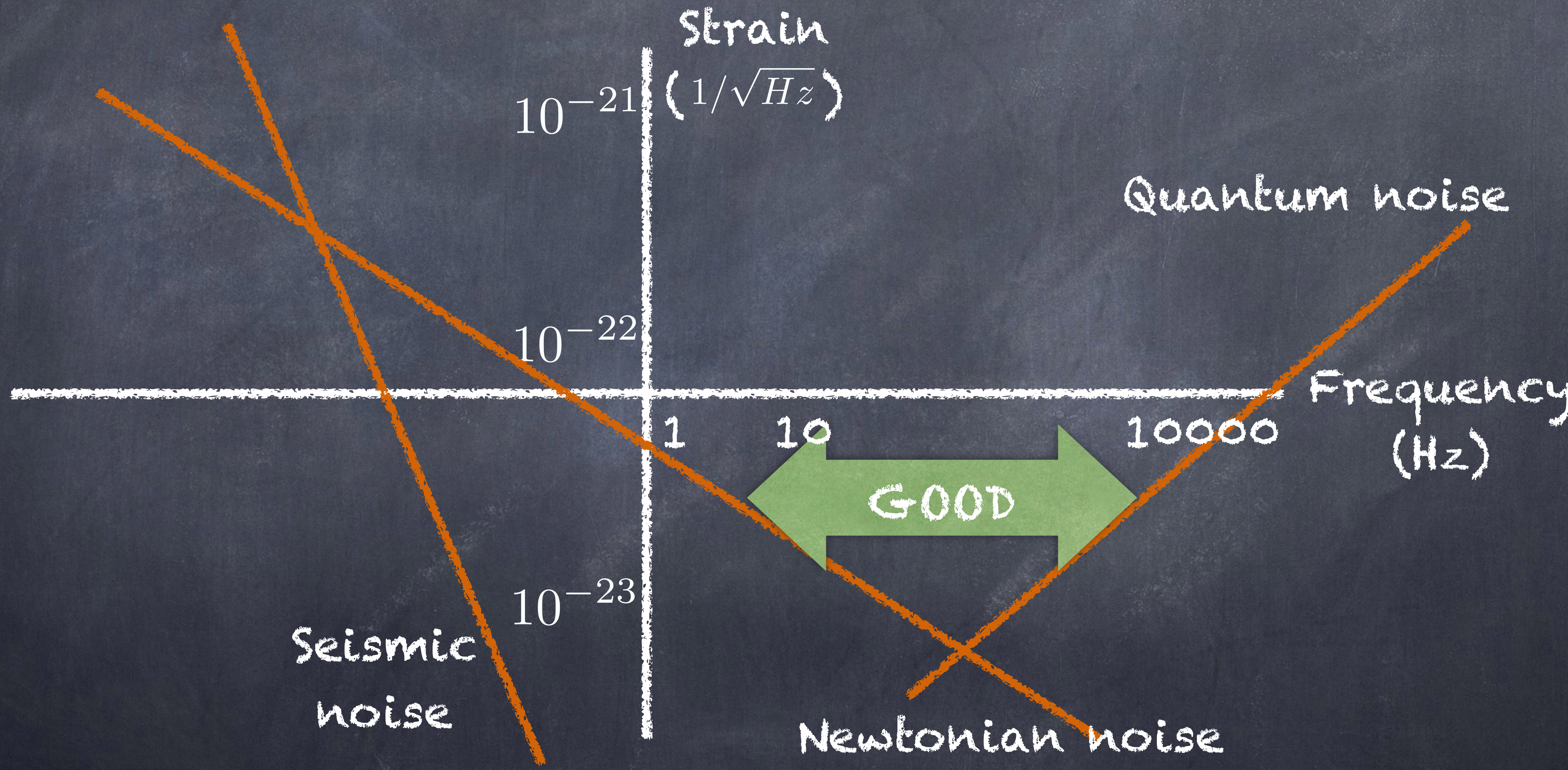
Geodesic equation: $\frac{d^2x^i}{d\tau^2} = -a^i - 2(\Omega \times v)^i + O(x^i)$

Earth gravity — Coriolis force

$O(\frac{r}{L_B})^2$ ⟶ GW and noise from varying gravity

In conclusion, a description in terms of Newtonian concept of forces is possible in the Earth detector frame.

Graviational Waves are subleading with respect to uncertainties in Earth gravity and local rotation, Newtonian and seismic noise.



Einstein Telescope: 2 possible sites: Sardaigna and Euregio

In conclusion, a description in terms of Newtonian concept of forces is possible in the Earth detector frame.

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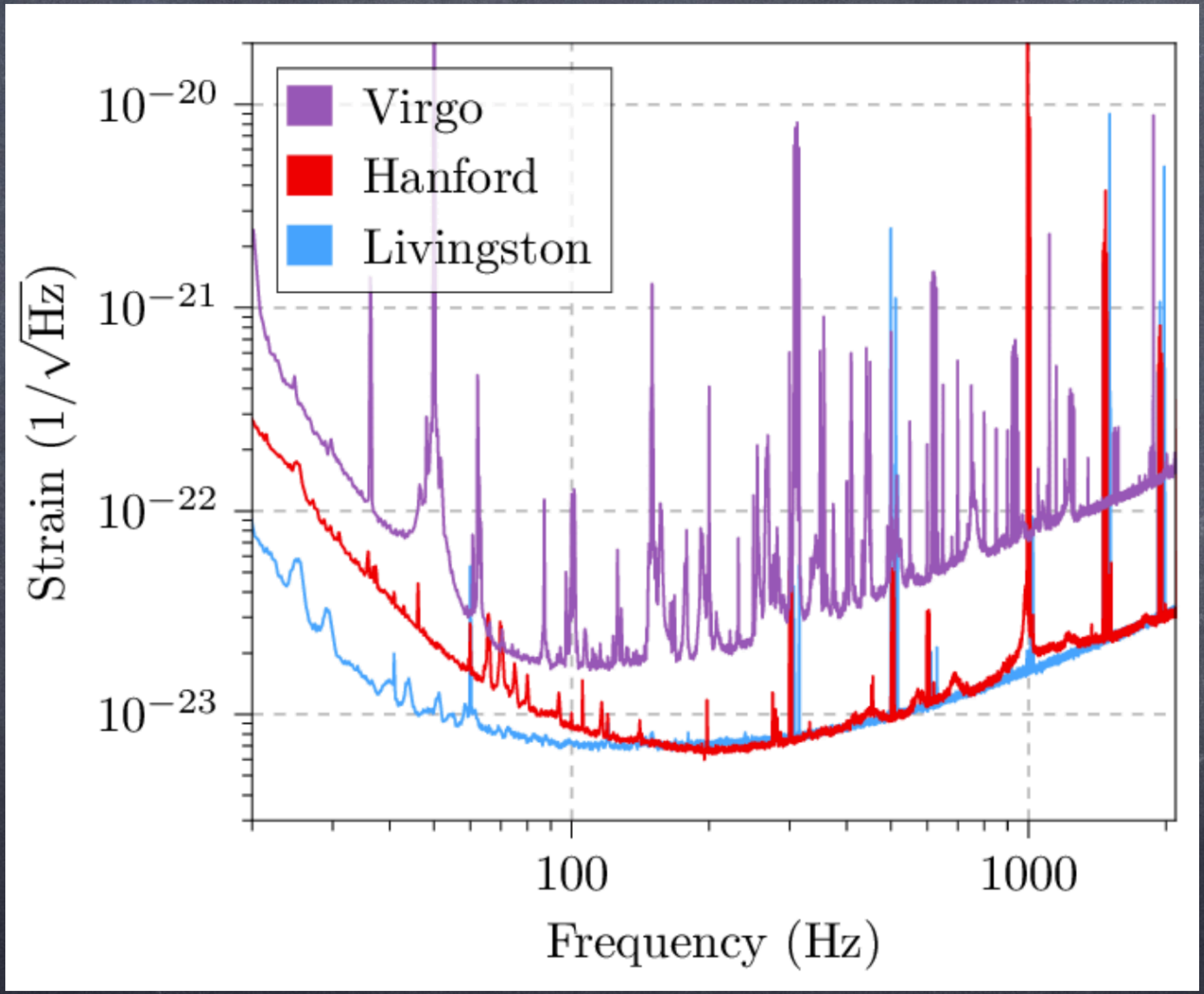


To isolate GW, we focus on the detector in its frequency window. Acceleration is compensated by suspenders.

Only Riemann terms matter and the expression reduces to in the freely falling frame, as far as we only look at components of $x^\mu(\tau)$ in the direction unconstrained by the suspension mechanism

If we look at the equation for geodesic deviation at the centre P of the local inertial frame

$$\Gamma^\alpha_{\mu\nu}|_P = 0 \qquad \frac{D^2\xi^i}{D\tau^2} = \frac{d^2\xi^i}{d\tau^2} = -R^i{}_{0j0}\xi^j\left(\frac{dx^0}{d\tau}\right)^2$$

Non relativistic motion $\frac{dx^i}{d\tau} \ll \frac{dx^0}{d\tau} \approx c$

$$\longrightarrow \quad \ddot{\xi}^i = -c^2 R^i{}_{0j0}\xi^j$$

$$= -c^2\left(-\frac{1}{2c^2}\ddot{h}^{TT}_{ij}\right)\xi^j$$

We can compute the Riemann in any frame including TT gauge.

Finally, in Earth detector frame in the directions unconstrained by suspenders:

$$\ddot{\xi}^i = \frac{1}{2}\ddot{h}^{TT}_{ij}\xi^j$$

In the Earth detector frame, the effect of GW on a point particle of mass m is described as a Newtonian force

$$F_i = \frac{m}{2}\ddot{h}^{TT}_{ij}\xi^j$$

without reference to GR.

Note that here we assumed $|\xi^i| \ll L_B$ which is true for LIGO.

Motion of test masses

- We consider a ring of test masses initially at rest in the Earth detector frame.
- We fix the origin at the centre of the ring
- Then ξ^i describes the distance w.r.t the origin (coordinate distance = proper distance)
- We consider GW propagating in the z direction. The ring is in the (x,y) plane

$$h_{ij}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - \frac{z}{c})]$$

$$\ddot{\xi}^i = \frac{1}{2}\ddot{h}_{ij}^{TT}\xi^j$$

$$h_{ij}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - \frac{z}{c})]$$

$\cos \downarrow \sin$

$$\ddot{\xi}^i = \frac{1}{2}\ddot{h}_{ij}^{TT}\xi^j$$

- If the particle is at z=0 at t=0 it will remain there.
- Therefore, GW are transverse in their physical effect: they displace the test masses transversally w.r.t. their direction of propagation

We write

$$\xi_i(t) = (x_0 + \delta x(t), y_0 + \delta y(t), 0)$$

(x_0, y_0) initial position

$$\delta x \ll x_0, \qquad \delta y \ll y_0$$

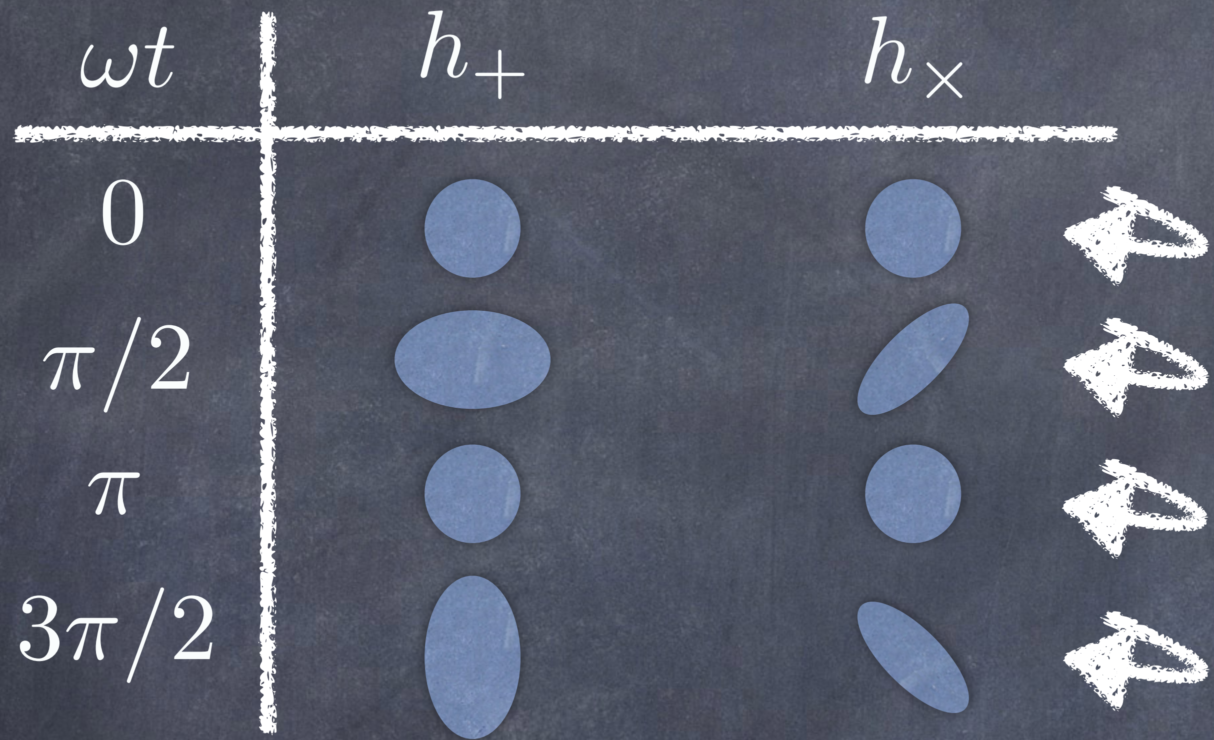
For h_+

$$\ddot{\delta x} = \frac{h_+}{2}(-\omega^2)\sin\omega t x_0 \qquad\longrightarrow\qquad \delta x = \frac{h_+}{2} x_0 \sin\omega\, t$$

$$\ddot{\delta y} = \frac{h_+}{2}\omega^2 \sin\omega t y_0 \qquad\qquad \delta y = -\frac{h_+}{2} y_0 \sin\omega\, t$$

ωt	h_+	$h_\times$
0		
$\pi/2$		
π		
$3\pi/2$		

Helicity of the graviton



Invariance under rotations of angle

$$\frac{2\pi}{2}$$

→ **Graviton has helicity 2**

Proof:

$$h_{\mu\nu}(x) \mapsto h'_{\mu\nu}(x') = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma h_{\rho\sigma}(x) \qquad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$h_{ij}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - \frac{z}{c})] \quad \mapsto \quad h_{ij}^{TT}(t,z) = \begin{pmatrix} h'_+ & h'_\times & 0 \\ h'_\times & h'_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t - \frac{z}{c})]$$

$$h'_+ = h_+ \cos 2\psi - h_\times \sin 2\psi$$
$$h'_\times = h_+ \sin 2\psi + h_\times \cos 2\psi$$

$$h_\times \pm i h_+ \mapsto e^{\mp 2i\psi}(h_\times \pm i h_+)$$

helicity eigenstates

Energy of GW

- It is clear that GW carry energy-momentum: they can accelerate masses ! (Derived by Bondi in 1961)
- According to GR, any form of energy induces curvature
- GW then backreacts at second order beyond linear order and that curvature allows to define energy.

Consistency of perturbation theory

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots \qquad |h_{\mu\nu}| \ll 1$$

Consistent if backreaction is smaller than the perturbation (far from masses)

In general, close to masses that generate GW:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu} + \dots \qquad |h_{\mu\nu}| \ll 1$$

We will distinguish the notion of background and perturbation by their frequency content :

background = low frequency
perturbation = high frequency

We consider the situation in which in some reference frame we can separate the metric into a background plus fluctuations where separation is based on a scale in time or space ("short-wave expansion")

$$\bar{\lambda} \ll L_B \qquad\qquad f \gg f_B$$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

Two small parameters: (1) $h \equiv O(h_{\mu\nu})$

(2) $\dfrac{\bar{\lambda}}{L_B}, \qquad \dfrac{f_B}{f}$

We now expand Einstein's equations to quadratic order in $h_{\mu\nu}$

$$R_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)$$

$$R_{\mu\nu}[g] = \bar{R}_{\mu\nu}[\bar{g}] + R^{(1)}_{\mu\nu}[h;\bar{g}] + R^{(2)}_{\mu\nu}[h,h;\bar{g}] + \dots$$

low freq modes — high freq modes — both modes $e^{\vec{k}\cdot\vec{x}}e^{-\vec{k}\cdot\vec{x}} \sim e^0$

Low mode eqs: $\bar{R}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{Low} + \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)^{low}$

High mode eqs: $R^{(1)}_{\mu\nu} = -[R^{(2)}_{\mu\nu}]^{High} + \frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)^{high}$

The low equation gives energy-momentum of GW. It can be obtained practically as follows. We introduce an intermediate time scale $\bar{t}$

$$\frac{1}{f_B} \gg \bar{t} \gg \frac{1}{f}$$

We then average over $\bar{t}$, i.e. over many periods of GW:

$$\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu}\rangle + \frac{8\pi G}{c^4}\langle T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\rangle$$

This was understood in the sixties. This is a renormalisation group flow. We integrated out high frequencies to describe physics of low frequencies.

We define the effective stress-tensor of GW

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle \qquad R^{(2)} \equiv \bar{g}^{\mu\nu}R^{(2)}_{\mu\nu}$$

$$t = \bar{g}^{\mu\nu}t_{\mu\nu} = +\frac{c^4}{8\pi G}\langle R^{(2)}\rangle$$

because $\bar{g}_{\mu\nu}$ is low frequency

We deduce

$$-\langle R^{(2)}_{\mu\nu}\rangle = \frac{8\pi G}{c^4}(t_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}t)$$

We define

$$\langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle \equiv \bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T}$$

The effective Einstein's equations at low frequencies

$$\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu} \rangle + \frac{8\pi G}{c^4} \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle$$

become

$$\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = \frac{8\pi G}{c^4} (\bar{T}_{\mu\nu} + t_{\mu\nu})$$

This is the coarse-grained form of Einstein's equations

We equated different orders in h. This is possible because there is a second small parameter $\frac{\bar{\lambda}}{L_B} \ll 1$.

Explicit expressions:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\bar{D}^\alpha \bar{D}_\mu h_{\nu\alpha} + \bar{D}^\alpha \bar{D}_\nu h_{\mu\alpha} - \bar{D}^\alpha \bar{D}_\alpha h_{\mu\nu} - \bar{D}_\nu \bar{D}_\mu h\right)$$

$$\begin{aligned} R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[&\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \\ &+ h_{\rho\alpha}(\bar{D}_\nu \bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta \bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta \bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta \bar{D}_\mu h_{\nu\sigma}) \\ &+ (\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big] \end{aligned}$$

Einstein equations' constraints on the small parameters

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{8\pi G}{c^4}(\bar{T}_{\mu\nu} + t_{\mu\nu})$$

We choose coordinates such that $\bar{g}_{\mu\nu} = O(1)$

$$\partial\bar{g}_{\mu\nu} \sim \frac{1}{L_B} \longrightarrow \bar{R}_{\mu\nu} \sim \partial^2\bar{g}_{\mu\nu} \sim \frac{1}{L_B^2}$$

$$\partial h_{\mu\nu} \sim \frac{h}{\bar{\lambda}} \longrightarrow R^{(2)}_{\mu\nu} \sim (\partial h)^2 \sim \frac{h^2}{\bar{\lambda}^2}$$

Einstein's equations give

$$\frac{1}{L_B^2} \sim \frac{h^2}{\bar{\lambda}^2} + (matter) \longleftrightarrow h \sim \frac{\bar{\lambda}}{L_B}$$ if no matter

If dominant matter $\frac{1}{L_B^2} \gg \frac{h^2}{\bar{\lambda}^2} \longleftrightarrow h \ll \frac{\bar{\lambda}}{L_B}$

Therefore, if background slow frequency perturbation are neglected, the perturbation series in h breaks down.

Gravitational energy -momentum tensor

The energy-momentum tensor can be computed assuming

$\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ (far from sources, ignoring Earth gravity)

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle$$

We do the computation in harmonic and $h = 0$ gauge

$$\partial_\mu h^{\mu\nu} = 0, \qquad h = 0$$

In harmonic gauge, a perturbation is a superposition of plane waves

$$h_{\mu\nu} = \int \epsilon_{\mu\nu} e^{i\omega(t - \frac{\hat{n}\cdot\vec{x}}{c})}, \qquad \vec{k} = \frac{\omega}{c}\hat{n}$$

We have

$$\partial_t h_{\mu\nu} = i\omega h_{\mu\nu}$$

$$\partial_i h_{\mu\nu} = -\frac{i\omega}{c} n_i h_{\mu\nu} \longrightarrow \partial_i h_{\mu\nu} = -\frac{1}{c} n_i \partial_t h_{\mu\nu}$$

Integrations by parts in $R^{(2)}_{\alpha\beta}$ are possible

∂_t total derivatives drop because of time average

∂_i can be expressed as ∂_t because waves

$$\longrightarrow \langle \partial_\alpha T^\alpha_{\mu\nu} \rangle = 0, \qquad \forall T^\alpha_{\mu\nu}$$

Exercise

Given

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\left[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})\right.$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$\left.+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\right]$$

Assume $g_{\mu\nu} = \eta_{\mu\nu} \quad \partial_\mu h^{\mu\nu} = 0, \quad h = 0 \quad \Box h_{\alpha\beta} = 0$

Use $\langle \partial_\alpha T^\alpha_{\mu\nu} \rangle = 0, \quad \forall T^\alpha_{\mu\nu}$ to prove

(i) $\langle R_{(2)} \rangle = 0$

(ii) $\langle R^{(2)}_{\mu\nu} \rangle = -\frac{1}{4}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} \rangle$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

Solution

$$
\begin{aligned}
R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[&\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma}) \\
&+ h_{\rho\alpha}(\bar{D}_\nu \bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta \bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta \bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta \bar{D}_\mu h_{\nu\sigma}) \\
&+ (\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]
\end{aligned}
$$

$$
\langle R^{(2)} \rangle = \frac{1}{2}\langle \frac{3}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} + h^{\alpha\beta}\partial_\mu \partial^\mu h_{\alpha\beta}\rangle
$$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

$$\langle R^{(2)}\rangle = \frac{1}{2}\langle\frac{3}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} + h^{\alpha\beta}\partial_\mu\partial^\mu h_{\alpha\beta}\rangle$$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

$$\langle R^{(2)}\rangle = \frac{1}{2}\langle\frac{3}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} + h^{\alpha\beta}\partial_\mu\partial^\mu h_{\alpha\beta}\rangle = 0$$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

$$\langle R^{(2)}\rangle = \frac{1}{2}\langle\frac{3}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} + h^{\alpha\beta}\partial_\mu\partial^\mu h_{\alpha\beta}\rangle = 0$$

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{1}{2}\langle\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + \partial_\alpha h_{\mu\beta}\partial^\alpha h^\beta_\nu - \partial_\alpha h_{\mu\beta}\partial^\beta h^\alpha_\nu$$
$$+h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} - h^{\alpha\beta}\partial_\alpha\partial_\mu h_{\nu\beta} - h^{\alpha\beta}\partial_\alpha\partial_\nu h_{\mu\beta}\rangle$$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

$$\langle R^{(2)}\rangle = \frac{1}{2}\langle\frac{3}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} + h^{\alpha\beta}\partial_\mu\partial^\mu h_{\alpha\beta}\rangle = 0$$

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{1}{2}\langle\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + \partial_\alpha h_{\mu\beta}\partial^\alpha h^\beta_\nu - \partial_\alpha h_{\mu\beta}\partial^\beta h^\alpha_\nu$$
$$+h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} - h^{\alpha\beta}\partial_\alpha\partial_\mu h_{\nu\beta} - h^{\alpha\beta}\partial_\alpha\partial_\nu h_{\mu\beta}\rangle$$

Solution

$$R^{(2)}_{\mu\nu} = \frac{1}{2}\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta}\Big[\frac{1}{2}\bar{D}_\mu h_{\rho\alpha}\bar{D}_\nu h_{\sigma\beta} + (\bar{D}_\rho h_{\nu\alpha})(\bar{D}_\sigma h_{\mu\beta} - \bar{D}_\beta h_{\mu\sigma})$$
$$+h_{\rho\alpha}(\bar{D}_\nu\bar{D}_\mu h_{\sigma\beta} + \bar{D}_\beta\bar{D}_\sigma h_{\mu\nu} - \bar{D}_\beta\bar{D}_\nu h_{\mu\sigma} - \bar{D}_\beta\bar{D}_\mu h_{\nu\sigma})$$
$$+(\frac{1}{2}\bar{D}_\alpha h_{\rho\sigma} - \bar{D}_\rho h_{\alpha\sigma})(\bar{D}_\nu h_{\mu\beta} + \bar{D}_\mu h_{\nu\beta} - \bar{D}_\beta h_{\mu\nu})\Big]$$

$$\langle R^{(2)}\rangle = \frac{1}{2}\langle\frac{3}{2}\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} - \partial_\alpha h_{\beta\gamma}\partial^\beta h^{\alpha\gamma} + h^{\alpha\beta}\partial_\mu\partial^\mu h_{\alpha\beta}\rangle = 0$$

$$\langle R^{(2)}_{\mu\nu}\rangle = \frac{1}{2}\langle\frac{1}{2}\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta} + \partial_\alpha h_{\mu\beta}\partial^\alpha h^\beta_\nu - \partial_\alpha h_{\mu\beta}\partial^\beta h^\alpha_\nu$$
$$+h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} + h^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} - h^{\alpha\beta}\partial_\alpha\partial_\mu h_{\nu\beta} - h^{\alpha\beta}\partial_\alpha\partial_\nu h_{\mu\beta}\rangle$$

$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{4}\langle\partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle$$

Gravitational energy -momentum tensor

Energy-momentum tensor is

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle \longrightarrow t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle$$

Do residual gauge transformation change the stress-tensor?

$$\delta_\xi h_{\mu\nu} = \delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$$\delta_\xi t_{\mu\nu} = ?$$

Exercise

Prove $\delta_\xi t_{\mu\nu} = 0$

Using $t_{\mu\nu} = \dfrac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle$

$$\delta_\xi h_{\mu\nu} = \delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Solution

Prove $\delta_\xi t_{\mu\nu} = 0$

Using $t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle$

$$\delta_\xi h_{\mu\nu} = \delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Remember
$$h = 0 \rightarrow \partial_\alpha \xi^\alpha = 0$$
$$\partial_\alpha h^{\alpha\mu} = 0 \rightarrow \Box \xi^\mu + \partial_\mu \partial_\alpha \xi^\alpha = 0$$

Therefore $\langle \partial_\alpha(\partial_{...} h_{\mu\nu} \partial_{...} \xi^\sigma)\rangle = 0$

Solution

Prove $\delta_\xi t_{\mu\nu} = 0$

Using $t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle$

$$\delta_\xi h_{\mu\nu} = \delta_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

Remember
$$h = 0 \to \partial_\alpha \xi^\alpha = 0$$
$$\partial_\alpha h^{\alpha\mu} = 0 \to \Box \xi^\mu + \partial_\mu \partial_\alpha \xi^\alpha = 0$$

Therefore $\langle \partial_\alpha (\partial_{\ldots} h_{\mu\nu} \partial_{\ldots} \xi^\sigma)\rangle = 0$

Compute
$$\delta_\xi t_{\mu\nu} \sim \langle \partial_\mu (\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha)\partial_\nu h^{\alpha\beta}\rangle + (\mu \leftrightarrow \nu)$$
$$\delta_\xi t_{\mu\nu} \sim \langle \partial_\mu \partial_\alpha \xi_\beta \partial_\nu h^{\alpha\beta}\rangle + (\mu \leftrightarrow \nu)$$
$$\delta_\xi t_{\mu\nu} \sim \langle \partial_\mu \xi_\beta \partial_\alpha \partial_\nu h^{\alpha\beta}\rangle + (\mu \leftrightarrow \nu) \sim 0$$

Gravitational energy -momentum tensor

Energy-momentum tensor is

$$t_{\mu\nu} \equiv -\frac{c^4}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\rangle \longrightarrow t_{\mu\nu} = \frac{c^4}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle$$

Question: do residual gauge transformation change the stress-tensor? Answer: no.

Therefore, we can replace $h_{\mu\nu}$ by $h^{TT}_{\mu\nu}$

In particular, the effective energy density is

$$t^{00} = \frac{c^2}{32\pi G}\langle \dot{h}^{TT}_{ij}\dot{h}^{TT}_{ij}\rangle = \frac{c^2}{16\pi G}\langle \dot{h}^2_+ + \dot{h}^2_\times\rangle$$

$$\dot{f} = \partial_t f = c\partial_0 f$$

$$h^{TT}_{ij} = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix} \cos[\omega(t - \frac{z}{c})]$$

Bianchi identities imply

$$\bar{D}^{\mu}(\bar{T}_{\mu\nu} + t_{\mu\nu}) = 0$$

This proves that one can associate to each Poincaré symmetry a conserved quantity (up to higher order perturbations)

Far from the sources

$$\partial^{\mu} t_{\mu\nu} = 0$$

$$\longrightarrow \int_V d^3x[\partial_0 t^{00} + \partial_i t^{i0}] = 0$$ on a given volume V

The effective energy of the volume is

$$E_V \equiv \int d^3x t^{00}$$

$$\frac{1}{c}\frac{dE_V}{dt} = -\int_V d^3x \partial_i t^{0i} = -\int dA n_i t^{0i}$$

Take S a spherical surface $dA = r^2 d\Omega$

$$\frac{dE_V}{dt} = -c\, r^2 \int d\Omega\, t^{0r}$$

$$= \frac{c^4}{32\pi G} \langle \partial^0 h_{ij}^{TT} \frac{\partial}{\partial r} h_{ij}^{TT} \rangle$$

Single plane waves are not realistic as global solutions at large distances. Instead, a GW propagating radially outwards has the following form at large distances from the source (proven later)

$$h_{ij}^{TT} = \frac{1}{r} f_{ij}(t - \frac{r}{c}) + O(r^{-2})$$

This is similar to electromagnetic waves, up to a spin-2 polarization

Exercise

Prove that $h_{ij}^{TT} = \frac{1}{r} f_{ij}(t - \frac{r}{c})$ obeys $\Box h_{ij}^{TT} = O(r^{-2})$

Solution

Prove that $h_{ij}^{TT} = \frac{1}{r} f_{ij}(t - \frac{r}{c})$ obeys $\Box h_{ij}^{TT} = O(r^{-2})$

Use $\Box\Phi = \frac{1}{\sqrt{|g|}}\partial_\mu \left(\sqrt{|g|} g^{\mu\nu} \partial_\mu \Phi\right)$

$$ds^2 = -c^2 dt^2 + dr^2 \qquad \sqrt{|g|} = c$$

$$ds^2 = -c^2 du^2 - 2c\,du\,dr \qquad \sqrt{|g|} = c$$

Drop c

$$x^\mu = (u, r) \qquad g^{\mu\nu} = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$$

$$V^\mu = g^{\mu\nu}\partial_\nu\left(\frac{f_{ij}(u)}{r}\right) = \left(-\frac{-1}{r^2} f_{ij}, -\left(\frac{1}{r}\right)\partial_u f_{ij} + \left(-\frac{1}{r^2}\right) f_{ij}\right)$$

$$\partial_\mu V^\mu = \frac{1}{r^2}\partial_u f_{ij} + \frac{1}{r^2}\partial_u f_{ij} + \frac{1}{r^3} f_{ij} = O(r^{-2})$$

$$\frac{\partial}{\partial r} h_{ij}^{TT} = -\frac{1}{r^2} f_{ij}(t - \frac{r}{c}) + \frac{1}{r}(-\frac{1}{c})\frac{\partial}{\partial t} f_{ij} = -\partial_0 h_{ij}^{TT} = \partial^0 h_{ij}^{TT}$$

$$\longrightarrow \quad t^{0r} = t^{00}$$

An observer sitting at large distances sees a plane wave front.

Since E_V decreases, GW carry away an energy flux

$$\frac{dE}{dAdt} = +c\, t^{00} = \frac{c^3}{32\pi G} \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle$$
$$= \frac{c^3}{16\pi G} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$

This is Einstein's formula for the flux-balance law of energy, with a factor of 2 corrected by Eddington.

Poincaré flux-balance laws

For each Killing vector of Minkowski, we have a conserved current :

$$J^\mu = t^{\mu\nu}\bar{\xi}_\nu \qquad \partial_\mu J^\mu = 0$$

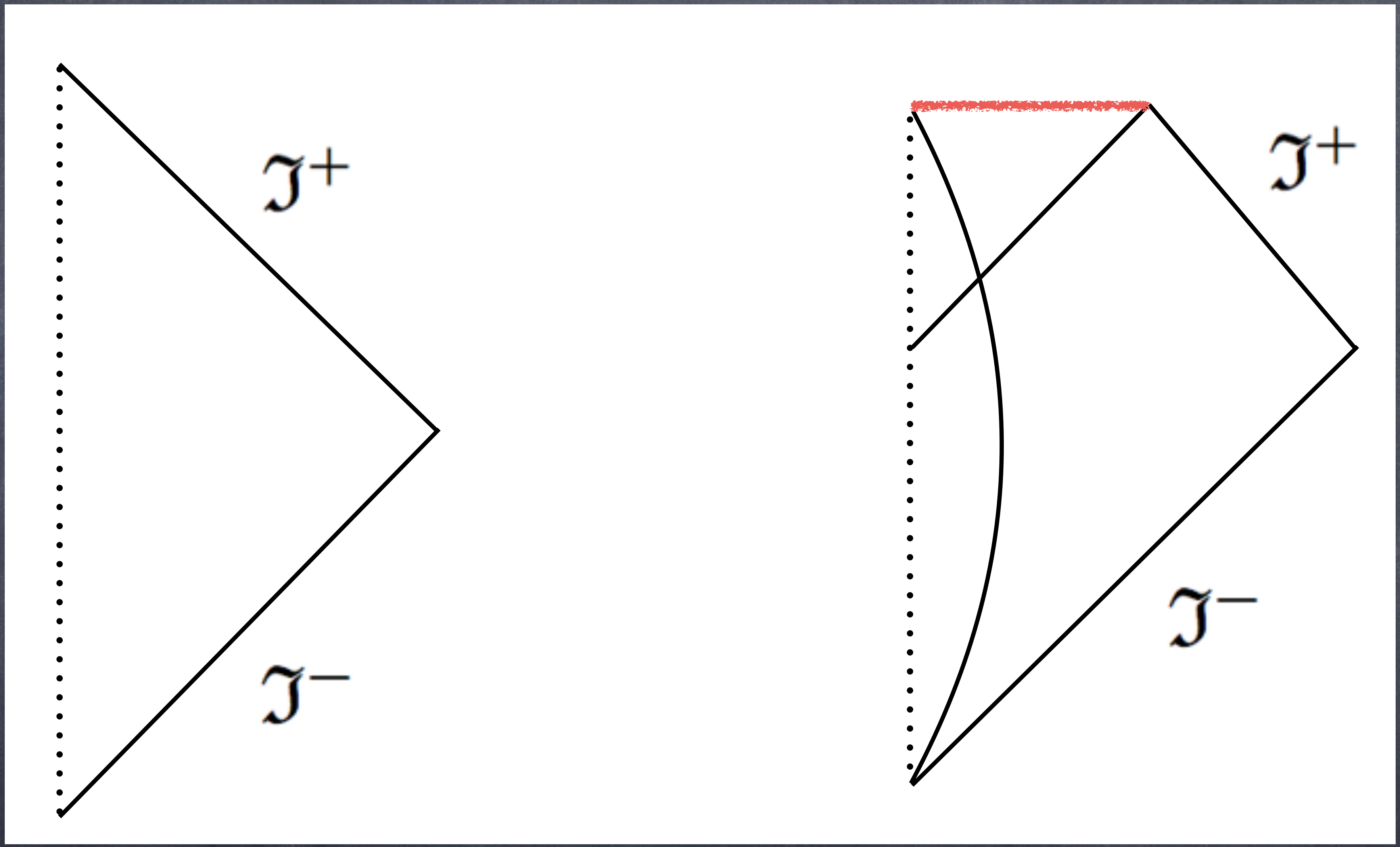
We can define the flux of momentum

$$\frac{dP^k}{dt} = -\frac{c^3}{32\pi G} r^2 \int d\Omega \langle \dot{h}_{ij}^{TT} \partial^k h_{ij}^{TT} \rangle$$

Also, the flux of angular momentum and centre of mass

Those are the flux-balance laws for Poincaré charges

Asymptotic view on flux-balance laws



Approach of Bondi, van den Burg, Metzner, Sachs (BMS), 1962

Einstein's solution

Radiative gauge [Newman-Unti gauge]

$$g_{rr} = -1, \qquad g_{ru} = g_{r\theta} = g_{r\phi} = 0$$

The asymptotic solution takes the form

$$\begin{aligned} ds^2 = &-c^2 du^2 - 2c du dr + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &+ \frac{2m}{r} du^2 + r C_{AB} dx^A dx^B + \dots \\ &+ \frac{N_A}{r} du dx^A + \dots \\ &+ \frac{1}{r^i} E^{(i)}_{AB} dx^A dx^B + \dots \end{aligned}$$

It depends on

$$C_{AB}, \quad m, N_A, E^{(i)}_{AB}, \qquad i = 1, 2, \dots$$

Unconstrained   They obey flux-balance laws

BMS flux-balance laws

Supermomentum $$Q_T(u) \equiv \int_S d^2\Omega\, m(u,\theta,\phi) T(\theta,\phi)$$

Super-Lorentz charge $$Q_R(u) \equiv \int_S d^2\Omega\, N_A(u,\theta,\phi) R^A(\theta,\phi)$$

Higher spin charges $$Q_S^{(i)}(u) \equiv \int_S d^2\Omega\, E_{AB}^{(i)}(u,\theta,\phi) S^{AB}(\theta,\phi)$$

$$\partial_u Q_T(u) = \text{flux on } \mathscr{I}^+ \text{dictated by Einstein's equations}$$

$$\partial_u Q_R(u) = \text{flux on } \mathscr{I}^+ \text{dictated by Einstein's equations}$$

$$\partial_u Q_S^{(i)}(u) = \text{flux on } \mathscr{I}^+ \text{dictated by Einstein's equations}$$

1.3. Generation of GW

Binaries as sources of GW

We consider a slowly moving source, which is weakly self-gravitating

Typical velocity → $\frac{v}{c} \ll 1$

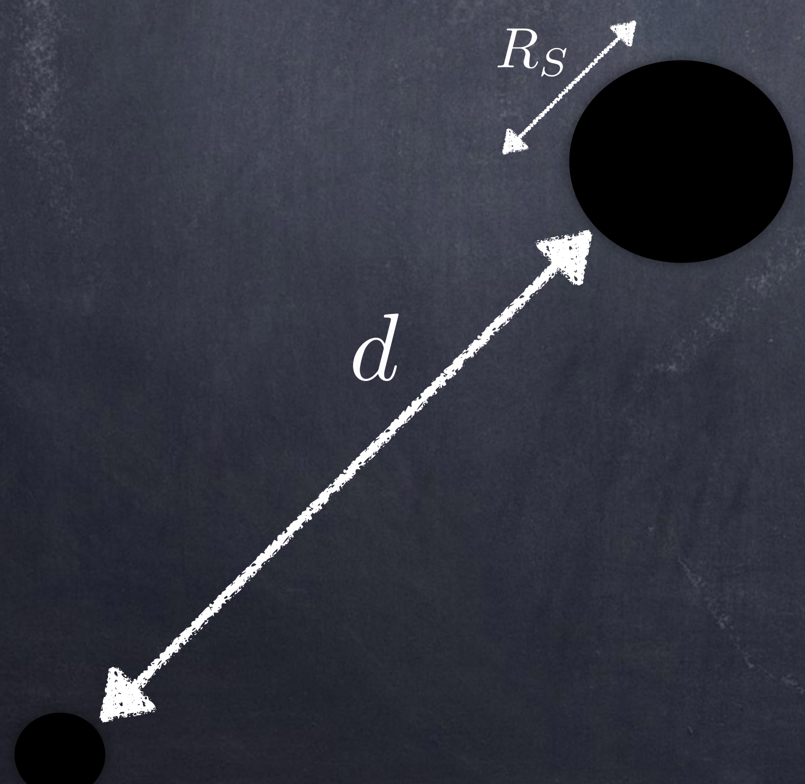
$\frac{R_S}{d} \ll 1$

Schwarzschild radius of the source → R_S

Size of the source → d

$R_S = \frac{2Gm}{c^2}$ ← Total mass

Typical example: a binary system in the early inspiral phase



Total Mass $m = m_1 + m_2$

Binaries admit only one small parameter as a result of gravitational binding

$$\frac{v}{c} \ll 1 \qquad \frac{R_S}{d} \ll 1$$

Reduced Mass $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$

Total Mass $m = m_1 + m_2$

Gravitational binding gives

$$\frac{1}{2}\mu v^2 \sim \frac{1}{2}\frac{G\mu m}{d}$$

$$\frac{v^2}{c^2} \sim \frac{R_S}{d}$$

As a result, for binary compact objects, corrections in v/c induce corrections in G.

GW wavelength in terms of the source size

- If ω_s is the typical frequency of motion inside the source and d is the source size, the typical velocities are

$$v \sim \omega_s d$$

- The frequency of radiation will also be of the order of

$$\omega_{gw} \sim 2\omega_s$$

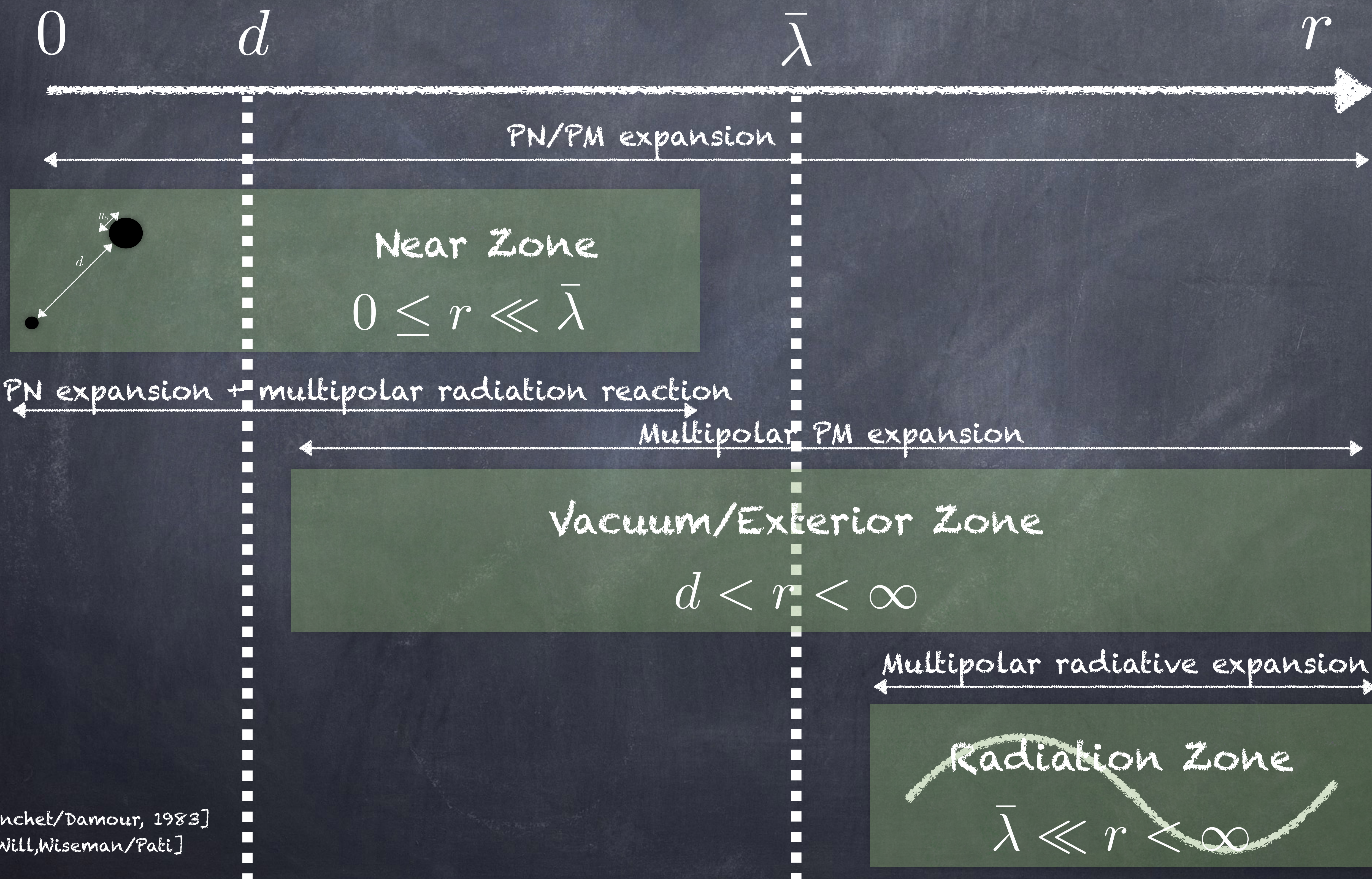
(Proof: later on !)

→ The reduced wavelength: $\bar{\lambda} = \dfrac{c}{\omega_{gw}} \sim \dfrac{c}{\omega_s} \sim \dfrac{c}{v} d$

For a non-relativistic system, $v \ll c$

→ $$d \ll \bar{\lambda}$$

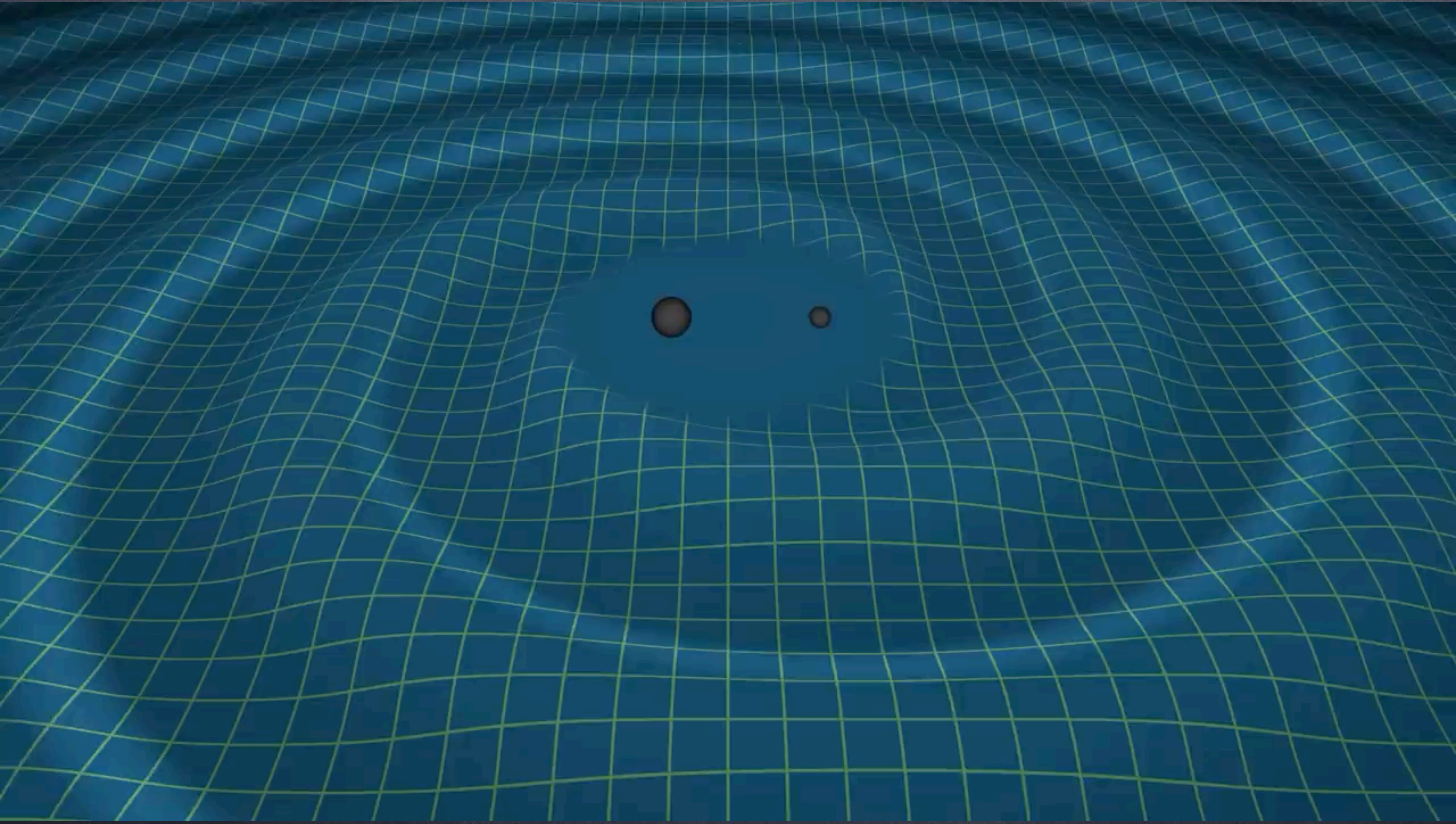
The 3 zones of the Post-Newtonian/ Post-Minkowskian formalism



[Blanchet/Damour, 1983]
[Will,Wiseman/Pati]

Multipolar PM expansion

PN expansion + multipolar radiation reaction



In the following, we will derive the motion at the lowest order:

- in the Near-Zone: Newtonian (0PN) + 2.5PN radiation-reaction
- in the exterior zone: Minkowski + linear GW (1PM)

Weak sources with arbitrary velocity

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

$$\partial^\mu \bar{h}_{\mu\nu} = 0, \qquad \partial^\nu T_{\mu\nu} = 0$$

We can solve it in terms of the retarded Green function:

$$\Box G(x - x') = \delta^4(x - x')$$

$$\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} \int d^4x' G(x - x') T_{\mu\nu}(x')$$

Explicitly,

$$G(x - x') = \frac{-1}{4\pi|\vec{x} - \vec{x}'|} \delta(x^0_{ret} - x'^0_{ret})$$

$$x'^0 = ct' \qquad x^0_{ret} = ct_{ret} \qquad t_{ret} = t - \frac{|\vec{x} - \vec{x}'|}{c}$$

Recall that outside the source, we can use TT gauge and we have

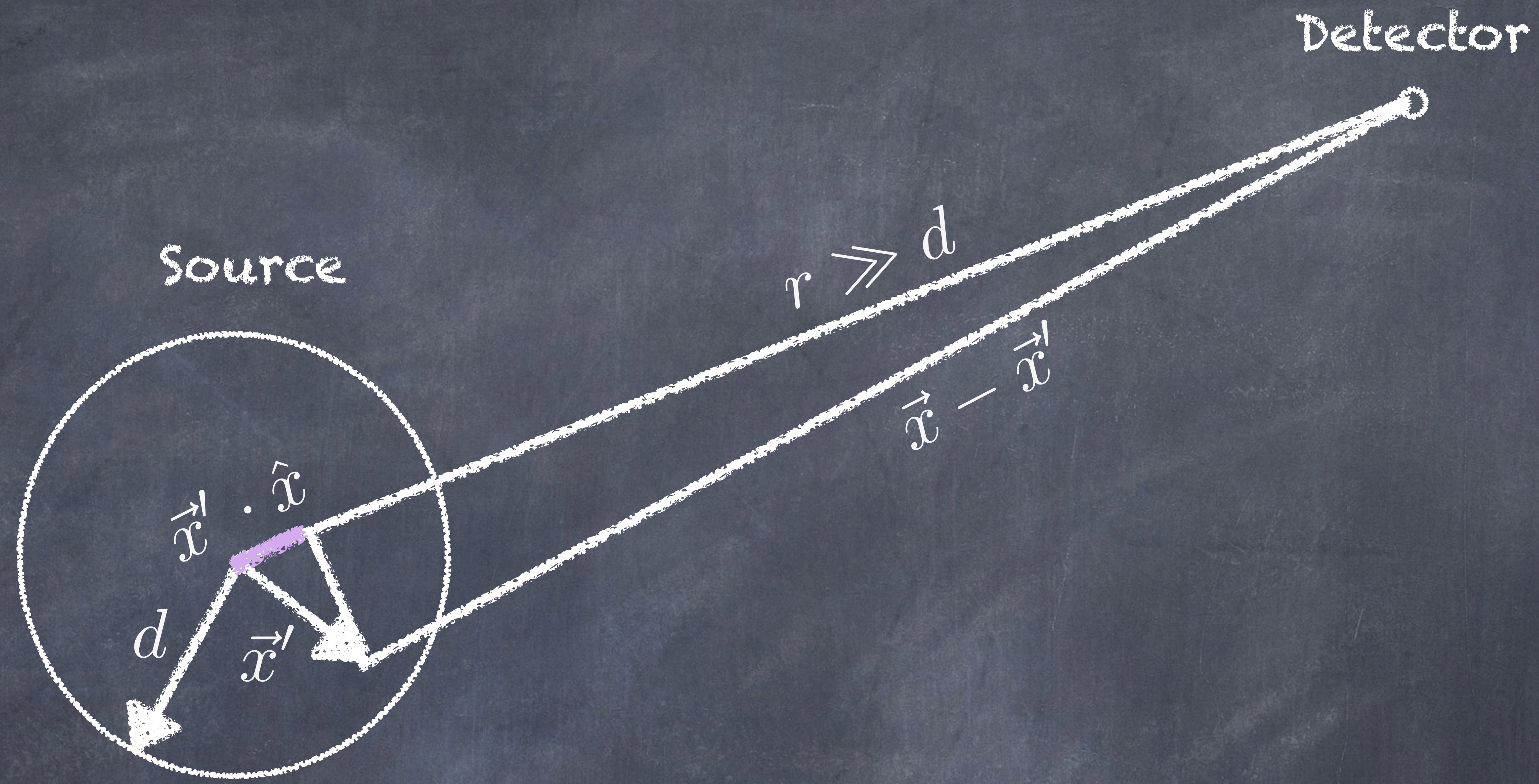
$$h_{ij}^{TT} = \Lambda_{ij,kl} h_{kl}$$

Therefore, outside the source we have

$$h_{ij}^{TT}(t, \vec{x}) = \frac{4G}{c^4} \Lambda_{ij,kl}(\vec{x}) \int d^3x' \frac{1}{|x - x'|} T_{kl}(t - \frac{|x - x'|}{c}, x')$$

Note that h_{ij}^{TT} depends upon the integrals of the spatial components of T_{kl}

(temporal components are related by conservation)



$$|\vec{x} - \vec{x}'| = r - \vec{x}' \cdot \hat{x} + O(d^2/r), \qquad \hat{x} \equiv \frac{\vec{x}}{r}$$

Therefore, for large r,

$$h_{ij}^{TT} = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{x}) \int d^3x' T_{kl}(t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{x}}{c}, \vec{x}')$$

Weak sources with low velocity

Fourier transform:

$$T_{kl}(t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{x}}{c}, \vec{x}') = \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{kl}(\omega, \vec{k}) e^{-i\omega(t - \frac{r}{c} + \frac{\vec{x}' \cdot \hat{x}}{c}) + i\vec{k}\cdot\vec{x}'}$$

For a non-relativistic system,

$v \ll c$ $\qquad$ $\bar{\lambda} \gg d$

$\omega_s d \ll c$

and $T_{kl} \neq 0$ only inside the source $|\vec{x}'| \leq d$

$$\longrightarrow \quad \omega \frac{\vec{x}' \cdot \hat{x}}{c} \leq \frac{\omega_s d}{c} \ll 1$$

We can therefore expand

$$e^{-i\omega(t-\frac{r}{c}+\frac{\vec{x}'\cdot\hat{x}}{c})} = e^{-i\omega(t-\frac{r}{c})}[1 - i\frac{\omega}{c}\vec{x}'\cdot\hat{x} + O(\frac{\omega}{c})^2]$$

This is equivalent in position space to expanding

$$T_{kl}(t-\frac{r}{c}+\frac{\vec{x}'\cdot\hat{x}}{c},\vec{x}') \approx T_{kl}(t-\frac{r}{c},\vec{x}') + \frac{\vec{x}'\cdot\hat{x}}{c}\partial_t T_{kl} + O(\partial_t^2 T_{kl})$$

We define the multipoles of the stress-tensor

$$S^{ij}(t) = \int d^3x\, T^{ij}(t,\vec{x})$$

$$S^{ij,k}(t) = \int d^3x\, T^{ij}(t,\vec{x})x^k$$

$$S^{ij,kl}(t) = \int d^3x\, T^{ij}(t,\vec{x})x^k x^l$$

We get

$$h_{ij}^{TT}(t,x) = \frac{1}{r}\frac{4G}{c}\Lambda_{ij,kl}(\hat{x})[S^{kl} + \frac{1}{c}\hat{x}_m\dot{S}^{kl,m} + \frac{1}{2c^2}\hat{x}_m\hat{x}_p\ddot{S}^{kl,mp} + \ldots]_{ret}$$

$$\frac{\omega_s d}{c} \sim \frac{v}{c} \qquad O(\frac{v}{c})^2$$

evaluated at t-r/c

Therefore, weak sources with low velocity emit gravitational radiation that is essentially determined by the lowest multipole moments

We do not need to know all the structure of the source

We only need to know its lowest multipole moments

$$h_{ij}^{TT}(t,x) = \frac{1}{r}\frac{4G}{c}\Lambda_{ij,kl}(\hat{x})[S^{kl} + \frac{1}{c}\hat{x}_m\dot{S}^{kl,m} + \frac{1}{2c^2}\hat{x}_m\hat{x}_p\ddot{S}^{kl,mp} + \dots]_{ret}$$

$$\frac{\omega_s d}{c} \sim \frac{v}{c} \qquad O(\frac{v}{c})^2$$

In terms of GW power:

2.5PN 3.5PN

[will be justified later]

Two sets of multipoles have a physical interpretation:

Mass multipoles:

$$M = \frac{1}{c^2}\int d^3x T^{00}(t,x)$$

$$M^i = \frac{1}{c^2}\int d^3x T^{00}(t,x)x^i$$

...

Momentum multipoles:

$$P^i = \frac{1}{c}\int d^3x\, T^{0i}(t,x)$$

$$P^{i,j} = \frac{1}{c}\int d^3x\, T^{0i}(t,x)x^j$$

...

The multipoles of T^{ij} are related to time derivatives of these multipoles by the conservation of the stress-tensor

Exercise

Obtain the time derivative of the lowest mass and momentum multipoles

$$M^L = \frac{1}{c^2}\int d^3x T^{00} x^L \qquad P^{i,L} = \frac{1}{c}\int d^3x T^{0i} x^L$$

(i) $\dot{M} = 0$

(ii) $\dot{M}^i = P^i$

(iii) $\dot{M}^{ij} = P^{i,j} + P^{j,i}$

(iv) $\dot{P}^i = 0$

(v) $\dot{P}^{i,j} = S^{ij}$

(vi) $\dot{P}^{i,jk} = S^{ij,k} + S^{ik,j}$

using $\partial_\mu T^{\mu\nu} = 0$.

Solution

(i) $$c\dot{M} = \int d^3x \partial_0 T^{00} = -\int d^3x \partial_i T^{0i} = -\int dS_i T^{0i} = 0$$

for a volume larger than the source.

In linear theory, back reaction of GW are absent, and the mass of matter is conserved.

(ii) $$c\dot{M}^i = \int d^3x \, x^i \partial_0 T^{00} = -\int d^3x \, x^i \partial_j T^{0j}$$
$$= \int d^3x \partial_j x^i \, T^{0j} = \int d^3x \, T^{0i} = c \, P^i$$

Solution

(iii) $\dot{M}^{ij} = P^{i,j} + P^{j,i}$

$$\dot{M}_{ij} = \frac{1}{c}\int d^3x \partial_0 T^{00} x^i x^j$$
$$= -\frac{1}{c}\int d^3x \partial_k T^{0k} x^i x^j$$
$$= \frac{2}{c}\int d^3x T^{0k} \partial_k x^{(i} x^{j)}$$
$$= P^{i,j} + P^{j,i}$$

(iv) $\dot{P}^i = 0$ This is the conservation of momentum

(v) $\dot{P}^{i,j} = S^{ij}$ [ij] -> conservation of angular momentum

(vi) $\dot{P}^{i,jk} = S^{ij,k} + S^{ik,j}$

We can combine these identities to express S^{ij}, $S^{ij,k}$ in terms of the mass and momentum multipole moments !

$$\dot{M}^{ij} = P^{i,j} + P^{j,i}$$

$$\dot{P}^{i,j} = S^{ij}$$

$$\longrightarrow \quad S^{ij} = \frac{1}{2}\ddot{M}^{ij}$$

The leading term of

$$h_{ij}^{TT}(t,x) = \frac{1}{r}\frac{4G}{c}\Lambda_{ij,kl}(\hat{x})[S^{kl} + \frac{1}{c}\hat{x}_m\dot{S}^{kl,m} + \frac{1}{2c^2}\hat{x}_m\hat{x}_p\ddot{S}^{kl,mp} + \dots]_{ret}$$

becomes

$$h_{ij}^{TT}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\vec{x})\ddot{M}^{kl}(t-\frac{r}{c})$$

The acceleration of mass quadrupole sources gravitational waves.

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \qquad P_{ij} \equiv \delta_{ij} - n_i n_j$$

Radiated energy

$$\frac{dE}{dtd\Omega} = \frac{c^3 r^2}{32\pi G}\langle \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT}\rangle$$

$$h_{ij}^{TT}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\vec{x})\ddot{M}^{kl}(t-\frac{r}{c})$$

$$= \frac{G}{8\pi c^5}\Lambda_{ij,kl}(\hat{n})\langle \dddot{M}_{ij}\dddot{M}_{kl}\rangle_{ret}$$

$$= \frac{G}{8\pi c^5}\Lambda_{ij,kl}(\hat{n})\langle \dddot{Q}_{ij}\dddot{Q}_{kl}\rangle_{ret}$$

$$Q_{ij} = M_{ij} - \frac{1}{3}\delta_{ij}M_{kk}$$

Angular integral can be done

$$\int d\Omega \Lambda_{ij,kl} = \frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$$

Exercise later on

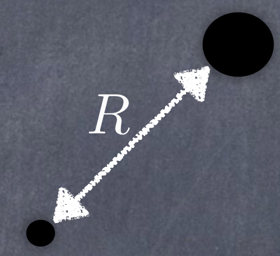
$$\longrightarrow \quad \frac{dE}{dt} = \frac{G}{5c^5}\langle \dddot{Q}_{ij}(t-\frac{r}{c})\dddot{Q}_{ij}(t-\frac{r}{c})\rangle$$

That is the Einstein quadrupole formula

By conservation of energy,

$$\frac{dE^{source}}{dt} = -\frac{dE}{dt}$$

For a binary source,



$$E^{source}_{0PN} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{G m_1 m_2}{R}$$

Effective one-body reduction

Reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Total mass $m = m_1 + m_2$

$$E^{source}_{0PN} = \frac{1}{2}m v^2_{c.m.} + \frac{1}{2}\mu \dot{R}^2 - \frac{G m \mu}{R}$$

$$\dot{R}^2 = \frac{Gm}{R}$$

Kepler law

$$E^{source}_{0PN} = \frac{1}{2}m v^2_{c.m.} - \frac{G m \mu}{2R}$$

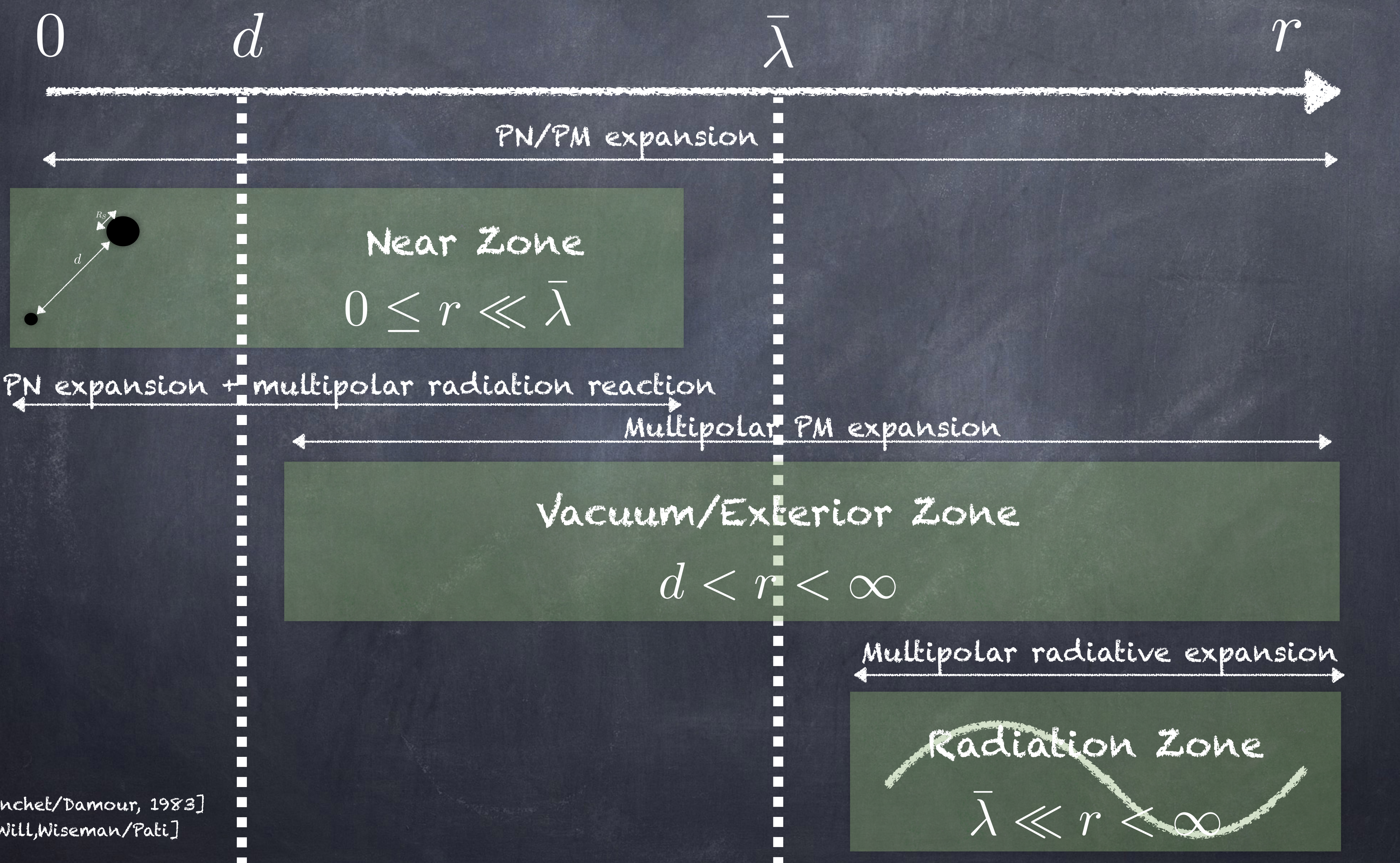
$$\frac{dE^{source}}{dt} = \frac{G m \mu}{2R^2}\dot{R}$$

$$\frac{dE}{dt} = \frac{G}{5c^5}\langle \dddot{Q}_{ij}(t - \frac{r}{c})\dddot{Q}_{ij}(t - \frac{r}{c})\rangle$$

$$\dot{R} \sim c^{-5}$$

The gravitational wave emission brings as first contribution a 2.5PN correction to the motion. It is called the radiation reaction.

The 3 zones of the Post-Newtonian/ Post-Minkowskian formalism



[Blanchet/Damour, 1983]
[Will,Wiseman/Pati]

$$h_{ij}^{TT}(t,x) = \frac{1}{r}\frac{4G}{c}\Lambda_{ij,kl}(\hat{x})[S^{kl} + \frac{1}{c}\hat{x}_m\dot{S}^{kl,m} + \frac{1}{2c^2}\hat{x}_m\hat{x}_p\ddot{S}^{kl,mp} + \dots]_{ret}$$

2.5PN 3.5PN

$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}\boldsymbol{n}_{12} + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \cdots\right]\boldsymbol{n}_{12} + \cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}} + \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN} \\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN} \\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN} \\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN

- [Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab] — ADM Hamiltonian
- [Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002] — Harmonic EOM
- [Itoh & Futamase 2003; Itoh 2004] — Surface integral method
- [Foffa & Sturani 2011] — Effective field theory

4PN

- [Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014] — ADM Hamiltonian
- [Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017abc] — Fokker Lagrangian
- [Foffa & Sturani 2013, 2019; Foffa, Porto, Rothstein & Sturani 2019] — Effective field theory

[review of Blanchet]

Exercise

Prove

$$\int d\Omega \Lambda_{ij,kl} = \frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$$

where

$$\Lambda_{ij,kl}(\hat{n}) \equiv P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \qquad P_{ij} \equiv \delta_{ij} - n_i n_j$$

$$n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

(i) Prove $$\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin|\vec{k}|}{|\vec{k}|}$$

(ii) Prove $$\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = 4\pi \lim_{|\vec{k}|\to 0} i^\ell \frac{\partial^\ell}{\partial k^{i_1}\cdots k^{i_\ell}}\left(\frac{\sin|\vec{k}|}{|\vec{k}|}\right)$$

(iii) Prove $$\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = \frac{4\pi}{\ell+1}\delta_{(i_1 i_2}\delta_{i_3 i_4}\cdots\delta_{i_{\ell-1}i_\ell)}$$ for ℓ even

(iv) Prove $$\int_S d\Omega \Lambda_{ij,kl} = \frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$$

Solution

(i) Prove $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin|\vec{k}|}{|\vec{k}|}$ $\quad n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

By SO(3) symmetry, we can assume that $\vec{k}$ points in the z direction

$$\int_S d\theta d\phi \sin\theta e^{-ik_z\cos\theta} = -2\pi \int_1^{-1} d\cos\theta e^{-ik_z\cos\theta}$$
$$= \frac{2\pi}{ik_z}(e^{+ik_z} - e^{-ik_z}) = \frac{4\pi\sin k_z}{k_z}$$

Since the right-hand side is SO(3) invariant (scalar), it is af function of the norm of $\vec{k}$.

Solution

(i) Prove $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin|\vec{k}|}{|\vec{k}|}$ $\qquad n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

(ii) Prove $\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = 4\pi \lim_{|\vec{k}|\to 0} i^\ell \frac{\partial^\ell}{\partial k^{i_1}\cdots k^{i_\ell}}\left(\frac{\sin|\vec{k}|}{|\vec{k}|}\right)$ $\qquad$ trivial

Solution

(i) Prove $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin|\vec{k}|}{|\vec{k}|}$ $\qquad n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

(ii) Prove $\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = 4\pi \lim_{|\vec{k}|\to 0} i^\ell \frac{\partial^\ell}{\partial k^{i_1}\cdots k^{i_\ell}}\left(\frac{\sin|\vec{k}|}{|\vec{k}|}\right)$ trivial

(iii) Prove $\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = \frac{4\pi}{\ell+1}\delta_{(i_1 i_2}\delta_{i_3 i_4}\cdots\delta_{i_{\ell-1}i_\ell)}$

Use Taylor series

$$\frac{\sin|\vec{k}|}{|\vec{k}|} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}|\vec{k}|^{2n}$$

Only one term survives: $2n = \ell$ $\longrightarrow$ $i^\ell = (-1)^n$, $\quad \frac{\ell!}{(2n+1)!} = \frac{1}{2n+1}$

Solution

(i) Prove $\int_S d\Omega e^{-i\vec{k}\cdot\vec{n}} = 4\pi \frac{\sin|\vec{k}|}{|\vec{k}|}$ $\quad n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

(ii) Prove $\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = 4\pi \lim_{|\vec{k}|\to 0} i^\ell \frac{\partial^\ell}{\partial k^{i_1}\cdots k^{i_\ell}}\left(\frac{\sin|\vec{k}|}{|\vec{k}|}\right)$ trivial

(iii) Prove $\int_S d\Omega\, n_{i_1}\cdots n_{i_\ell} = \frac{4\pi}{\ell+1}\delta_{(i_1 i_2}\delta_{i_3 i_4}\cdots\delta_{i_{\ell-1}i_\ell)}$

Use Taylor series

$$\frac{\sin|\vec{k}|}{|\vec{k}|} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}|\vec{k}|^{2n}$$

Only one term survives: $2n = \ell$ ⟶ $i^\ell = (-1)^n$, $\quad \frac{\ell!}{(2n+1)!} = \frac{1}{2n+1}$

(iv) Prove $\int_S d\Omega\Lambda_{ij,kl} = \frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$

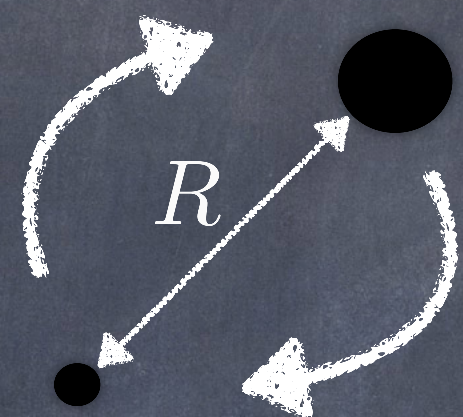
$\frac{4!}{2\cdot 2\cdot 2} = 3$ ↓

Use $\int_S d\Omega 1 = 4\pi,\quad \int_S d\Omega n_i n_j = \frac{4\pi}{3}\delta_{ij}\quad \int_S d\Omega n_i n_j n_k n_l = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})$

1.4. Quasi-circular inspiral of compact binaries

- Neglect conservative PN corrections (Keplerian motion + radiation reaction)
- Assume quasi-circularity

Kepler law $\omega_s^2 = \frac{Gm}{R^3}$



Total Mass $m = m_1 + m_2$

Reduced Mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$

Law of motion given by the flux-balance law of energy:

$$\frac{dE^{source}}{dt} = -\frac{dE}{dt}$$

$$\frac{dE^{source}}{dt} = \frac{Gm\mu}{2R^2}\dot{R}$$

$$\frac{dE}{dt} = \frac{G}{5c^5}\langle \dddot{Q}_{ij}(t-\frac{r}{c})\dddot{Q}_{ij}(t-\frac{r}{c})\rangle$$

Exercise

Obtain the quadrupole radiation from a mass in a circular orbit

(i) Starting from the relativistic expression

$$T^{\mu\nu} = \sum_{A=1,2} \int d\tau_A m_A \frac{dx^\mu_A}{d\tau} \frac{dx^\nu_A}{d\tau} \delta^{(4)}(x - x_A(\tau_A))$$

prove that at Newtonian order:

$$T^{00}(t, \vec{x}) = \sum_{A=1,2} m_A c^2 \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

(ii) Deduce the quadrupole formula

$$Q^{ij}(t) = \left[\frac{1}{c^2} \int d^3x T^{00}(t, \vec{x}) x^i x^j\right]^{STF} = \mu \left(x^i_0(t) x^j_0(t) - \frac{1}{3} r^2_0(t) \delta^{ij} \right)$$

where $\vec{x}_0 \equiv \vec{x}_1 - \vec{x}_2$

Solution

(i)

$$T^{\mu\nu} = \sum_{A=1,2} \int d\tau_A m_A \frac{dx^\mu_A}{d\tau_A} \frac{dx^\nu_A}{d\tau_A} \delta^{(4)}(x - x_A(\tau_A))$$

$$c^2 d\tau_A^2 = -\eta_{\mu\nu} dx^\mu_A dx^\nu_A = c^2(1 - \frac{v_A^2}{c^2})dt^2 = c^2 \gamma_A^{-2} dt^2 \longrightarrow d\tau_A = \frac{dt}{\gamma_A}$$

$$T^{\mu\nu} = \sum_{A=1,2} \gamma_A m_A \frac{dx^\mu_A}{dt} \frac{dx^\nu_A}{dt} \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

$$T^{00}(t, \vec{x}) = \sum_{A=1,2} \gamma_A m_A c^2 \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

Newtonian limit:

$$T^{00}(t, \vec{x}) = \sum_{A=1,2} m_A c^2 \delta^{(3)}(\vec{x} - \vec{x}_A(t))$$

(ii)

$$T^{00}(t,\vec{x}) = \sum_{A=1,2} m_A c^2 \delta^{(3)}(\vec{x}-\vec{x}_A(t))$$

$$M^{ij} = \frac{1}{c^2}\int d^3x T^{00}(t,\vec{x}) x^i x^j$$

$$M^{ij} = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = m x^i_{c.m.} x^j_{c.m.} + \mu x_0^i x_0^j$$

$$\vec{x}_0 \equiv \vec{x}_1 - \vec{x}_2 \qquad \vec{x}_{c.m.} \equiv \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1+m_2} = 0$$

$$Q^{ij}(t) = \left[\frac{1}{c^2}\int d^3x T^{00}(t,\vec{x}) x^i x^j\right]^{STF}$$

$$Q^{ij}(t) = \mu\left(x_0^i(t) x_0^j(t) - \frac{1}{3} r_0^2(t)\delta^{ij}\right)$$

Explicit expression

Assume orbital motion is exactly circular (not elliptic)

$$z = 0$$

$$\begin{cases} x_0(t) = R\cos(\omega_s t + \frac{\pi}{2}) \\ y_0(t) = R\sin(\omega_s t + \frac{\pi}{2}) \\ z_0(t) = 0 \end{cases}$$

$$M^{ij} = \mu x_0^i(t) x_0^j(t)$$

$$M^{11} = \mu R^2 \cos^2(\omega_s t + \frac{\pi}{2})$$

$$M^{12} = \mu R^2 \sin(\omega_s t + \frac{\pi}{2}) \cos(\omega_s t + \frac{\pi}{2})$$

$$M^{22} = \mu R^2 \sin^2(\omega_s t + \frac{\pi}{2})$$

Metric perturbation

$$h_{ij}^{TT}(t,\vec{x}) = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\vec{x})\ddot{M}^{kl}(t-\frac{r}{c})$$

For a generic direction $\hat{n} = (\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta)$

we find

$$\begin{aligned} h_+(t,\theta,\phi) = \frac{1}{r}\frac{G}{c^4}[&\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta)] \\ &+ \ddot{M}_{22}(\sin^2\phi - \cos^2\phi\cos^2\theta) \\ &- \ddot{M}_{33}\sin^2\theta \\ &- \ddot{M}_{12}\sin 2\phi(1+\cos^2\theta) \\ &+ \ddot{M}_{13}\sin\phi\sin 2\theta \\ &+ \ddot{M}_{23}\cos\phi\sin 2\theta] \end{aligned}$$

$$\begin{aligned} h_\times(t,\theta,\phi) = \frac{1}{r}\frac{G}{c^4}[&(\ddot{M}_{11} - \ddot{M}_{22})\sin 2\phi\cos\theta] \\ &+ 2\ddot{M}_{12}\cos 2\phi\cos\theta \\ &- 2\ddot{M}_{13}\cos\phi\sin\theta \\ &+ 2\ddot{M}_{23}\sin\phi\sin\theta] \end{aligned}$$

$$h_+(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{ret} + 2\phi)$$

$$h_\times(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\cos\theta\sin(2\omega_s t_{ret} + 2\phi)$$

Quadrupole radiation is at twice the frequency of the source.

$$\omega_{gw} \sim 2\omega_s$$ [as announced earlier]

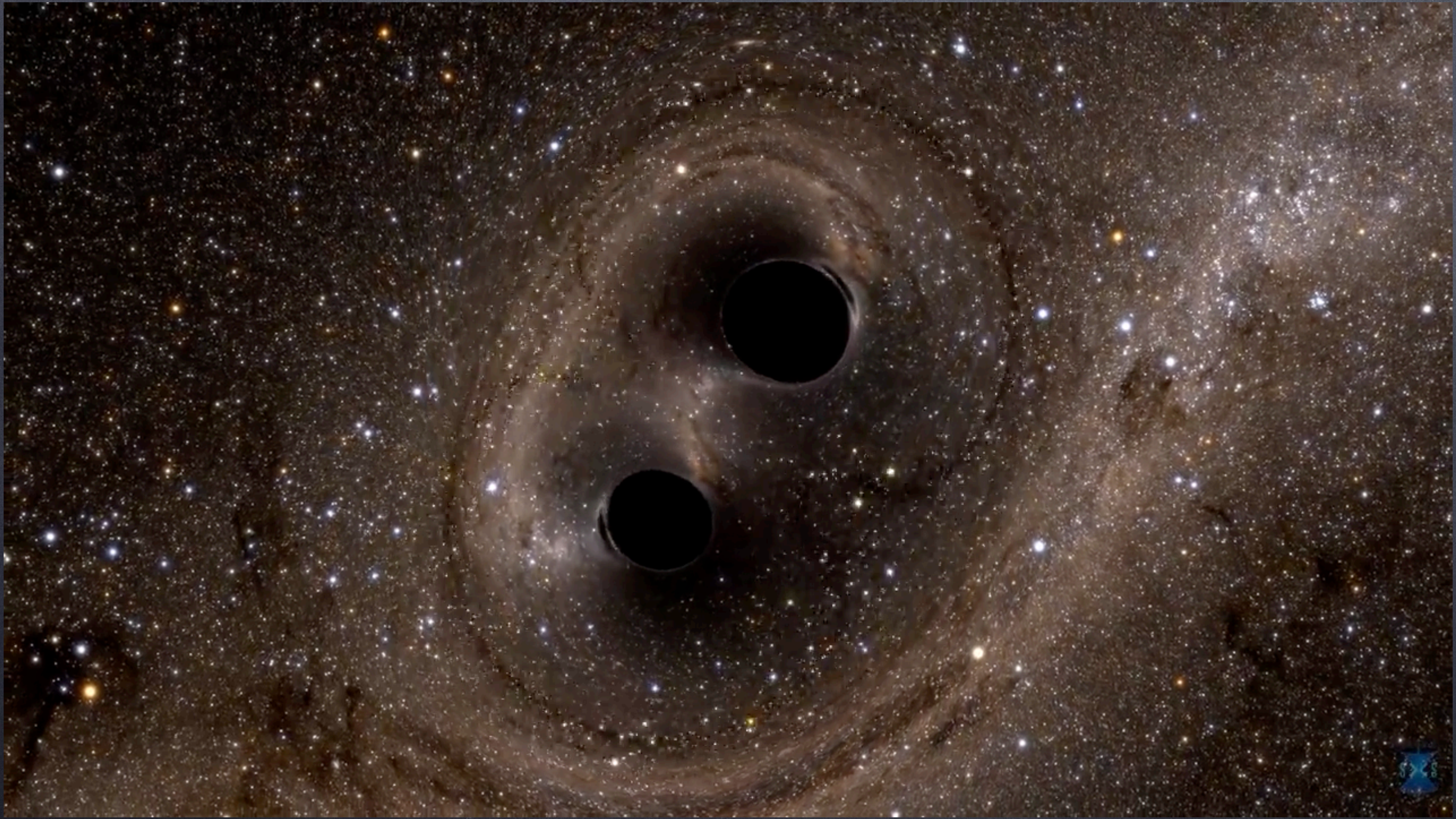
Helicoidal structure

Performing a rotation is equivalent to shifting time

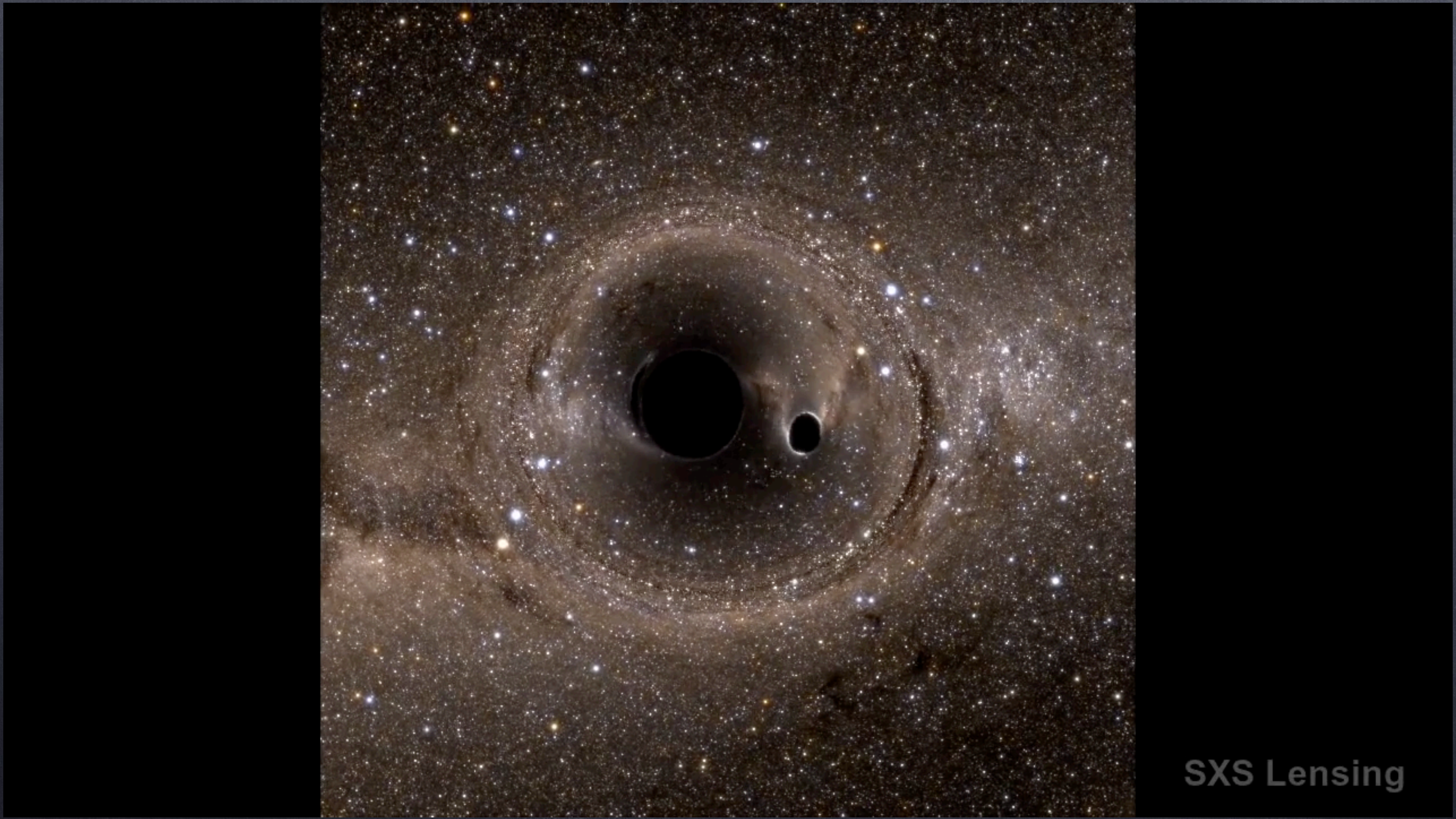
The angle θ is the angle between the normal to the orbit and the line-of-sight

The distance r is a constant

Face-on $\theta = 0$



Edge-on $\theta = \frac{\pi}{2}$



Compute the angular distribution of radiated power in the quadrupole approximation

$$\frac{dE}{dtd\Omega} = \frac{r^2 c^3}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$

using

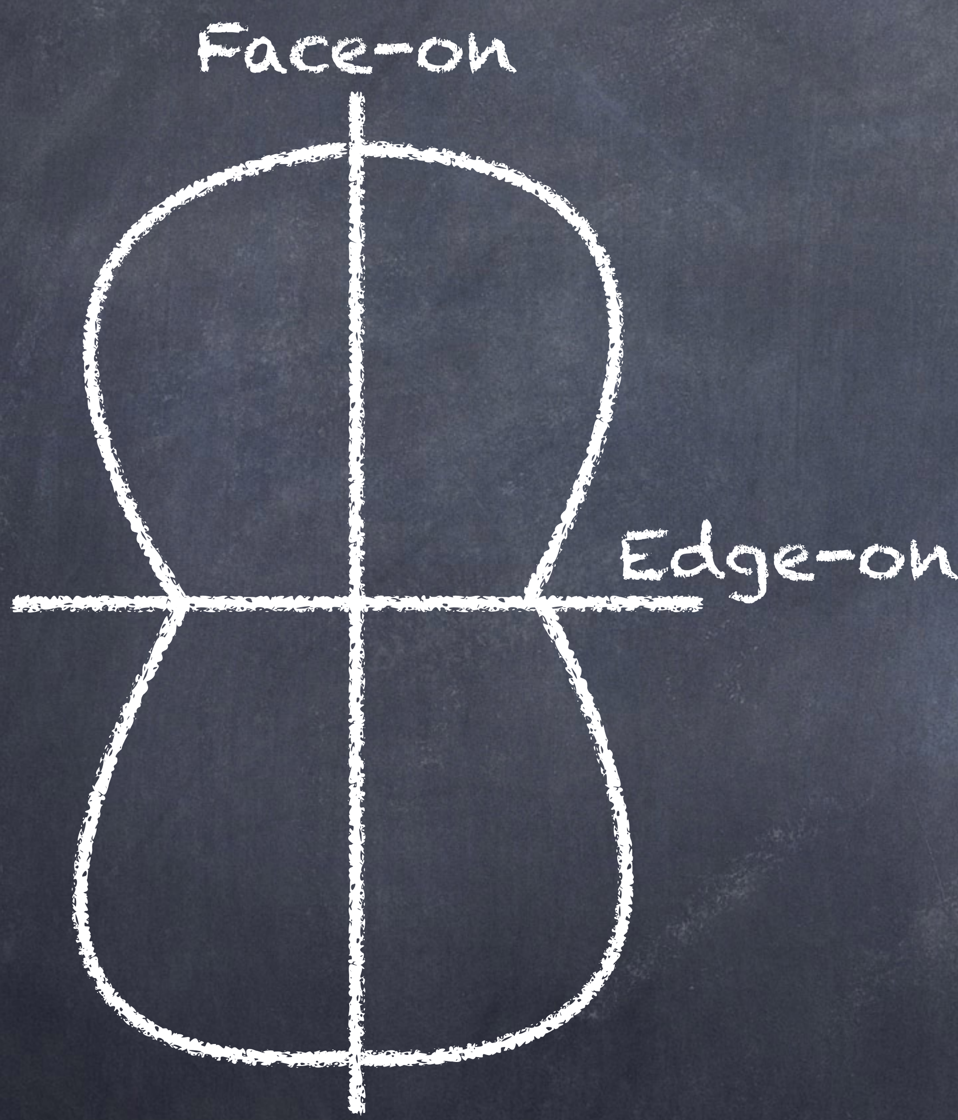
$$h_+(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}(\frac{1+\cos^2\theta}{2})\cos(2\omega_s t_{ret} + 2\phi)$$

$$h_\times(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2 R^2}{c^4}\cos\theta\sin(2\omega_s t_{ret} + 2\phi)$$

Use

$$\langle \cos^2(2\omega_s t_{ret} + \phi)\rangle = \langle \sin^2(2\omega_s t_{ret} + \phi)\rangle = \frac{1}{2}$$

$$\frac{dE}{dt d\Omega} = \frac{2G\mu^2 R^4 \omega_s^6}{\pi c^5} g(\theta)$$



$$g(\theta) = (\frac{1+\cos^2\theta}{2})^2 + \cos^2\theta$$

$$\int \frac{d\Omega}{4\pi} g(\theta) = \frac{4}{5}$$

$$P_{rad} = \frac{dE}{dt} = \frac{32}{5}\frac{G\mu^2}{c^5} R^4 \omega_s^6 = \frac{32}{5}\frac{G\mu^2}{R^2}(\frac{v}{c})^6$$

Inspiral motion to coalescence as a consequence of energy flux-balance

Emission of GW costs energy, taken from the system's total energy

$$\frac{dE^{source}}{dt} = \frac{Gm\mu}{2R^2}\dot{R} \qquad \frac{dE^{source}}{dt} = -\frac{dE}{dt}$$

⟶ R decreases with time

⟶ $\omega_s^2 = \frac{Gm}{R^3}$ increases with time

⟶ power radiated in GW increases

⟶ Runaway process leading to coalescence

Exercise The motion is quasi-circular as long as $\dot{R} \ll v$

Prove that it is equivalent to $\frac{\dot{f}_{gw}}{f_{gw}^2} \ll 1$

Solution

$$R = \left(\frac{Gm}{\omega_s^2}\right)^{1/3}$$ Kepler's law

$$\dot{R} = -\frac{2}{3} R \frac{\dot{\omega}_s}{\omega_s}$$
$$= -\frac{2}{3} (\omega_s R) \frac{\dot{\omega}_s}{\omega_s^2}$$
$$= -\frac{2}{3} v \frac{\dot{\omega}_s}{\omega_s^2}$$

$$\dot{R} \ll v \Leftrightarrow \frac{\dot{\omega}_s}{\omega_s^2} \ll 1 \Leftrightarrow \frac{\dot{\omega}_{gw}}{\omega_{gw}^2} \ll 1 \Leftrightarrow \frac{\dot{f}_{gw}}{f_{gw}^2} \ll 1$$

Exercise

Rewrite

$$h_+(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}(\frac{1+\cos^2\theta}{2})\cos(2\omega_s t_{ret}+2\phi)$$

$$h_\times(t,\theta,\phi) = \frac{1}{r}\frac{4G\mu\omega_s^2R^2}{c^4}\cos\theta\sin(2\omega_s t_{ret}+2\phi)$$

using

$$R = (\frac{Gm}{\omega_s^2})^{1/3} \qquad \omega_s = \frac{1}{2}\omega_{gw} = \frac{1}{2}2\pi f_{gw} = \pi f_{gw}$$

and the chirp mass

$$M_c = \mu^{3/5}m^{2/5} = \frac{(m_1m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

Keep M_c , m and f_{gw}. What do you notice?

Solution

$$h_+(t) = \frac{4}{r}(\frac{GM_c}{c^2})^{5/3}(\frac{\pi f_{gw}}{c})^{2/3}\frac{1+\cos^2\theta}{2}\cos(2\pi f_{gw}t_{ret}+2\phi)$$

$$h_\times(t) = \frac{4}{r}(\frac{GM_c}{c^2})^{5/3}(\frac{\pi f_{gw}}{c})^{2/3}\cos\theta\sin(2\pi f_{gw}t_{ret}+2\phi)$$

All dependence in the mass is through the chirp mass at this lowest PN order!

Power radiated:

$$\frac{dP}{d\Omega} = \frac{2}{\pi}\frac{c^5}{G}(\frac{GM_c\omega_{gw}}{2c^3})^{10/3}g(\theta)$$

$$P = \frac{32}{5}\frac{c^5}{G}(\frac{GM_c\omega_{gw}}{2c^3})^{10/3}$$

Exercise

(i) Prove

$$E^{source} = -\frac{G\mu m}{2R} = -\left(\frac{G^2 M_c^5 \omega_{gw}^2}{32}\right)^{1/3}$$

using

$$\omega_{gw} = 2\omega_s \qquad R = \left(\frac{Gm}{\omega_s^2}\right)^{1/3} \qquad M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

(ii) Derive the evolution of the GW frequency $f_{gw} = \frac{\omega_{gw}}{2\pi}$

using the energy flux-balance law with GW power

$$P = \frac{32}{5}\frac{c^5}{G}\left(\frac{G M_c \omega_{gw}}{2c^3}\right)^{10/3}$$

Solution

$$\dot{f}_{gw} = \frac{96}{5}\pi^{8/3}\left(\frac{GM_c}{c^3}\right)^{5/3} f_{gw}^{11/3}$$

It integrates to

$$f_{gw}(t) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{t_{coal}-t}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$

time of coalescence

Validity of the quasi-circularity hypothesis:

$$\frac{\dot{f}_{gw}}{f_{gw}^2} \ll 1 \Leftrightarrow \left(\frac{96}{5}\pi^{8/3}\right)^{3/5}\frac{GM_c}{c^3} f_{gw} \ll 1$$

To get the waveform, we need the orbital motion with 2.5PN radiation-reaction (Matching between Near Zone and Exterior zone)

$z = 0$

$$\begin{cases} x_0(t) = R\cos(\omega_s t + \frac{\pi}{2}) \longrightarrow R(t)\cos\frac{\Phi(t)}{2} \\ y_0(t) = R\sin(\omega_s t + \frac{\pi}{2}) \longrightarrow R(t)\sin\frac{\Phi(t)}{2} \\ z_0(t) = 0 \end{cases}$$

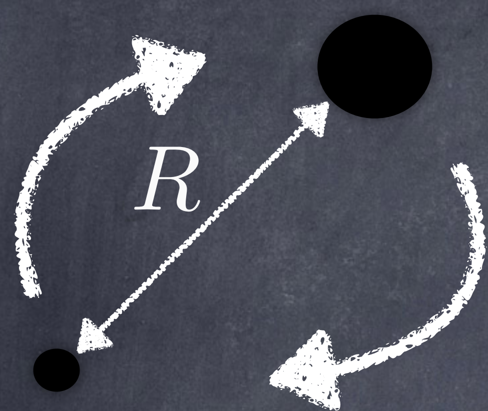
where

$$\Phi(t) = \int_{t_0}^{t} dt' \omega_{gw}(t') = 2\int_{t_0}^{t} dt' \omega_s(t')$$

$$\Phi(t) = \Phi_0 - 2\left(\frac{5GM_c}{c^3}\right)^{-5/8} (t_{coal} - t)^{5/8}$$

-> Quadrupole moment/Waveform gets corrections.

Quadrupole moment with radiation-rection



$$M^{ij} = \mu x_0^i(t) x_0^j(t)$$

$$M^{11} = \mu R^2 \cos^2(\omega_s t + \frac{\pi}{2})$$

$$M^{12} = \mu R^2 \sin(\omega_s t + \frac{\pi}{2}) \cos(\omega_s t + \frac{\pi}{2})$$

$$M^{22} = \mu R^2 \sin^2(\omega_s t + \frac{\pi}{2})$$

Replace

$$R \to R(t)$$

$$\omega_s t + \frac{\pi}{2} \to \Phi(t)/2$$

However, $\dot{R}$ is negligible as long as the orbit is quasi-circular. Therefore, we can ignore these terms.

Therefore, the waveform $h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r} \frac{2G}{c^4} \Lambda_{ij,kl}(\vec{x}) \ddot{M}^{kl}(t - \frac{r}{c})$

is simply obtained by making the replacement $\omega_s t + \frac{\pi}{2} \to \Phi(t)/2$

Final 0PN+2.5PN rad-react/1PM GW waveform

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}(t_{\cancel{ret}})}{c}\right)^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos\Phi(t_{\cancel{ret}})$$

$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}(t_{\cancel{ret}})}{c}\right)^{2/3}(\cos\theta)\sin\Phi(t_{\cancel{ret}})$$

$$\Phi(t) = \Phi_0 - 2\left(\frac{5GM_c}{c^3}\right)^{-5/8}(t_{coal}-t)^{5/8}$$

where

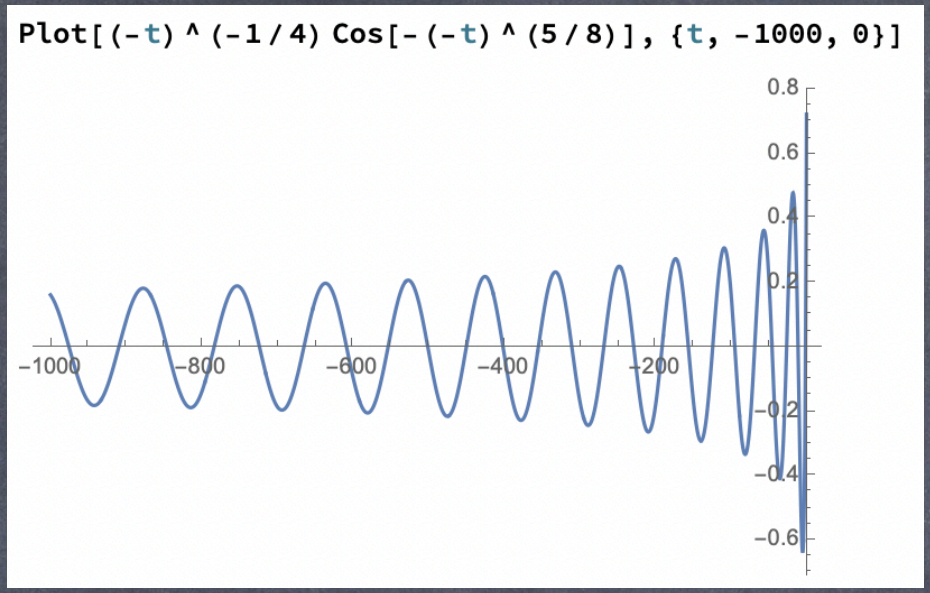
$$f_{gw}(t) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{t_{coal}-t}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$

The retardation is a fixed time shift. Instead we express the waveform as a function of time before coalescence.

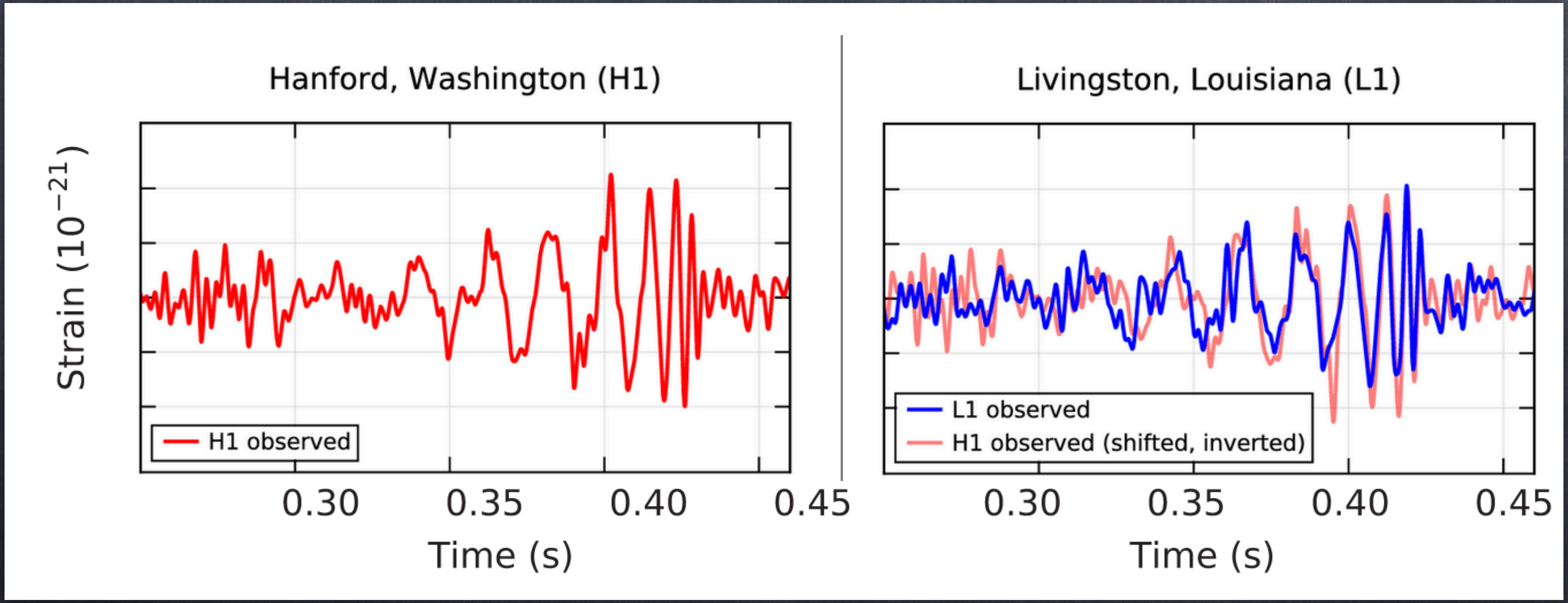
$$h_+(t) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c(t_{coal}-t)}\right)^{1/4}\left(\frac{1+\cos^2\theta}{2}\right)\cos\Phi(t_{coal}-t)$$

$$h_\times(t) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c(t_{coal}-t)}\right)^{1/4}\cos\theta\sin\Phi(t_{coal}-t)$$

Compare 0PN+2.5PN rad-react/1PM GW waveform

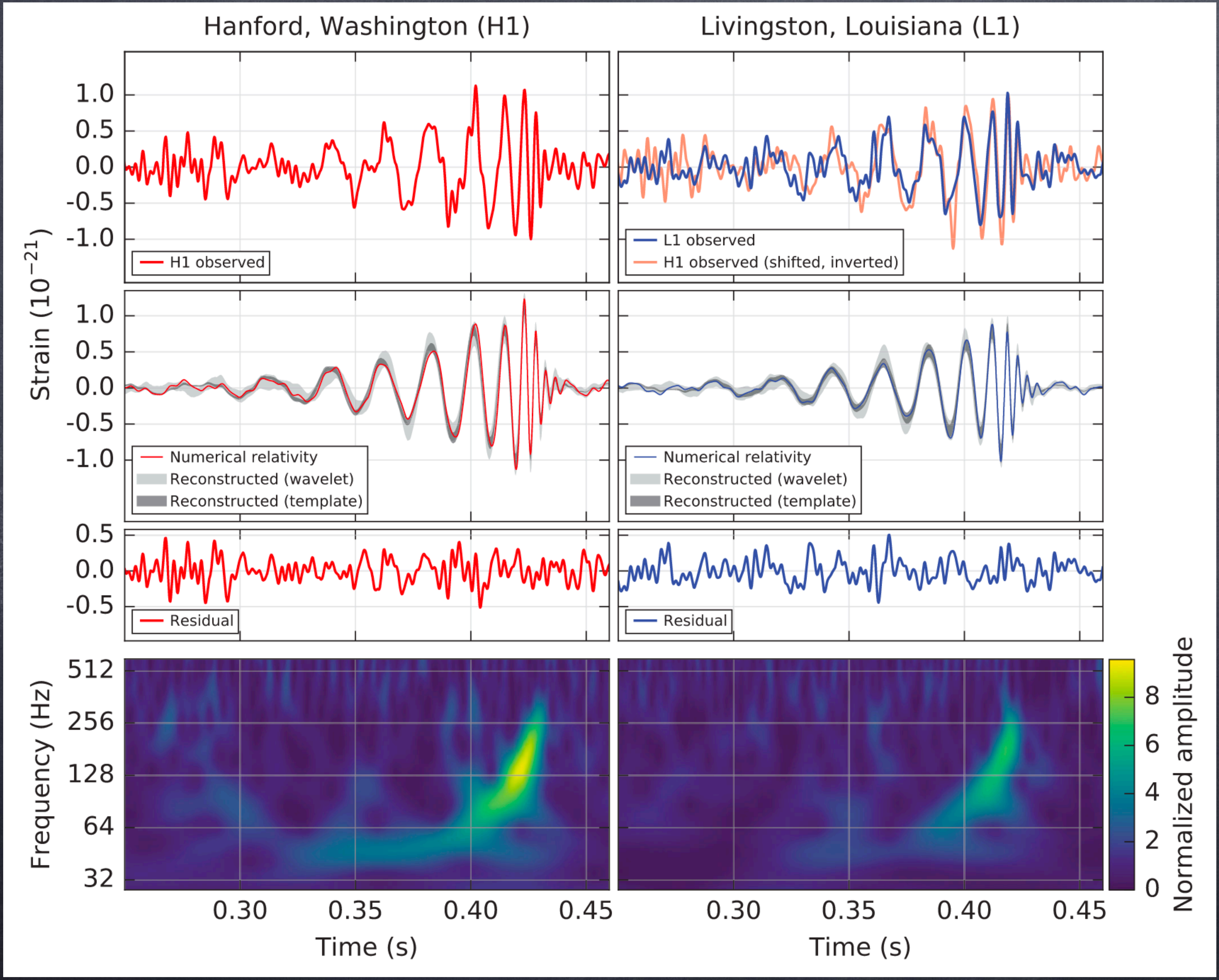


with GW150914



What do you observe?

Actual GW template used



Exercise

Match with GW150914

Estimated source parameters

Quantity	Value	Upper/Lower error estimate	Unit
Primary black hole mass	**36.2**	+5.2 -3.8	M sun
Secondary black hole mass	**29.1**	+3.7 -4.4	M sun
Final black hole mass	**62.3**	+3.7 -3.1	M sun
Final black hole spin	**0.68**	+0.05 -0.06	
Luminosity distance	**420**	+150 -180	Mpc
Source redshift, *z*	**0.09**	+0.03 -0.04	
Energy radiated	**3.0**	+0.5 -0.5	M sun

$$\frac{2GM_\odot}{c^2} = 3 \text{ km}$$

$$1 \text{ pc} = 3.3 \text{ light-years}$$

$$1 \text{ light-year} = 9.5 \times 10^{15} m$$

$$M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

(i) Obtain the GW amplitude 0.2 sec before merger

$$A(t_{coal} - t) = \frac{1}{r}\left(\frac{GM_c}{c^2}\right)^{5/4}\left(\frac{5}{c(t_{coal} - t)}\right)^{1/4}$$

(ii) Obtain the GW frequency 0.2 sec before merger

$$f_{gw}(t_{coal} - t) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{t_{coal} - t}\right)^{3/8}\left(\frac{GM_c}{c^3}\right)^{-5/8}$$

Solution

(i)

$$M_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} \longrightarrow M_c = 28 M_\circ$$

$$\frac{2GM_\circ}{c^2} = 3 \text{ km} \longrightarrow \frac{GM_c}{c^2} = 42 \text{ km}$$

$$420 \text{ Mpc} = 1.3 \times 10^{22} \text{ km} \qquad \frac{(42)^{5/4} 5^{1/4}}{1.3 \times 10^{22} (3 \times 10^5)^{1/4} (0.2)^{1/4}} = 7.8 \times 10^{-22}$$

$$\mathcal{A}(t) = 7.8 \times 10^{-22} \left(\frac{420 \text{ Mpc}}{r}\right) \left(\frac{M_c}{28 M_\circ}\right)^{5/4} \left(\frac{0.2 \text{ sec}}{t_{coal} - t}\right)^{1/4}$$

(ii)

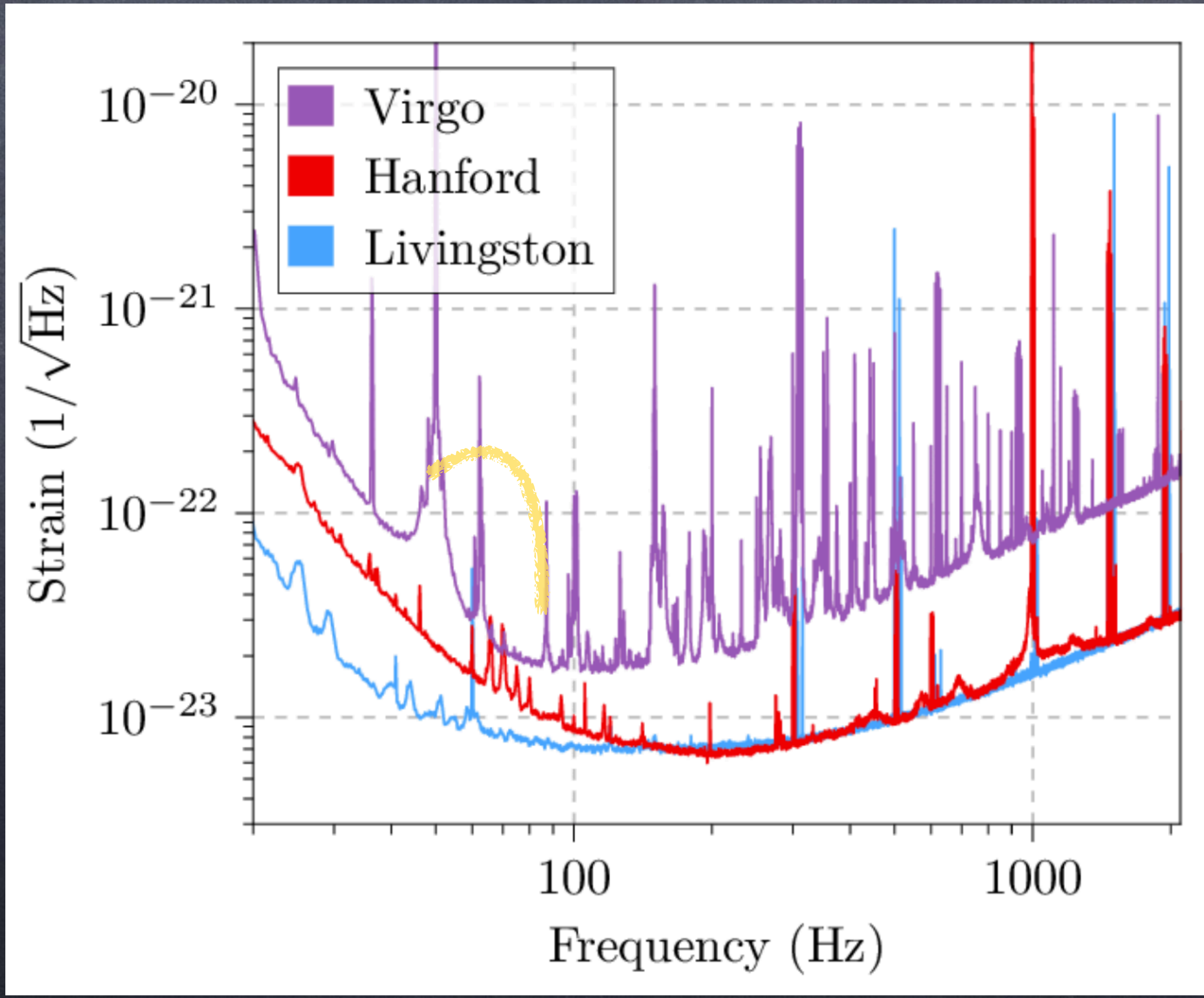
$$\frac{1}{\pi} \left(\frac{5}{256}\right)^{3/8} \frac{42^{-5/8} (3 \times 10^5)^{5/8}}{0.2^{3/8}} = 34$$

$$f_{gw}(t) = 34 \,\text{Hz} \left(\frac{M_c}{28 M_\circ}\right)^{-5/8} \left(\frac{0.2 \text{ sec}}{t_{coal} - t}\right)^{3/8}$$

Model versus LIGO sensitivity

$$\mathcal{A}(t) = 7.8 \times 10^{-22} \left(\frac{420\ \text{Mpc}}{r}\right) \left(\frac{M_c}{28 M_\odot}\right)^{5/4} \left(\frac{0.2\ \text{sec}}{t_{coal} - t}\right)^{1/4}$$

$$f_{gw}(t) = 34\,\text{Hz} \left(\frac{M_c}{28 M_\odot}\right)^{-5/8} \left(\frac{0.2\ \text{sec}}{t_{coal} - t}\right)^{3/8}$$



Limit of the PN/PM formalism

The PN/PM formalism leads to a perturbation series around a Keplerian orbit. After taking into account the backreaction of the central black hole, relativistic effects come in.

Schwarzschild admits an innermost stable circular orbit (ISCO)

$$r_{ISCO} = \frac{6Gm}{c^2}$$

The quasi-circularity breaks down because there are no more stable circular orbits beyond that radius. This regime is called the transition to merger. The maximal frequency where the PN/PM formalism is valid can be approximated to

$$f_{gw}^{MAX} = 2f_s = \frac{2\omega_s}{2\pi} = \frac{1}{\pi}\sqrt{\frac{Gm}{R^3}} = \frac{1}{6\sqrt{6}\pi}\frac{c^3}{Gm} \sim 34 \text{ Hz}\frac{65M_\circ}{m}$$

Note that it occurs before the end of the quasi-circularity hypothesis computed earlier:

$$f_{gw}^{MAX,QC} = \left(\frac{96}{5}\pi^{8/3}\right)^{-3/5}\frac{c^3}{GM_c} \sim 194\text{Hz}\frac{28M_\circ}{M_c}$$

1.5. Post-Newtonian corrections

Post-Newtonian corrections to the motion

$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}\boldsymbol{n}_{12} + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \frac{4G^2m_2^2}{r_{12}^3} + \cdots\right]\boldsymbol{n}_{12} + \cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}}$$
$$+ \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\ \text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

2PN : [Ohta et al.,1975],[Deruelle, Damour, 1981]

3PN
- [Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab] — ADM Hamiltonian
- [Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002] — Harmonic EOM
- [Itoh & Futamase 2003; Itoh 2004] — Surface integral method
- [Foffa & Sturani 2011] — Effective field theory

4PN
- [Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014] — ADM Hamiltonian
- [Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017abc] — Fokker Lagrangian
- [Foffa & Sturani 2013, 2019; Foffa, Porto, Rothstein & Sturani 2019] — Effective field theory

[review of Blanchet]

Newtonian sources

$$\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}\int d^4x' G(x-x')T_{\mu\nu}(x')$$

$$G(x-x') = \frac{-1}{4\pi|\vec{x}-\vec{x}'|}\delta(x^0_{ret} - x'^0_{ret})$$

For Newtonian sources: $T_{00} \gg |T_{0i}|, |T_{ij}|$

$$\bar{h}_{00} = -4\phi(\mathbf{x}) \qquad \bar{h}_{0i} = 0, \qquad \bar{h}_{ij} = 0.$$

$$\phi(\mathbf{x}) = -G\int \frac{c^{-3}T_{00}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x'$$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + (\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h})$$

$$g_{00} = -(1+2\phi), \qquad g_{0i} = 0, \qquad g_{ij} = (1-2\phi)\delta_{ij}$$

$$g_{00} = -(1+2\phi), \qquad g_{0i} = 0, \qquad g_{ij} = (1-2\phi)\delta_{ij}$$

$$\phi(\mathbf{x}) = -G \int \frac{c^{-3} T_{00}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x'$$

In the monopolar approximation,

$$\phi = -\frac{GM}{c^2 r} = -\frac{m}{r}$$

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 + \frac{2m}{r})(dx^2 + dy^2 + dz^2)$$

Multipolar expansion:

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} \stackrel{r>r'}{=} \frac{4\pi}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \left(\frac{r'}{r}\right)^l Y^*_{lm}(\hat{\mathbf{x}}') Y_{lm}(\hat{\mathbf{x}})$$

$$\phi(\mathbf{x}) = -4\pi G \sum_{l,m} \frac{Q^*_{lm}}{2l+1} \frac{1}{r^{l+1}} Y_{lm}(\hat{\mathbf{x}}), \qquad Q_{lm} = \int \rho(\mathbf{x}') r'^l Y_{lm}(\mathbf{x}') d^3x'$$

The "effacement" principle or "skeletonization"

[Brillouin, Levi-Civita, Damour]

"As far as gravitational effects are concerned, the internal structure of slowly moving bodies leads to very small corrections with respect to the point particle approximation."

Assume two nearly spherically symmetric bodies of linear dimension L, of mass M, separated by a distance R.

Self-gravitational force on one body: $\frac{GM}{L^2}$

Tidal gravitational force on one body caused by the other: $L\frac{d}{dr}\left(\frac{GM}{r^2}\right) = \frac{GM}{R^3}L$

Tidal distortion ellipticity : $\varepsilon \sim \frac{tidal}{self} \sim \frac{L^3}{R^3}$

Tidally induced quadrupole moment $J_2 = \varepsilon M L^2$

Force induced by the quadrupole : $J_2 \dfrac{GM}{R^3} \sim \dfrac{G\epsilon M^2 L^2}{R^4}$

$$\frac{\text{Force due to tidal forces}}{\text{Newtonian force}} \qquad \frac{\frac{G\epsilon M^2 L^2}{R^4}}{\frac{GM^2}{R^2}} \sim \frac{\varepsilon L^2}{R^2} \sim \left(\frac{L}{R}\right)^5$$

For a compact object, $L \sim \dfrac{GM}{c^2}$

Slow motion/Weak field: $\dfrac{v^2}{c^2} \sim \dfrac{GM}{Rc^2}$

$\longrightarrow$ $$\frac{\text{Force due to tidal forces}}{\text{Newtonian force}} \sim \left(\frac{v^2}{c^2}\right)^5$$ 5PN effect

Relativistic tidal deformability

$$\frac{1-g_{tt}}{2} = -\frac{m}{r} - \frac{3Q_{ij}}{2r^3}\left(n^i n^j - \frac{1}{3}\delta^{ij}\right) + O(r^{-4}) + \frac{1}{2}\mathcal{E}_{ij}x^i x^j + O(r^3),$$

Tidal deformability

$$\lambda \equiv -\frac{Q_{ij}}{\mathcal{E}_{ij}}.$$

Love number

$$\lambda = \frac{2}{3}k_2 R^5,$$

Dimensionless tidal deformability

$$\Lambda \equiv \frac{\lambda}{m^5}$$

It is 0 for black holes [due to hidden symmetries], non-zero for neutron stars.

GW depend at leading 5PN order only on the combination:

$$\tilde{\Lambda} \equiv \frac{16}{3}\frac{(m_1+12m_2)m_1^4\Lambda_1 + (m_2+12m_1)m_2^4\Lambda_2}{(m_1+m_2)^5},$$

[Flanagan, Hinderer, 2007]
[Favata, 2013]

Of interest to extract information on the composition of Neutron Stars.

Field equations in the PN approximation: the 1PN Einstein-Infeld-Hoffmann equations

board

1.6. Effective one-body resummation

board

Challenges within the PN/PM formalism

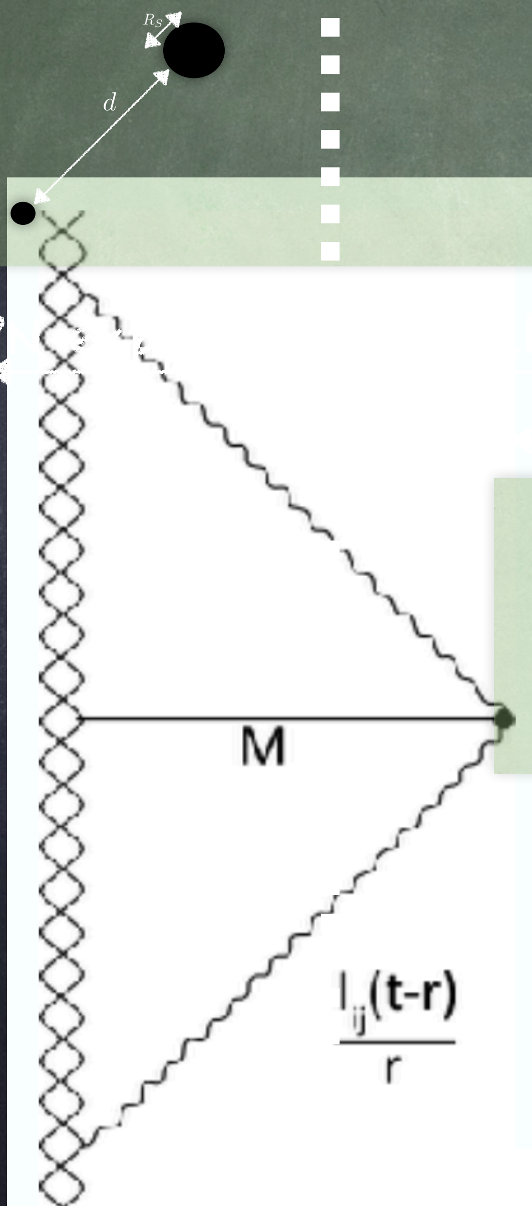
[Blanchet/Damour, 1988]

0 d $\bar{\lambda}$ r

PN/PM expansion

Near Zone

$$0 \leq r \ll \bar{\lambda}$$



P ... ltipolar radiation reaction

Multipolar PM expansion

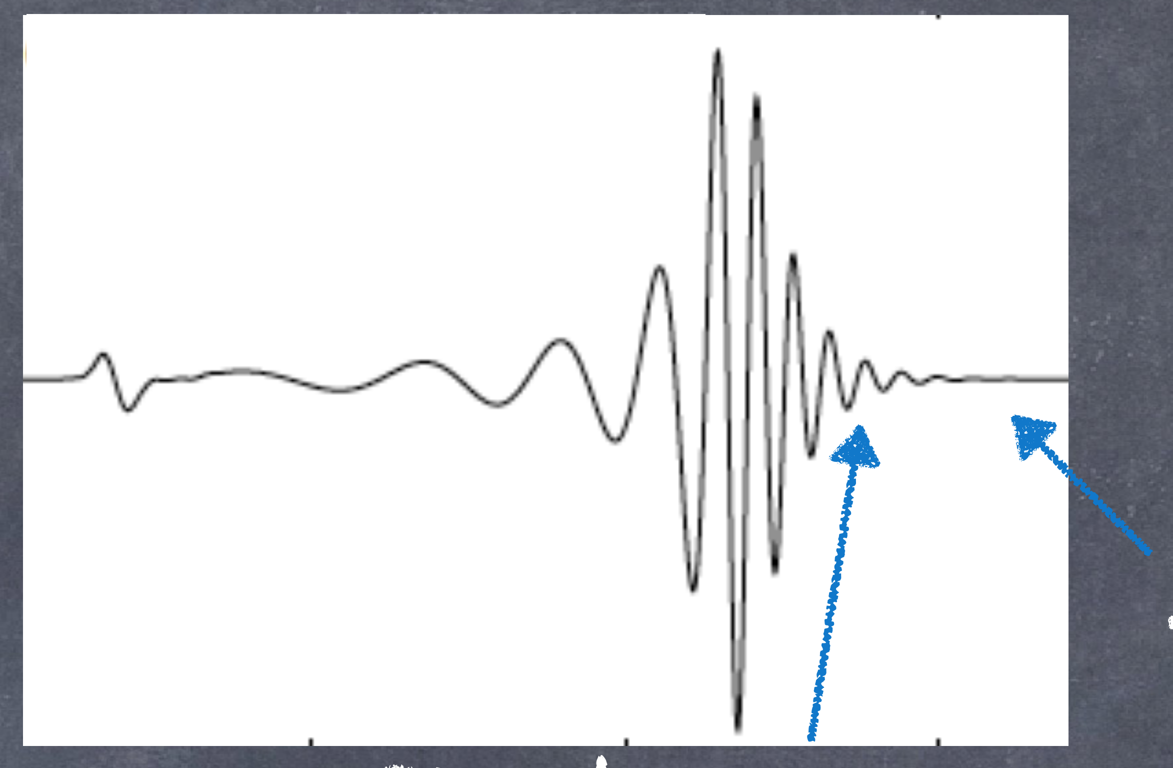
Vacuum/Exterior Zone

$$d < r < \infty$$

There is a "hereditary" PM2/PN4 term in the equation of motion of the source.
It is non-local in time.
The mathematics of such evolution equations is not yet fully understood.

Chapter 2. Black hole perturbation theory

Ringtones of black holes



We would like to know

- Frequencies (quasi-normal modes)
- Amplitudes (as a function of the binary source parameters)

Uniqueness theorems: Kerr(M,J) is the universal final stage of collapse

- Frequencies (M,J) : all is known (1972 Teukolsky; 1985 Leaver)
- Amplitudes : estimates for detectability (1997 Flanagan-Hughes)

Plan

2. Black hole Perturbation theory

    2.1. Regge-Wheeler and Zerilli equations

    2.2. Quasi-normal modes of Schwarzschild - Black Hole Spectroscopy

    2.3. Newman-Penrose formalism, Petrov's classification, Teukolsky equation

    2.4. Quasi-normal modes of Kerr

    2.5. Mathisson-Papapetrou-Dixon theory

2.1. Regge-Wheeler and Zerilli equations

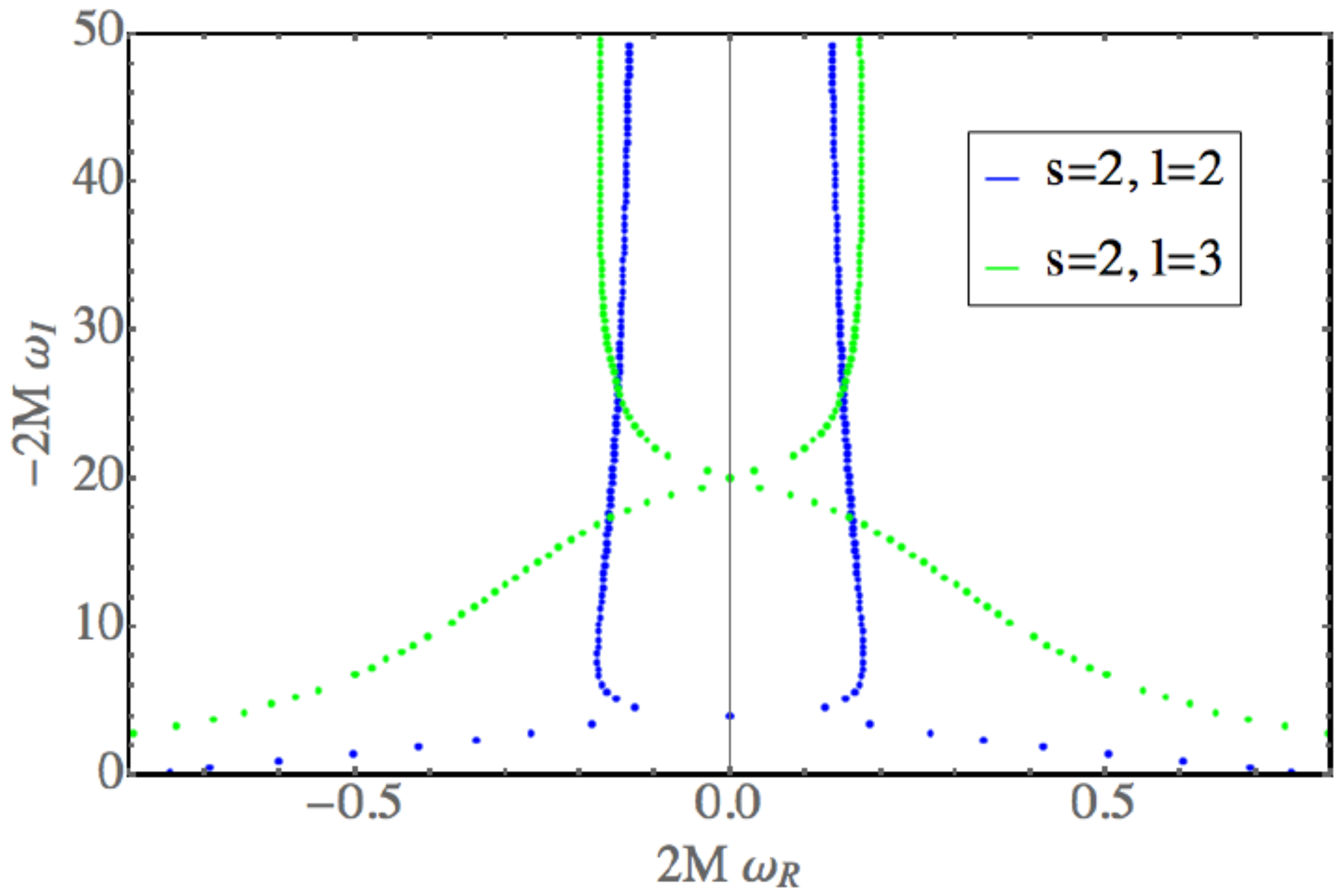
board

2.2. Quasi-normal modes of Schwarzschild

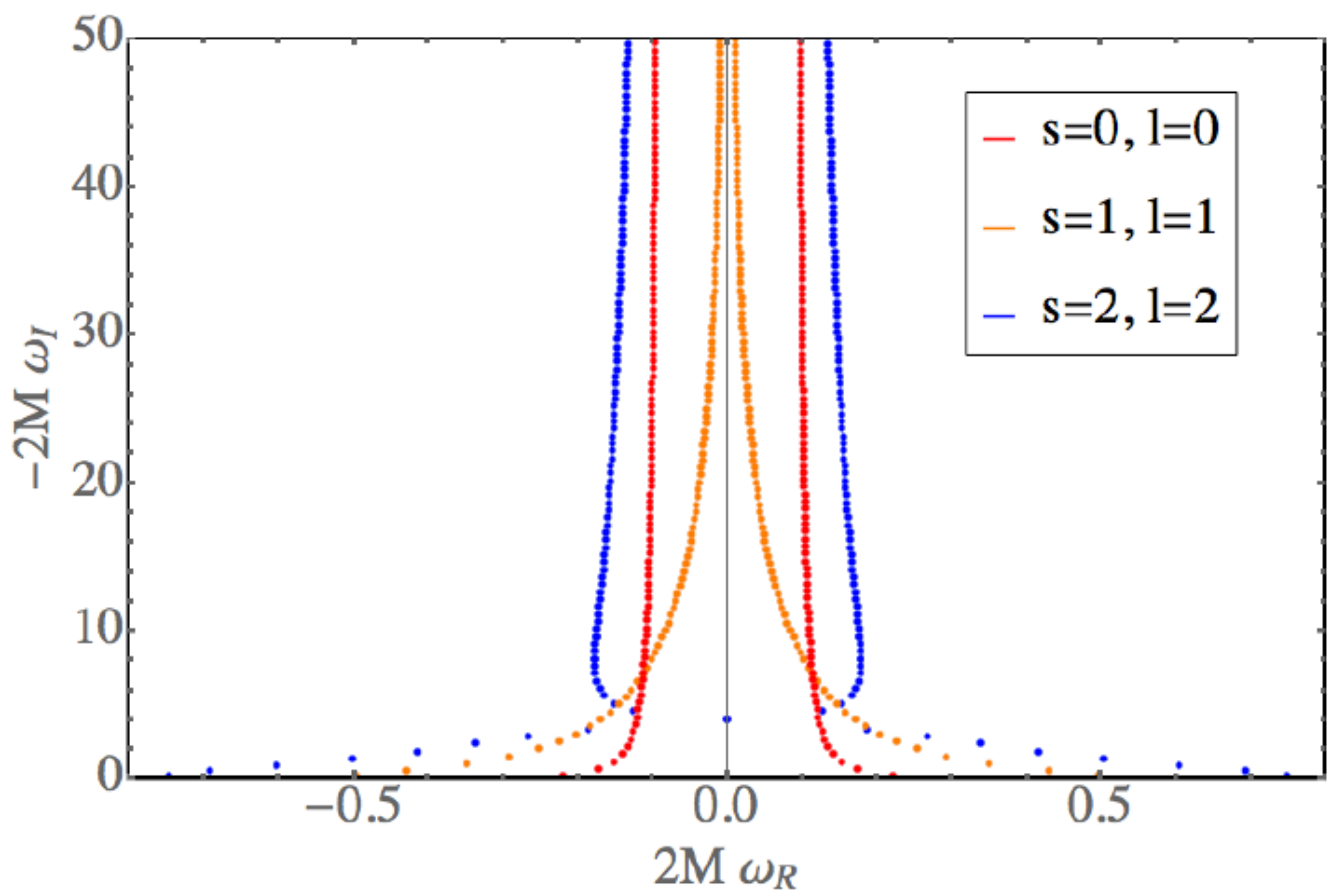
Black hole spectroscopy

board

Schwarzschild spectroscopy



(a) *Quasi-normal modes frequencies for gravitational perturbations* ($s = 2$).



(b) *Comparison of quasi-normal modes fundamental spectra* $l = |s|$ *for scalar, vector, and gravitational perturbations* ($s = 0, 1, 2$).

Most weakly damped s=2 mode $M\omega = 0.3737 - 0.0890i.$

Adapted from

E. Berti, V. Cardoso, and A. O. Starinets, "Quasinormal modes of black holes and black branes", *Class. Quant. Grav.* **26** (2009) 163001, arXiv:0905.2975 [gr-qc].

"E. Berti's homepage, Ringdown." https://pages.jh.edu/~eberti2/ringdown/.

2.3. Newman-Penrose formalism, Petrov's classification, Teukolsky equation

Newman-Penrose formalism

Tetrad : $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$

useful for

- Fermions
- Cartan formalism $e^a = e^a_\mu dx^\mu$

The Newman-Penrose formalism is a tetrad formalism with complex tetrads where the tangent space Minkowski metric η_{ab} is chosen at each point to be

$$\eta_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The tetrad frame is chosen to be a set of 4 *null vectors* $l_\mu, n_\mu, m_\mu, \bar{m}_\mu$ with

$$g_{\mu\nu} = -l_\mu n_\nu - n_\mu l_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu \,.$$

Null directions suitable to study GW propagation

Reformulation of the connection

4 ∇_μ ⟶

$$D = l^\mu \nabla_\mu \quad ; \quad \Delta = n^\mu \nabla_\mu \quad ; \quad \delta = m^\mu \nabla_\mu \quad ; \quad \bar{\delta} = \bar{m}^\mu \nabla_\mu .$$

24=4x6 $\Gamma^\mu_{\nu\rho}$ ⟶

$$\begin{aligned}
\kappa &= -m^\mu l^\nu \nabla_\nu l_\mu \, ; \qquad \sigma = -m^\mu m^\nu \nabla_\nu l_\mu \, ; \\
\lambda &= -n^\mu \bar{m}^\nu \nabla_\nu \bar{m}_\mu \, ; \qquad \nu = -n^\mu n^\nu \nabla_\nu \bar{m}_\mu \, ; \\
\rho &= -m^\mu \bar{m}^\nu \nabla_\nu l_\mu \, ; \qquad \mu = -n^\mu m^\nu \nabla_\nu \bar{m}_\mu \, ; \\
\tau &= -m^\mu n^\nu \nabla_\nu l_\mu \, ; \qquad \varpi = -n^\mu l^\nu \nabla_\nu \bar{m}_\mu \, ; \\
\epsilon &= -\frac{1}{2}(n^\mu l^\nu \nabla_\nu l_\mu + m^\mu l^\nu \nabla_\nu \bar{m}_\mu) \, ; \\
\gamma &= -\frac{1}{2}(n^\mu n^\nu \nabla_\nu l_\mu + m^\mu n^\nu \nabla_\nu \bar{m}_\mu) \, ; \\
\alpha &= -\frac{1}{2}(n^\mu \bar{m}^\nu \nabla_\nu l_\mu + m^\mu \bar{m}^\nu \nabla_\nu \bar{m}_\mu) \, ; \\
\beta &= -\frac{1}{2}(n^\mu m^\nu \nabla_\nu l_\mu + m^\mu m^\nu \nabla_\nu \bar{m}_\mu).
\end{aligned}$$

Reformulation of the curvature

$R_{\alpha\beta\mu\nu}$ $R_{\mu\nu}$

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - g_{\mu[\rho}R_{\sigma]\nu} + g_{\nu[\rho}R_{\sigma]\mu} + \frac{1}{3}Rg_{\mu[\rho}g_{\sigma]\nu}.$$

20 10 10

10 $W_{\alpha\beta\mu\nu}$ →

$$\Psi_0 = W_{\alpha\beta\gamma\delta}l^\alpha m^\beta l^\gamma m^\delta,$$
$$\Psi_1 = W_{\alpha\beta\gamma\delta}l^\alpha n^\beta l^\gamma m^\delta,$$
$$\Psi_2 = W_{\alpha\beta\gamma\delta}l^\alpha m^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_3 = W_{\alpha\beta\gamma\delta}l^\alpha n^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_4 = W_{\alpha\beta\gamma\delta}n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta.$$

5 Weyl-Newman-Penrose scalars

Reformulation of gauge invariance

$$\Psi_0 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta,$$
$$\Psi_1 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta,$$
$$\Psi_2 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_3 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_4 = W_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta.$$

They are diffeomorphism invariant but dependent upon the choice of tetrad.

One can perform flips $l^\mu \leftrightarrows n^\mu$ and local Lorentz transformation at each spacetime point (6 real functions = 3 complex functions).

- Rotations of type I which leave l^μ unchanged ($a \in \mathbb{C}$);

$$l^\mu \mapsto l^\mu, \qquad n^\mu \mapsto n^\mu + a^* m^\mu + a\bar{m}^\mu + aa^* l^\mu, \qquad m^\mu \mapsto m^\mu + a l^\mu \qquad \bar{m}^\mu \mapsto \bar{m}^\mu + a^* l^\mu.$$

- Rotations of type II which leave n^μ unchanged ($b \in \mathbb{C}$);

$$n^\mu \mapsto n^\mu, \qquad l^\mu \mapsto l^\mu + b^* m^\mu + b\bar{m}^\mu + bb^* n^\mu, \qquad m^\mu \mapsto m^\mu + b n^\mu, \qquad m^\mu \mapsto m^\mu + b^* n^\mu.$$

- Rotations of type III which leave the directions of l^μ and n^μ unchanged and rotate m^μ by an angle in the $m^\mu, \bar{m}^\mu$ plane ($A, \theta \in \mathbb{R}$);

$$l^\mu \mapsto A^{-1} l^\mu, \qquad n^\mu \mapsto A n^\mu, \qquad m^\mu \mapsto e^{i\theta} m^\mu \qquad \bar{m}^\mu \mapsto e^{-i\theta} \bar{m}^\mu.$$

$$\Psi_0 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta,$$
$$\Psi_1 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta,$$
$$\Psi_2 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_3 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_4 = W_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta.$$

Exercise

Prove that under ... we have

(i) Rotations of type I which leave l^μ unchanged ($a \in \mathbb{C}$);

$$l^\mu \mapsto l^\mu, \qquad n^\mu \mapsto n^\mu + a^* m^\mu + a\bar{m}^\mu + aa^* l^\mu, \qquad m^\mu \mapsto m^\mu + a l^\mu \qquad \bar{m}^\mu \mapsto \bar{m}^\mu + a^* l^\mu.$$

$$\Psi_0 \mapsto \Psi_0,$$
$$\Psi_1 \mapsto \Psi_1 + a^*\Psi_0,$$
$$\Psi_2 \mapsto \Psi_2 + 2a^*\Psi_1 + (a^*)^2\Psi_0,$$
$$\Psi_3 \mapsto \Psi_3 + 3a^*\Psi_2 + 3(a^*)^2\Psi_1 + (a^*)^3\Psi_0,$$
$$\Psi_4 \mapsto \Psi_4 + 4a^*\Psi_3 + 6(a^*)^2\Psi_2 + 4(a^*)^3\Psi_1 + (a^*)^4\Psi_0;$$

(ii) Rotations of type II which leave n^μ unchanged ($b \in \mathbb{C}$);

$$n^\mu \mapsto n^\mu, \qquad l^\mu \mapsto l^\mu + b^* m^\mu + b\bar{m}^\mu + bb^* n^\mu, \qquad m^\mu \mapsto m^\mu + b n^\mu, \qquad m^\mu \mapsto m^\mu + b^* n^\mu.$$

$$\Psi_0 \mapsto \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4,$$
$$\Psi_1 \mapsto \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4,$$
$$\Psi_2 \mapsto \Psi_2 + 2b\Psi_3 + b^2\Psi_4,$$
$$\Psi_3 \mapsto \Psi_3 + b\Psi_4,$$
$$\Psi_4 \mapsto \Psi_4;$$

(iii) Rotations of type III which leave the directions of l^μ and n^μ unchanged and rotate m^μ by an angle in the $m^\mu, \bar{m}^\mu$ plane ($A, \theta \in \mathbb{R}$);

$$l^\mu \mapsto A^{-1} l^\mu, \qquad n^\mu \mapsto A n^\mu, \qquad m^\mu \mapsto e^{i\theta} m^\mu \qquad \bar{m}^\mu \mapsto e^{-i\theta}\bar{m}^\mu.$$

$$\Psi_0 \mapsto A^2 e^{-2i\theta}\Psi_0,$$
$$\Psi_1 \mapsto A^{-1} e^{i\theta}\Psi_1,$$
$$\Psi_2 \mapsto \Psi_2,$$
$$\Psi_3 \mapsto A e^{-i\theta}\Psi_3,$$
$$\Psi_4 \mapsto A^2 e^{-2i\theta}\Psi_4.$$

Exercise

Remember

$$\Psi_0 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta,$$
$$\Psi_1 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta,$$
$$\Psi_2 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_3 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_4 = W_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta.$$

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$$

$$\eta_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We define the 3 anti-symmetric bivectors

$$X_{\mu\nu} = -2n_{[\mu}\bar{m}_{\nu]} \quad ; \quad Y_{\mu\nu} = 2l_{[\mu}m_{\nu]} \quad ; \quad Z_{\mu\nu} = 2m_{[\mu}\bar{m}_{\nu]} - 2l_{[\mu}n_{\nu]}.$$

Prove that the Weyl tensor is a linear combination of these 3 bivectors

$$W_{\alpha\beta\gamma\delta} = \Psi_0\, X_{\alpha\beta}X_{\gamma\delta} + \Psi_1\left(X_{\alpha\beta}Z_{\gamma\delta} + Z_{\alpha\beta}X_{\gamma\delta}\right) + \Psi_2\left(Y_{\alpha\beta}X_{\gamma\delta} + X_{\alpha\beta}Y_{\gamma\delta} + Z_{\alpha\beta}Z_{\gamma\delta}\right)$$
$$+ \Psi_3\left(Y_{\alpha\beta}Z_{\gamma\delta} + Z_{\alpha\beta}Y_{\gamma\delta}\right) + \Psi_4\, Y_{\alpha\beta}Y_{\gamma\delta} + c.c.$$

Exercise

Remember

$$\Psi_0 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta,$$
$$\Psi_1 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta,$$
$$\Psi_2 = W_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_3 = W_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta,$$
$$\Psi_4 = W_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta.$$

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$$

$$\eta_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We define the 3 anti-symmetric bivectors

$$X_{\mu\nu} = -2n_{[\mu}\bar{m}_{\nu]} \quad ; \quad Y_{\mu\nu} = 2l_{[\mu}m_{\nu]} \quad ; \quad Z_{\mu\nu} = 2m_{[\mu}\bar{m}_{\nu]} - 2l_{[\mu}n_{\nu]}.$$

Prove that the Weyl tensor is a linear combination of these 3 bivectors

$$W_{\alpha\beta\gamma\delta} = \Psi_0 X_{\alpha\beta}X_{\gamma\delta} + \Psi_1 \left(X_{\alpha\beta}Z_{\gamma\delta} + Z_{\alpha\beta}X_{\gamma\delta}\right) + \Psi_2 \left(Y_{\alpha\beta}X_{\gamma\delta} + X_{\alpha\beta}Y_{\gamma\delta} + Z_{\alpha\beta}Z_{\gamma\delta}\right) + \Psi_3 \left(Y_{\alpha\beta}Z_{\gamma\delta} + Z_{\alpha\beta}Y_{\gamma\delta}\right) + \Psi_4 Y_{\alpha\beta}Y_{\gamma\delta} + c.c.$$

Hint: compute

$X_{\gamma\delta} l^\gamma m^\delta$

$Y_{\gamma\delta} l^\gamma m^\delta$

$Z_{\gamma\delta} l^\gamma m^\delta$

$X_{\gamma\delta} \bar{m}^\gamma n^\delta$

$Y_{\gamma\delta} \bar{m}^\gamma n^\delta$

$Z_{\gamma\delta} \bar{m}^\gamma n^\delta$

Exercise

Use

$$W_{\alpha\beta\gamma\delta} = \Psi_0\, X_{\alpha\beta}X_{\gamma\delta} + \Psi_1\left(X_{\alpha\beta}Z_{\gamma\delta} + Z_{\alpha\beta}X_{\gamma\delta}\right) + \Psi_2\left(Y_{\alpha\beta}X_{\gamma\delta} + X_{\alpha\beta}Y_{\gamma\delta} + Z_{\alpha\beta}Z_{\gamma\delta}\right) + \Psi_3\left(Y_{\alpha\beta}Z_{\gamma\delta} + Z_{\alpha\beta}Y_{\gamma\delta}\right) + \Psi_4\, Y_{\alpha\beta}Y_{\gamma\delta} + c.c.$$

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$$

$$\eta_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

to prove that

$$l_{[\alpha}W_{\beta]\gamma\delta[\rho}l_{\sigma]}l^{\gamma}l^{\delta} = \Psi_0 l_{[\alpha}\bar{m}_{\beta]}l_{[\rho}\bar{m}_{\sigma]} + \Psi_0^* l_{[\alpha}m_{\beta]}l_{[\rho}m_{\sigma]}.$$

Petrov's Classification

[Petrov, 1954][Géhéniau, 1957]

Higher dimensions: 2004

Petrov classified the Weyl tensor by the number of degenerate local eigenvalues and (antisymmetric) eigenbivectors of the Weyl tensor. The eigenvalue equation reads as

$$W^{\mu\nu}{}_{\alpha\beta}X^{\alpha\beta} = \lambda\, X^{\mu\nu}.$$

Distinction: $G_{\mu\nu} \sim T_{\mu\nu}, \qquad W_{\mu\nu\alpha\beta}$

Classification of the Weyl tensor categorizes both vacuum and non-vacuum solutions to Einstein's equations

$W_{\mu\nu\alpha\beta}$ is conformally invariant $\longrightarrow$ $W_{\mu\nu\alpha\beta}$ identical for AdS and Minkowski

Three equivalent classifications

(1)

Petrov classified the Weyl tensor by the number of degenerate local eigenvalues and (antisymmetric) eigenbivectors of the Weyl tensor. The eigenvalue equation reads as

$$W^{\mu\nu}{}_{\alpha\beta} X^{\alpha\beta} = \lambda\, X^{\mu\nu}.$$

(2)

A non-trivial result due to Penrose in 1960 shows that solving this eigenvalue problem is equivalent to classify spacetimes according to the degeneracy of *principal null directions* of the Weyl tensor. Such directions are spanned by null vectors k^μ obeying

$$k_{[\alpha} W_{\beta]\gamma\delta[\rho} k_{\sigma]} k^\gamma k^\delta = 0.$$

(3)

Yet another equivalent formulation of the classification is the following. We have just seen that with respect to a chosen tetrad, the Weyl tensor is completely determined by the five Weyl-Newman-Penrose scalars. The third formulation of the classification consists in determining how many of these scalars can be made to vanish for a given spacetime by choosing a suitable orientation of the tetrad frame.

For the proof (1) <-> (2) : see [Stephani, Kramer, MacCallum, Hoenselaers, Herlt, 2004]

Petrov classification using formulation (3)

Type I

$$\Psi_0 \mapsto \Psi_0,$$
$$\Psi_1 \mapsto \Psi_1 + a^*\Psi_0,$$
$$\Psi_2 \mapsto \Psi_2 + 2a^*\Psi_1 + (a^*)^2\Psi_0,$$
$$\Psi_3 \mapsto \Psi_3 + 3a^*\Psi_2 + 3(a^*)^2\Psi_1 + (a^*)^3\Psi_0,$$
$$\Psi_4 \mapsto \Psi_4 + 4a^*\Psi_3 + 6(a^*)^2\Psi_2 + 4(a^*)^3\Psi_1 + (a^*)^4\Psi_0;$$

Type II

$$\Psi_0 \mapsto \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4,$$
$$\Psi_1 \mapsto \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4,$$
$$\Psi_2 \mapsto \Psi_2 + 2b\Psi_3 + b^2\Psi_4,$$
$$\Psi_3 \mapsto \Psi_3 + b\Psi_4,$$
$$\Psi_4 \mapsto \Psi_4;$$

Type III

$$\Psi_0 \mapsto A^2 e^{-2i\theta}\Psi_0,$$
$$\Psi_1 \mapsto A^{-1} e^{i\theta}\Psi_1,$$
$$\Psi_2 \mapsto \Psi_2,$$
$$\Psi_3 \mapsto A e^{-i\theta}\Psi_3,$$
$$\Psi_4 \mapsto A^2 e^{-2i\theta}\Psi_4.$$

Given a NP tetrad $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$ we compute the scalars $\{\Psi_i\}_{i=1,\dots,5}$

If Weyl vanishes, we are done. Let us assume it is not vanishing.

We assume $\Psi_4 \neq 0$. Otherwise do a Type I rotation.

We consider a Type II rotation with complex parameter b.

Ψ_0 can be made to vanish if b is a solution to

$$\Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 = 0.$$

There are exactly 4 roots. The corresponding Type II rotations lead to

$$n^\mu \mapsto n^\mu, \quad l^\mu \mapsto l^\mu + b^* m^\mu + b\bar{m}^\mu + bb^* n^\mu, \quad m^\mu \mapsto m^\mu + bn^\mu, \quad m^\mu \mapsto m^\mu + b^* n^\mu.$$

They define the 4 principal null directions of the Weyl tensor

Using $$l_{[\alpha}W_{\beta]\gamma\delta[\rho}l_{\sigma]}l^\gamma l^\delta = \Psi_0 l_{[\alpha}\bar{m}_{\beta]}l_{[\rho}\bar{m}_{\sigma]} + \Psi_0^* l_{[\alpha}m_{\beta]}l_{[\rho}m_{\sigma]}.$$

the principal null directions obey $l_{[\alpha}W_{\beta]\gamma\delta[\rho}l_{\sigma]}l^\gamma l^\delta = 0.$ This proves (3) -> (2)

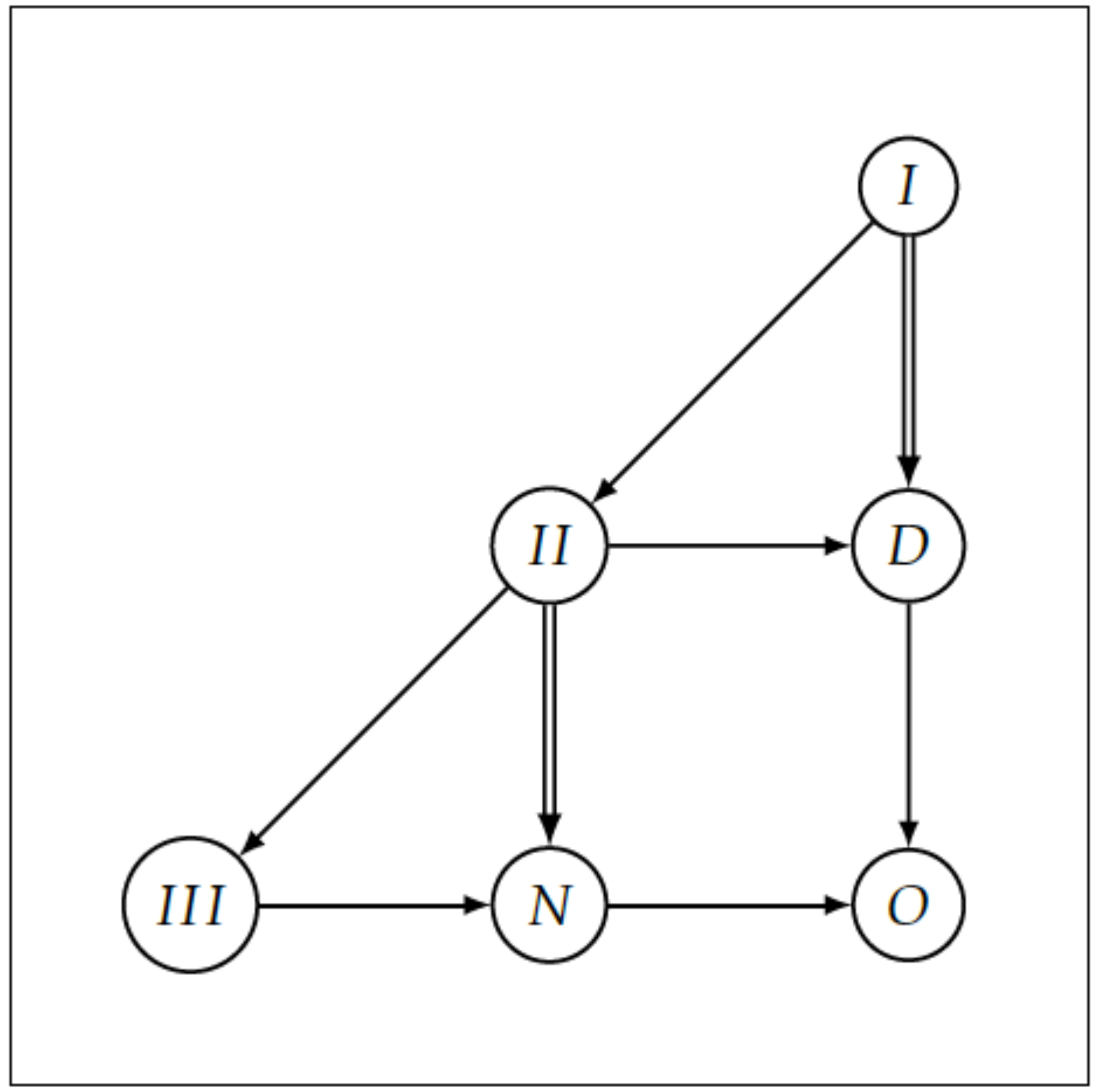
The degeneracy structure of the roots of a quartic polynomial lead to the classification.

Petrov's classification

Table and diagram adapted from
[Stephani, Kramer, MacCallum, Hoenselaers, Herlt, 2004]

Petrov type	Multiplicity of p.n.d.	Vanishing Weyl components	Criterion on $W_{\alpha\rho\sigma\beta}$
I	$(1,1,1,1)$	$\Psi_0 = 0$	$k_{[\gamma}W_{\alpha]\rho\sigma[\beta}k_{\delta]}k^\rho k^\sigma = 0$
II	$(2,1,1)$	$\Psi_0 = \Psi_1 = 0$	$k_{[\gamma}W_{\alpha]\rho\sigma\beta}k^\rho k^\sigma = 0$
D	$(2,2)$	$\Psi_0 = \Psi_1 = 0$	$k_{[\gamma}W_{\alpha]\rho\sigma\beta}k^\rho k^\sigma = 0$
III	$(3,1)$	$\Psi_0 = \Psi_1 = \Psi_2 = 0$	$k_{[\gamma}W_{\alpha]\rho\sigma\beta}k^\rho = 0$
N	(4)	$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$	$W_{\alpha\rho\sigma\beta}k^\alpha = 0$
O	$\varnothing$	$\Psi_i = 0, \forall i$	$W_{\alpha\rho\sigma\beta} = 0$

(a) *Characterisation of Petrov types. k^μ is always the most degenerate principal null direction (p.n.d.).*



(b) *The Penrose graph summarizing the degeneracy growth in Petrov's classification. Each arrow indicates one additional degeneracy.*

Kerr is Type D: it admits 2 distinct principal null directions.

A Newman-Penrose tetrad adapted to these null directions and such that Ψ_3, Ψ_4 vanish is called a Kinnersley tetrad. Only Ψ_2 is non-vanishing in a Kinnersley tetrad.

The Goldman-Sachs theorem implies that for a Type D spacetime, the principal null directions are shear-free geodesic congruences. The Newman-Penrose formalism is therefore well-adapted for the study of GW in Kerr!

Quasi-normal modes: definition

We consider linear perturbations of matter and the metric around the Kerr geometry.
All linear perturbations are collectively denoted as $\Phi^i(t,r,\theta,\phi)$

Thanks to the 2 Killing vectors of Kerr, there is no explicit time or angular dependence in the field equations. Therefore, the linear solution can be decomposed in isolated Fourier modes:

$$\Phi^i(t,r,\theta,\phi) = \frac{1}{2\pi}\int d\omega e^{-i\omega t}\sum_{m\in\mathbb{Z}} e^{im\phi}F^i(r,\theta)$$

The linear equations are partial differential equations in r,θ, which dependence upon (M,a=J/M) and m,ω

The boundary conditions are

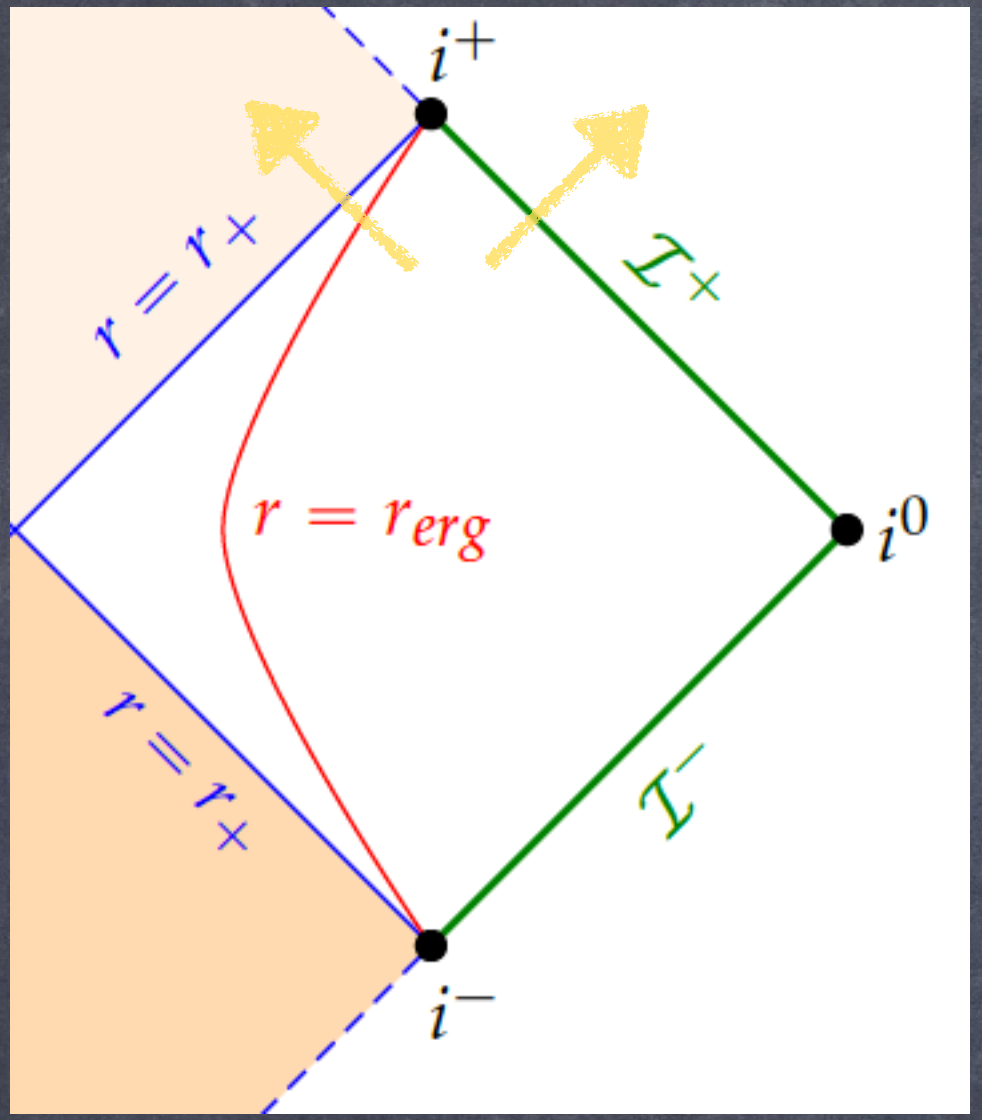
- "In" : Ingoing at the horizon $e^{-i\omega t+im\phi}F(r,\theta) \xrightarrow{r\to r_+} e^{-i\omega v_\star+im\phi_\star}F(\theta)$

where $r_\star$ is the *tortoise coordinate*, $v_\star = t + r_\star$ the advanced time and $\phi_\star$ the angular coordinate which define the regular ingoing Eddington-Finkelstein coordinates $v_\star, r_\star, \theta, \phi_\star$ (or in other words, which resolves the geometry near the horizon).

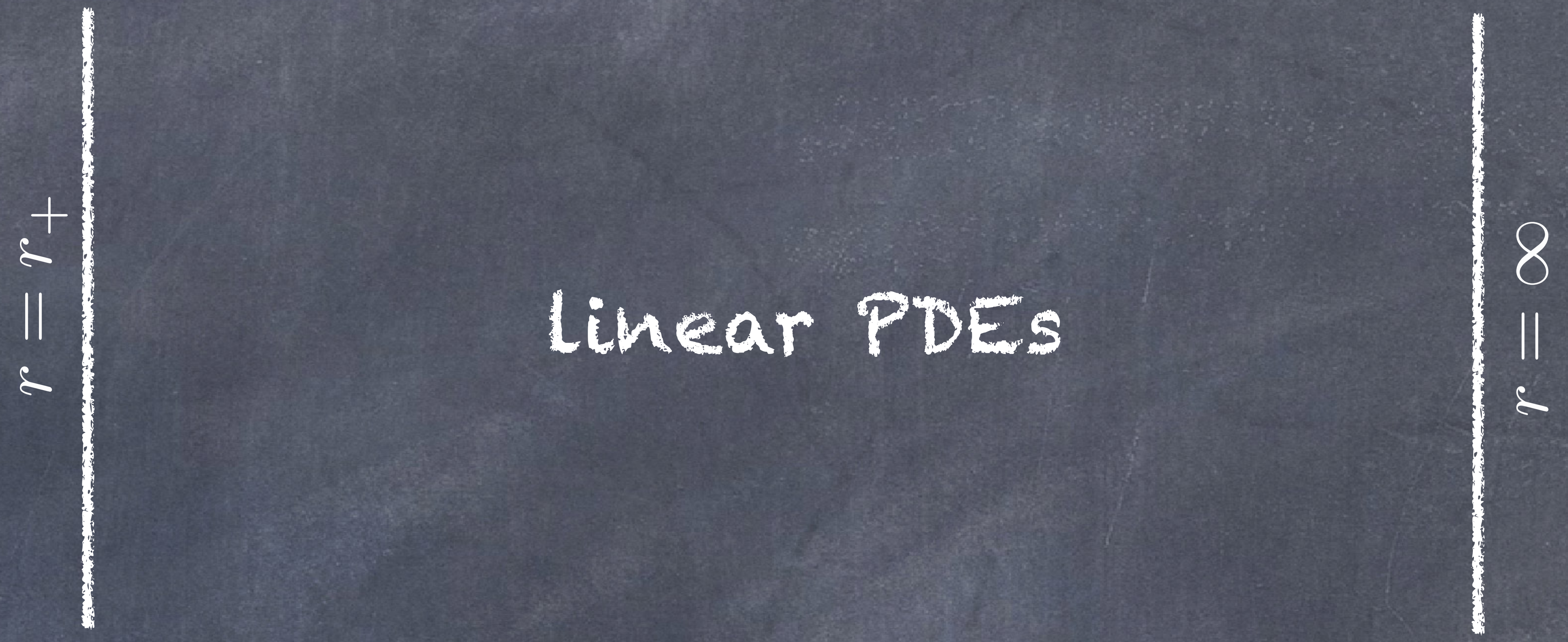
- "Up" : Outgoing at infinity $e^{-i\omega t+im\phi}F(r,\theta) \xrightarrow{r\to\infty,\, u \text{ fixed}} e^{-i\omega u+im\phi}\tilde{F}(\theta)$

where $u = t - r$ is the asymptotically flat retarded time.

Quasi-normal modes: qualitative solution



Penrose diagram of the domain of outer communication of Kerr



Radial interval at fixed Boyer-Linquist time

This is a boundary value problem. It admits an infinite set of solutions labelled by

– Spheroidal harmonic numbers l, m

– Overtone $N = 0, 1, 2, \ldots$

The frequencies are $\omega_{lmN} = \mathrm{Re}(\omega_{lmN}) + i\,\mathrm{Im}(\omega_{lmN})$ Linear stability is equivalent to $\mathrm{Im}(\omega_{lmN}) < 0$

Teukolsky equation

Teukolsky found during his PhD thesis with Kip Thorne in 1972 how to separate the radial and polar equations for the Weyl components.

He started to write down the linear perturbation equations in terms of the 5 Weyl-Newman-Penrose complex scalars for a Kinnersley tetrad. In Kerr (in Boyer-Linquist coordinates), only

$$\Psi_2 = -\frac{M}{(r - ia\cos\theta)^3}.$$

is non-vanishing. It turns out that the linear perturbations $\{\delta\Psi_i\}_i$ can be expressed in terms of either $\delta\Psi_0$ or $\delta\Psi_4$ up to the change of M,a (in $\delta\Psi_2$).

Insight: the equations for $(r - ia\cos\theta)^4\delta\Psi_4$ and $\delta\Psi_0$ are separable. We call it ψ for s=-2 or +2. It obeys

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\phi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\phi^2}$$
$$- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\phi}$$
$$- 2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = T$$

Radial and polar Teukolsky equations

$$\psi(t,r,\theta,\phi) = \frac{1}{2\pi}\int d\omega e^{-i\omega t}\sum_{l=|s|}^{\infty}\sum_{m=-l}^{+l} e^{im\phi}R^s_{lm\omega}(r)S^s_{lm\omega}(\cos\theta).$$

The equation for $S^s_{lm\omega}(\cos\theta)$ is called the *spin weighted spheroidal harmonic* equation

$$\left[\frac{d}{dx}(1-x^2)\frac{d}{dx}\right]S^s_{lm\omega}(x) + \left[a^2\omega^2x^2 - 2a\omega sx + \mathcal{E}^s_{lm\omega} - \frac{m^2+2msx+s^2}{1-x^2}\right]S^s_{lm\omega}(x) = 0$$

where $x = \cos\theta$ and $\mathcal{E}^s_{lm\omega}$ is the separation constant. When $a = 0$, the dependence in ω drops out and the functions $S^s_{lm}(\cos\theta)$ reduce to spin-weighted spherical harmonics $Y^s_{lm}(\theta,\phi) = S^s_{lm}(\cos\theta)e^{im\phi}$ after inclusion of the Fourier ϕ factor. In this case, the angular separation constants $\mathcal{E}^s_{lm\omega} = \mathcal{E}^s_{lm}$ are known analytically to be $\mathcal{E}^s_{lm} = l(l+1)$.

The radial equation is the *radial Teukolsky equation*:

$$\Delta^{-s}\frac{\partial}{\partial r}(\Delta^{s+1}\frac{\partial R_{lm\omega}}{\partial r}) - V(r)R_{lm\omega}(r) = T_{lm\omega}(r)$$

with source $T_{lm\omega}(r)$ and potential

$$\begin{aligned} V(r) &= -\frac{(K_{m\omega})^2 - 2si(r-M)K_{m\omega}}{\Delta} - 4si\omega r + \lambda_{\ell m\omega}, \\ K_{m\omega} &\triangleq (r^2+a^2)\omega - ma, \\ \lambda_{\ell m\omega} &\triangleq \mathcal{E}_{lm\omega} - 2am\omega + a^2\omega^2 - s(s+1). \end{aligned}$$

When $a = 0$, the m dependence drops out. This is a consequence of $SO(3)$ symmetry.

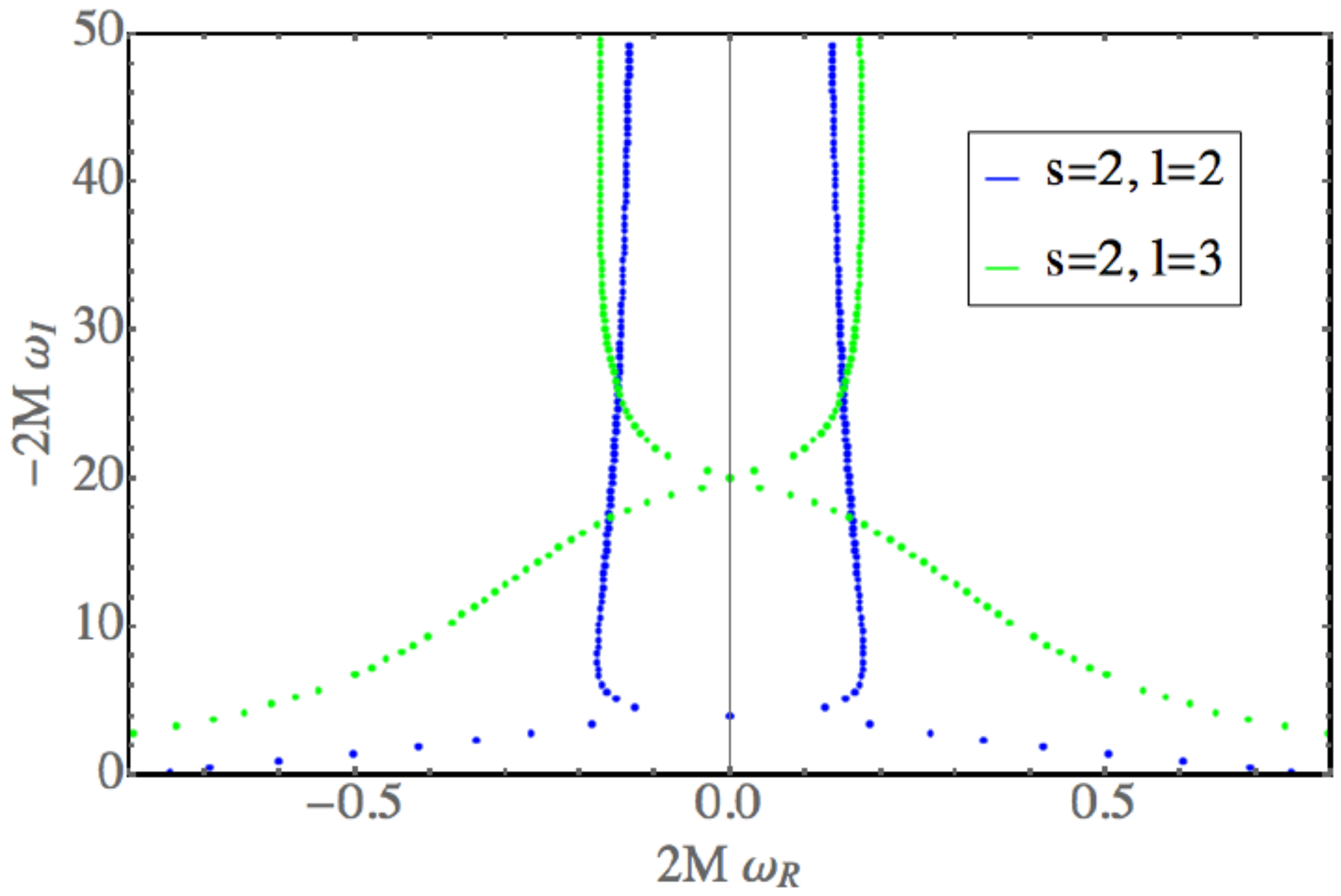
Radial and polar Teukolsky solutions

- Leaver method: continued fractions [1985]
- Any method that solves a boundary value ODE !
- Implementations in “Black Hole Perturbation Toolkit”

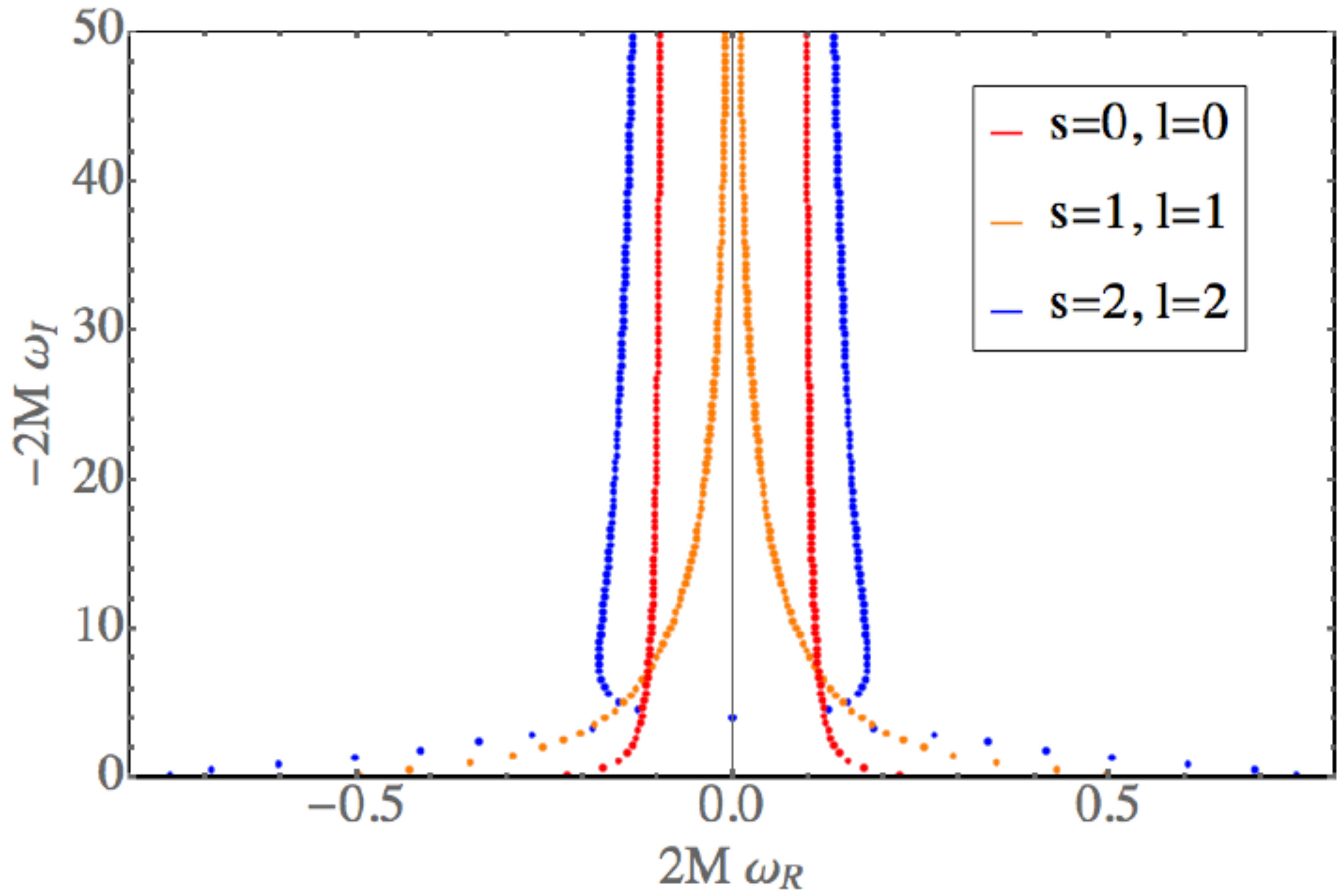
http://bhptoolkit.org/

- Mathematica 12.1. now implements HeunG which solves the Spin weighted spheroidal harmonic equation.

Schwarzschild spectroscopy



(a) *Quasi-normal modes frequencies for gravitational perturbations* ($s = 2$).



(b) *Comparison of quasi-normal modes fundamental spectra* $l = |s|$ *for scalar, vector, and gravitational perturbations* ($s = 0, 1, 2$).

Most weakly damped s=2 mode $M\omega = 0.3737 - 0.0890i.$

Adapted from

E. Berti, V. Cardoso, and A. O. Starinets, "Quasinormal modes of black holes and black branes", *Class. Quant. Grav.* **26** (2009) 163001, `arXiv:0905.2975 [gr-qc]`.

"E. Berti's homepage, Ringdown." `https://pages.jh.edu/~eberti2/ringdown/`.

Kerr spectroscopy

- Most weakly damped s=2 mode $M\omega_{020} \approx 0.4437 - 0.0739(1 - a/M)^{0.3350}$

- Zeeman splitting (dependence upon m)

- Highly spinning behavior (Split between non-damped and zero-damped modes with half-integer real frequency). This is due to the near-horizon Kerr region with angular velocity $M\Omega = \frac{1}{2}$

Video on

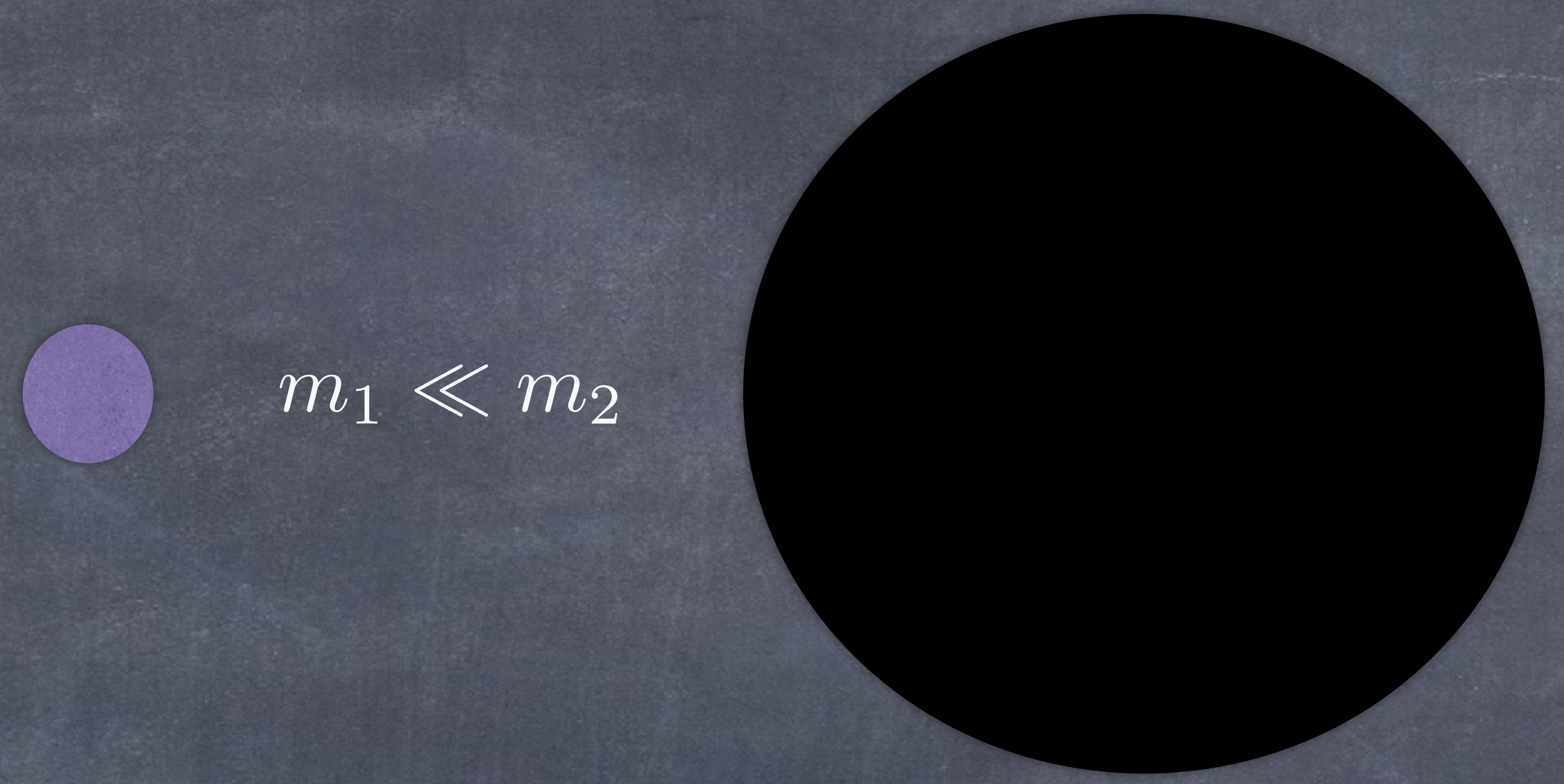
https://www.youtube.com/watch?v=lmXqtM4Ke9Q

[Cook, Zalutskiy, 1410.7698]

2.4. Mathisson-Papapetrou-Dixon theory

Or how to model finite objects without their gravitational backreaction?

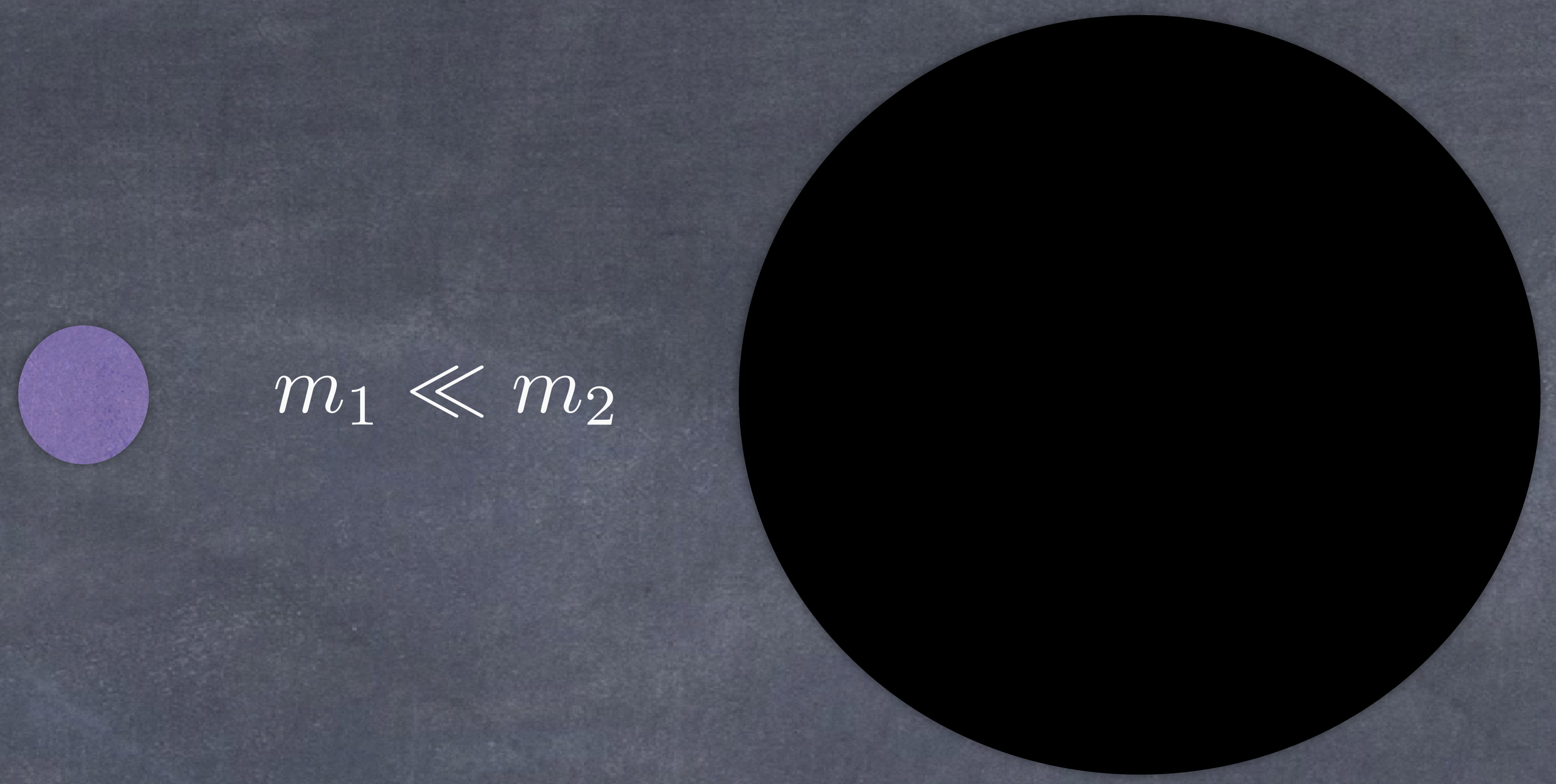
We consider the small ratio limit of the two-body problem



If body 1 is a point particle without any internal structure not gravitational backreaction, its motion is determined by geodesic motion in the metric generated by the body 2. [Einstein-Grommer, 1927]

This is not a postulate but a consequence of Einstein's equations. For a modern proof, see [Wald-Gralla, 2008]

This is very useful if the body 2 is a black hole because the metric is the Kerr metric depending only on M and J. It is also useful for a stationary metric like a neutron star, which is determined by two infinite sets of multipole moments.

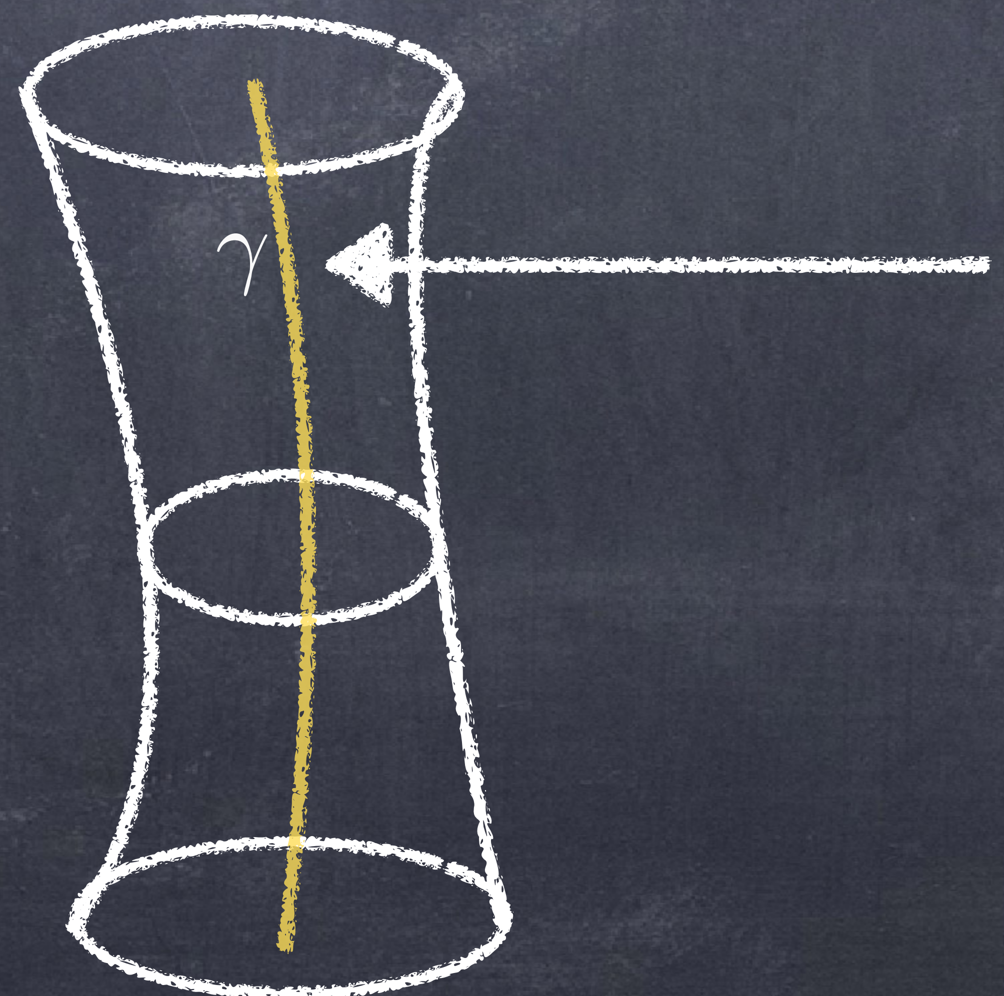


If body 1 is not a point particle but an extended object, a more general theory of motion is required

In addition, the body 1 gravitational backreacts and "gravitational self-force" corrections are also required.

In this lecture, we ignore the gravitational self-force. We assume that the background metric is the Kerr metric determined by the black hole (body 2).

We consider the space-time diagram



Worldline inside the object
(typically the center-of-mass)

Stress-tensor for timelike geodesics

$$S = -m \int_{-\infty}^{\infty} d\tau$$

$$d\tau = \sqrt{-g_{\mu\nu}(x_*) dx_*^\mu dx_*^\nu} = \frac{d\tau}{d\lambda} d\lambda$$

$$S = -m \int_{-\infty}^{\infty} d\lambda \frac{d\tau}{d\lambda} \delta^{(4)}(x - x_*)$$

$$= -m \int_{-\infty}^{\infty} d\lambda \sqrt{-g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \delta^{(4)}(x - x_*)$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}(x)}$$

$$= \frac{2}{\sqrt{-g}} (-m) \int_{-\infty}^{\infty} d\lambda \frac{1}{\frac{d\tau}{d\lambda}} \frac{1}{2} (-) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \delta^{(4)}(x - x_*)$$

$$u^\mu \equiv \frac{dx^\mu}{d\tau}$$

$$= m \int_{-\infty}^{\infty} \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

$$T^{\mu\nu} = m \int_{-\infty}^{\infty} \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}} u^\mu(\tau) u^\nu(\tau)$$

Exercise

Show that the conservation of

$$T^{\mu\nu} = m \int_{-\infty}^{\infty} d\tau \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}} u^\mu(\tau) u^\nu(\tau)$$

$$\nabla_\nu T^{\mu\nu}(x) = 0$$

is equivalent to the geodesic equation.

Hint: use $$\nabla_\nu T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} (\sqrt{-g} T^{\mu\nu}) + \Gamma^\mu_{\alpha\nu} T^{\alpha\nu}$$

Solution

$$\nabla_\nu T^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}(\sqrt{-g}T^{\mu\nu}) + \Gamma^\mu_{\alpha\nu}T^{\alpha\nu}$$

$$T^{\mu\nu} = m\int_{-\infty}^{\infty} d\tau \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}} u^\mu(\tau)u^\nu(\tau)$$

$$\nabla_\nu T^{\mu\nu} = m\int_{-\infty}^{\infty} \frac{d\tau}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\delta^{(4)}(x - x_*(\tau))u^\mu u^\nu + m\int_{-\infty}^{\infty} d\tau \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}}\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta$$

$$\frac{\partial}{\partial x^\mu}\delta^{(4)}(x - x_*(\tau)) = -\frac{\partial}{\partial x_*^\mu}\delta^{(4)}(x - x_*(\tau))$$

$$u^\mu(\tau)\frac{\partial}{\partial x^\mu}\delta^{(4)}(x - x_*(\tau)) = -\frac{d}{d\tau}\delta^{(4)}(x - x_*(\tau))$$

$$\nabla_\nu T^{\mu\nu} = m\int_{-\infty}^{\infty} \frac{d\tau}{\sqrt{-g}}\delta^{(4)}(x - x_*(\tau))\frac{d}{d\tau}u^\mu + m\int_{-\infty}^{\infty} d\tau \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}}\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta$$

$$\nabla_\nu T^{\mu\nu}(x) = 0 \quad \longleftrightarrow \quad \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0 \quad \longleftrightarrow \quad u^\nu \nabla_\nu u^\mu = 0$$

Extended objects contain additional multipoles:

- Spin
- Quadrupole
- 2^N pole

Their equation of motion follows from

$$\nabla_a T^{ab}_{body} = 0$$

$$P_t[\xi] = \int_{B_t} T^{ab}_{body}(x')\xi_a(x')dS_b$$

↑

??

Killing vector = Symmetry of spacetime

$$\mathcal{L}_\xi g_{\mu\nu} = 0 \qquad \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\rho\mu} = 0$$

By covariance: $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$

In Minkowski, using Cartesian coordinates, $(ct, \vec{x})$

$$\partial_{(\mu}\xi_{\nu)} = 0 \qquad \xi^\mu = a^\mu + b^{[\mu\nu]}x^\nu \qquad [\xi^\mu_{(a)}, \xi^\nu_{(b)}] = C^{(c)}_{(a)(b)}\xi^\mu_{(c)}$$

translations Lorentz

Poincaré algebra under the Lie bracket

In AdS_4/dS_4 (other two maximally symmetric spacetimes) -> 10 as well

$AdS_4 : \; SO(2,3)$ $\qquad$ $dS_4 : \; SO(1,4)$

Schwarzschild: $\mathbb{R} \times SO(3)$

Kerr: $\mathbb{R} \times SO(2) \times \mathbb{Z}_2$

Exercise

Prove that a Killing vector obeys

$$\nabla_a \nabla_b \xi_c = R_{cbad} \xi^d$$

using the Misner-Thorne-Wheeler/Wald convention $[\nabla_a, \nabla_b]\xi^c = R^c{}_{dab}\xi^d$

Solution

Prove that a Killing vector obeys

$$\nabla_a \nabla_b \xi_c = R_{cbad}\xi^d$$

using the Misner-Thorne-Wheeler/Wald convention $[\nabla_a, \nabla_b]\xi^c = R^c{}_{dab}\xi^d$

$$\nabla_a(\nabla_b\xi_c + \nabla_c\xi_b) = 0 \longrightarrow \nabla_a\nabla_b\xi_c + \nabla_c\nabla_a\xi_b + [\nabla_a, \nabla_c]\xi_b = 0$$

$$\nabla_a\nabla_b\xi_c + \nabla_c\nabla_a\xi_b + R_{bdac}\xi^d = 0 \quad (1)$$

$a \mapsto c$, $b \mapsto a$, $c \mapsto b$

$$\nabla_c\nabla_a\xi_b + \nabla_b\nabla_c\xi_a + R_{adcb}\xi^d = 0 \quad (2)$$

$a \mapsto c$, $b \mapsto a$, $c \mapsto b$

$$\nabla_b\nabla_c\xi_a + \nabla_a\nabla_b\xi_c + R_{cdba}\xi^d = 0 \quad (3)$$

(1)-(2)+(3) =

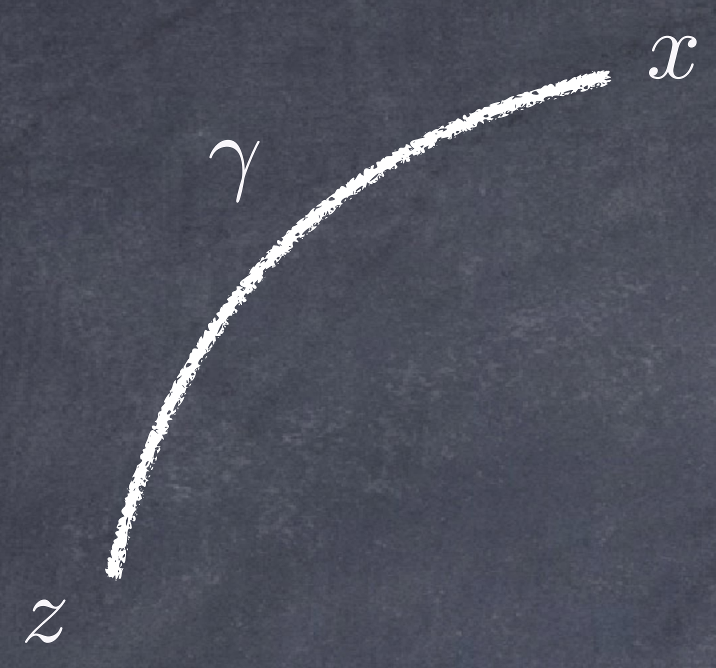
$$2\nabla_a\nabla_b\xi_c + (R_{bdac} - R_{adcb} + R_{cdba})\xi^d = 0$$

$$2\nabla_a\nabla_b\xi_c + (\underline{R_{acbd}} - R_{cbad} + \underline{R_{bacd}})\xi^d = 0$$

$$R_{cbad} + \underline{R_{bacd}} + \underline{R_{acbd}} = 0 \longrightarrow 2\nabla_a\nabla_b\xi_c + 2(-R_{cbad})\xi^d = 0$$

Tool 1: Killing transport along a curve

[Geroch, 1969]



Given $\xi_a(z)$, $L_{ab}(z)$ we can build $\xi_a(x)$, $L_{ab}(x)$ at a point x

by integrating along the curve the following ordinary differential equations,

$$v^a(x')\nabla_a \xi_b(x') = v^a(x') L_{ab}(z)$$
$$v^a(x')\nabla_a L_{bc}(x') = -R_{bcad}(x')\xi^d(z) v^a(x')$$

Tangent vector along the curve

$$v^a(x') = \frac{dx'^a}{d\lambda}$$

The equation is trivially obeyed at z. By construction, $L_{ab}(x') = L_{[ab]}(x')$ if $L_{(ab)}(x_0) = 0$ at any x_0

This defines $\xi_a(z)$, $L_{ab}(z)$ along the curve. There is also a relationship among these quantities along the curve,

which are compatible with the Killing equation.

Tool 2: Bitensors

[Poisson, Pound, Vega, 1102.0529]
Note: They use $\sigma(z,x)$ instead of $\sigma(x,z)$.

Although there is no vector which preserves geodesic distances between all pairs of points in a general spacetime, there are vectors which preserve geodesic distances from a given worldline.

$\sigma(z,x')$ Synge function = 1/2 square of geodesic distance between z and x'

$$\sigma(z,x') = \frac{1}{2}(\lambda_{x'} - \lambda_z) \underbrace{\int_{\lambda_z}^{\lambda_{x'}} g_{\mu\nu}(y) \frac{dy^\mu}{d\lambda} \frac{dy^\nu}{d\lambda}}_{\text{Constant: } \epsilon} d\lambda = \frac{1}{2}\epsilon(\lambda_{x'} - \lambda_z)^2$$

Definition invariant under reparametrizations $\lambda \mapsto a\lambda + b$

It is a bitensor: scalar with respect to both z and x.

We can define vectors/tensors with respect to each point:

$$\sigma_a = \nabla_a = \partial_a \sigma(z,x'), \qquad \sigma_{ab} = \nabla_b \nabla_a \sigma(z,x')$$
$$\sigma_{a'} = \nabla_{a'}\sigma = \partial_{a'} \sigma(z,x'), \qquad \sigma_{aa'} = \nabla_{a'} \nabla_a \sigma(z,x')$$

In the coincident limit, $z \mapsto x'$

$[\sigma] \equiv \lim_{z \mapsto x'} \sigma(z, x') = 0$ Geodesic distance between a point and itself is 0

$[\sigma_a] \equiv \lim_{z \mapsto x'} \sigma_a(z, x') = 0$ because no odd tensor exists.

$[\sigma_{a'}] = 0$

$[\sigma_{a'ab}] = 0$

...

$[\sigma_{ab}] = g_{ab}$

$[\sigma_{a'b'}] = g_{a'b'}$ We need to prove it

$[\sigma_{a'b}] = -g_{a'b}$

Exercise

Use

$$\sigma(z,x') = \frac{1}{2}(\lambda_{x'}-\lambda_z)\int_{\lambda_z}^{\lambda_{x'}} g_{\mu\nu}(y)\frac{dy^\mu}{d\lambda}\frac{dy^\nu}{d\lambda}d\lambda = \frac{1}{2}\epsilon(\lambda_{x'}-\lambda_z)^2$$

to prove that

(i) $$\sigma(z+\delta z,x') - \sigma(z,x') = -(\lambda_{x'}-\lambda_z)g_{\mu\nu}\dot{z}^\mu\delta z^\nu$$
$$\sigma(z,x'+\delta x') - \sigma(z,x') = +(\lambda_{x'}-\lambda_z)g_{\mu'\nu'}\dot{x}'^{\mu'}\delta x'^{\nu'}$$

(ii) $$g^{ab}\sigma_a\sigma_b = 2\sigma$$

(iii) $$\sigma^a(\sigma_{ab}-g_{ab}) = 0$$

(iv) $$[\sigma_{ab}] = g_{ab}$$

(v) $$[\sigma_{a'b}] = -g_{a'b}$$

Solution

[Poisson, Pound, Vega, 1102.0529, pg 35-37]

(i)

$$\sigma(z,x') = \frac{1}{2}(\lambda_{x'}-\lambda_z)\int_{\lambda_z}^{\lambda_{x'}} g_{\mu\nu}(y)\frac{dy^\mu}{d\lambda}\frac{dy^\nu}{d\lambda}d\lambda = \frac{1}{2}\epsilon(\lambda_{x'}-\lambda_z)^2$$

Adjust parameter such that $\Delta\lambda \equiv \lambda_{x'}-\lambda_z = \lambda_{x'}-\lambda_{z+\delta z}$

$$\delta\sigma(z,x') = \Delta\lambda\int_{\lambda_z}^{\lambda_{x'}}\left(g_{\mu\nu}(y)\frac{dy^\mu}{d\lambda}\frac{d\delta y^\nu}{d\lambda} + \frac{1}{2}\partial_\lambda g_{\mu\nu}\frac{dy^\mu}{d\lambda}\frac{dy^\nu}{d\lambda}\delta y^\lambda\right)d\lambda$$

$$\delta\sigma(z,x') = \Delta\lambda\left[g_{\mu\nu}\frac{dy^\mu}{d\lambda}\delta y^\nu\right]_z^{x'} - \Delta\lambda\int_{\lambda_z}^{\lambda_{x'}}(g_{\mu\nu}\frac{d^2y^\mu}{d\lambda^2} + \Gamma^\sigma_{\mu\nu}g_{\sigma\lambda}\frac{dy^\mu}{d\lambda}\frac{dy^\lambda}{d\lambda})d\delta y^\nu d\lambda$$

$$\sigma(z+\delta z,x') - \sigma(z,x') = -(\lambda_{x'}-\lambda_z)g_{\mu\nu}\dot{z}^\mu\delta z^\nu$$

$$\sigma(z,x'+\delta x') - \sigma(z,x') = +(\lambda_{x'}-\lambda_z)g_{\mu'\nu'}\dot{x}'^{\mu'}\delta x'^{\nu'}$$

(ii)

$$\sigma_a(z,x') = -(\lambda_{x'}-\lambda_z)g_{ab}\dot{z}^b$$

$$g^{ab}\sigma_a\sigma_b = \Delta\lambda^2\dot{z}^a g_{ab}\dot{z}^b = \Delta\lambda^2\epsilon = 2\sigma$$

(iii) $g^{ab}\sigma_a\sigma_b = 2\sigma$ $\rightarrow$ $2g^{ab}\sigma_{ac}\sigma_b = 2\sigma_c$ $\rightarrow$ $\sigma^a\sigma_{ac} = \sigma_c$ $\rightarrow$ $\sigma^a(\sigma_{ac} - g_{ac}) = 0$

(iv) $\sigma_a(z,x') = -(\lambda_{x'} - \lambda_z)g_{ab}\dot{z}^b$ $\rightarrow$ $\dot{z}^a(\sigma_{ac} - g_{ac}) = 0$ $\rightarrow$ $[\sigma_{ab}] = g_{ab}$

(v) $g^{ab}\sigma_a\sigma_b = 2\sigma$ $\rightarrow$ $g^{ab}\sigma_{ac'}\sigma_b = \sigma_{c'}$ $\rightarrow$ $\sigma^a\sigma_{ac'} = \sigma_{c'}$

$\sigma_{a'}(z,x') = +(\lambda_{x'} - \lambda_z)g_{a'b'}\dot{x}'^{b'}$ $\rightarrow$ $-\dot{z}^a\sigma_{ac'} = +\dot{x}_{c'}$

$\rightarrow$ $[\sigma_{a'b}] = -g_{a'b}$

Generalized Killing vector

[Harte, 2008]

We define a foliation $\{B_s\}_{s\in\mathbb{R}}$ around the curve γ

Given a pair $\begin{matrix}\xi_a(z)\\ L_{ab}(z)\end{matrix}$ along the curve γ we define ξ^a in the vicinity of γ as

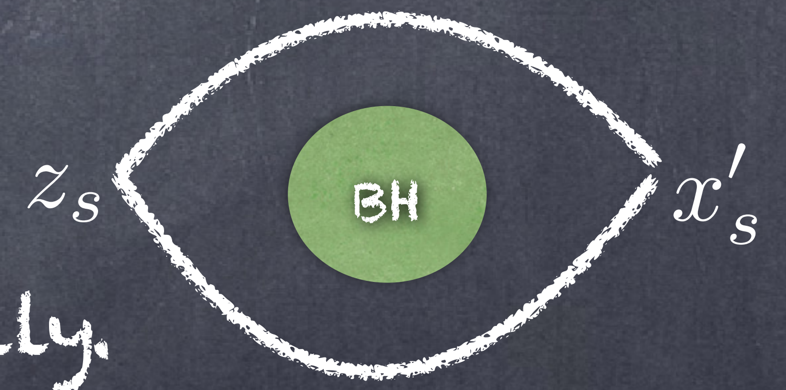
$$\mathcal{L}_\xi \sigma(z_s, x'_s) = 0 \qquad \forall z_s = \gamma \cap B_s \qquad \forall x'_s \in B_s$$

This vector exists. Proof: $\mathcal{L}_\xi \sigma(z_s, x'_s) = 0 \longrightarrow \mathcal{L}_\xi \sigma^a(z_s, x'_s) = 0$

$$\xi^{a'}(x'_s)\nabla_{a'}\sigma^a + \xi^b(z_s)\nabla_b\sigma^a - \nabla_b\xi^a(z_s)\sigma^b = 0$$

$$\xi^{a'}(x'_s)\nabla_{a'}\sigma^a = -\xi^b(z_s)\nabla_b\sigma^a + \nabla_b\xi^a(z_s)\sigma^b$$

Define the inverse $H^{b'}{}_a(x'_s, z_s)$ such that $H^{b'}{}_a(x'_s, z_s)\nabla_{a'}\sigma^a(z_s, x'_s) = -\delta^{b'}_{a'}$



This inverse exists and is unique locally around γ . Due to caustics, it is not unique globally.

$$\xi^{b'}(x'_s) = H^{b'}{}_a(x'_s, z_s)\left(\xi^b(z_s)\nabla_b\sigma^a - \nabla_b\xi^a(z_s)\sigma^b\right)$$

$$\xi^{a'}(x') = \Xi^{a'a}(x', z)\xi_a(z) + \Xi^{a',ab}(x', z)\nabla_a\xi_b(z)$$

$$\Xi^{a'a}(x', z) = H^{a'}_b(x', z)\sigma^{ba}(z, x')$$

$$\Xi^{a',ab}(x', z) = -H^{a'b}(x', z)\sigma^a(z, x')$$

We can write

$$\xi^{a'}(x') = \Xi^{a'a}(x',z)\xi_a(z) + \Xi^{a',ab}(x',z)\nabla_a\xi_b(z)$$

$$H^{b'}{}_a(x'_s,z_s)\nabla_{a'}\sigma^a(z_s,x'_s) = -\delta^{b'}_{a'}$$

$$\Xi^{a'a}(x',z) = H^{a'}{}_b(x',z)\sigma^{ba}(z,x')$$

$$\Xi^{a',ab}(x',z) = -H^{a'b}(x',z)\sigma^a(z,x')$$

equivalently by

$$\xi^{a'}(x') = \Xi^{a'a}(x',z)\xi_a(z) + \Xi^{a',ab}(x',z)L_{ab}(z)$$

Indeed, in the coincident limit, $z \mapsto x'$

$$[\sigma^a{}_{a'}] = -\delta^a_{a'} \quad [H^{b'}{}_a] = \delta^{b'}_a \quad [\sigma^{ba}] = g^{ba} \quad [\sigma^a] = 0$$

$$\longrightarrow \quad [\Xi^{a'a}] = g^{a'a} \quad [\Xi^{a',ab}] = 0 \quad \longrightarrow \quad \xi^{a'}(x') = \xi^{a'}(x')$$

$$\nabla_{b'}\xi^{a'}(x') = \nabla_{b'}\Xi^{a'a}(x',z)\xi_a(z) + \nabla_{b'}\Xi^{a',ab}(x',z)L_{ab}(z)$$

$$[\nabla_{b'}\Xi^{a'a}] = 0 \quad [\sigma^a] = 0 \quad [\nabla_{b'}\Xi^{a',ab}] = -[H^{a'b}][\sigma^a{}_{b'}] = \delta^{a'b}\delta^a_{b'}$$

$$\longrightarrow \quad \nabla_{a'}\xi_{b'}(x') = L_{a'b'}(x')$$

Finally, using the property of Killing transport $L_{ab}(z) = L_{[ab]}(z)$ after choosing $\nabla_{(a}\xi_{b)}(z_0) = 0$ for any $z_0 \in \gamma$

$$\xi^{a'}(x') = \Xi^{a'a}(x',z)\xi_a(z) + \Xi^{a',[ab]}(x',z)L_{ab}(z)$$

[Harte, private communication, 2020]

Exercise

Check that the Killing vectors of Minkowski spacetime obey

$$\xi^{a'}(x') = \Xi^{a'a}(x',z)\xi_a(z) + \Xi^{a',[ab]}(x',z)\nabla_a\xi_b(z)$$

with

$$\Xi^{a',a}(x',z) = \delta^{a',a}$$
$$\Xi^{a',ab}(x',z) = (x'-z)^{[a}\delta^{b]a'}$$

Solution

Check that the Killing vectors of Minkowski spacetime obey

$$\xi^{a'}(x') = \Xi^{a'a}(x',z)\xi_a(z) + \Xi^{a',[ab]}(x',z)\nabla_a\xi_b(z)$$

with

$$\Xi^{a',a}(x',z) = \delta^{a',a}$$
$$\Xi^{a',ab}(x',z) = (x'-z)^{[a}\delta^{b]a'}$$

Explicitly,

$$\xi^a(z) = A^a + B^{[ab]}z^b$$

$$\xi^{a'}(x') = A^a + B^{[a'b']}x'^{b'} = A^a + B^{[a'b']}z^{b'} + (x'^{b'} - z^{b'})B^{a'b'}$$

Therefore,

$$\xi^{a'}(x') = \xi^{a'}(z) + (x'-z)^a\nabla_a\xi^{a'}(z)$$

Generalized Killing charges

$$P_t[\xi] = \int_{B_t} T^{a'b'}_{body}(x')\xi_{a'}(x')dS_{b'}$$

Using the decomposition

$$\xi^{a'}(x') = \Xi^{a'a}(x', z)\xi_a(z) + \Xi^{a',[ab]}(x', z)\nabla_a \xi_b(z)$$

we obtain

$$P_t[\xi] = p^a(z_t, t)\xi_a(z_t) + \frac{1}{2}S^{ab}(z_t, t)\nabla_{[a}\xi_{b]}(z_t)$$

where

$$p^a(z_t, t) = \int_{B_t} T^{a'b'}_{body}(x')\Xi_{a'}{}^{a}(x', z_t)dS_{b'}$$

is the momentum vector with respect to the reference worldline at foliation time t

$$S^{ab}(z_t, t) = \int_{B_t} T^{a'b'}_{body}(x')\Xi_{a'}{}^{ab}(x', z_t)dS_{b'}$$

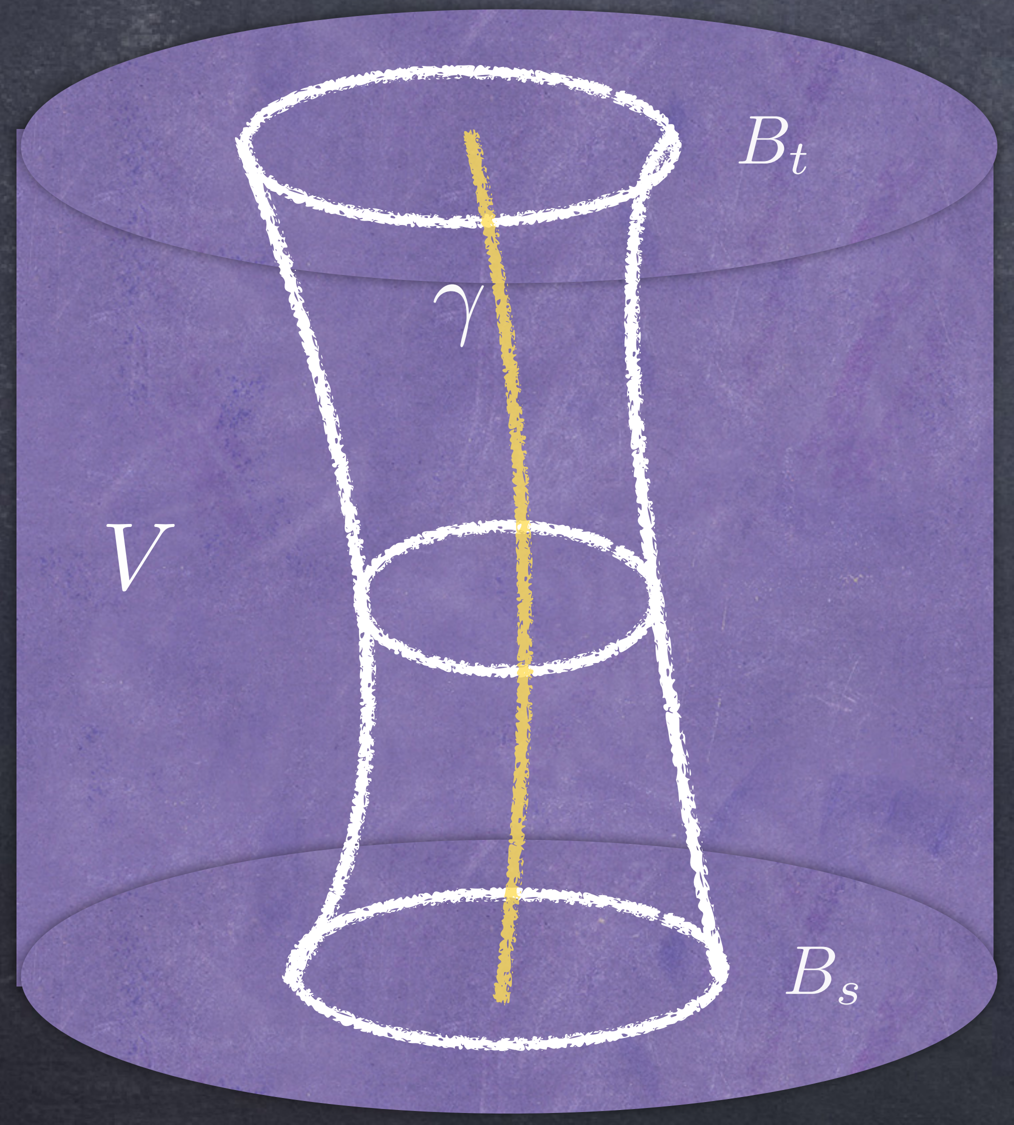
is the Lorentz charge tensor with respect to the reference worldline at foliation time t

Flux-balance law

The current $J^a \equiv T^{ab}_{body}(x)\xi_b(x)$ obeys $\nabla_a J^a = T^{ab}_{body}(x)\nabla_a\xi_b(x) = \frac{1}{2}T^{ab}_{body}\mathcal{L}_\xi g_{ab}$

Stokes' theorem implies

$$\int_V \sqrt{-g}\nabla_a J^a = \int_V \partial_a(\sqrt{-g}J^a) = \int_{\partial V}\sqrt{-g}J^a dS_a$$



$$\int_s^t dt' \int_{B_{t'}} d^3x\sqrt{-g}T^{ab}_{body}\mathcal{L}_\xi g_{ab} = P_t[\xi] - P_s[\xi]$$

Infinitesimally,

$$\frac{d}{dt}P_t[\xi] = \int_{B_t} d^3x\sqrt{-g}T^{ab}_{body}\mathcal{L}_\xi g_{ab} \equiv F_t[\xi]$$

These are the Mathisson-Papapetrou-Dixon equations

Mathisson-Papapetrou-Dixon equations

$$\frac{d}{dt}P_t[\xi] = \int_{B_t} d^3x\sqrt{-g}T^{ab}\mathcal{L}_\xi g_{ab} \equiv F_t[\xi] \qquad P_t[\xi] = p^a(z_t,t)\xi_a(z_t) + \frac{1}{2}S^{ab}(z_t,t)\nabla_{[a}\xi_{b]}(z_t)$$

Using the decomposition:

$$\frac{Dp^a}{Dt}\xi_a(z_t) + p^{[a}\dot{z}_t^{b]}\nabla_b\xi_a(z_t) + \frac{1}{2}\frac{DS^{ab}}{Dt}\nabla_a\xi_b(z_t) + \frac{1}{2}S^{ab}\dot{z}_t^c\nabla_c\nabla_a\xi_b(z_t) = F^a\xi_a(z_t) + \frac{1}{2}N_{ab}\nabla_{[a}\xi_{b]}(z_t)$$

use

$$\nabla_c\nabla_a\xi_b = -R_{bacd}\xi^d$$

True independently for $\xi_a(z_t),\ \nabla_{[a}\xi_{b]}(z_t)$

We obtain

$$\frac{Dp^a}{Dt} = \frac{1}{2}S^{bc}R_{bcd}{}^a\dot{z}_t^d + F^a$$

$$\frac{DS^{ab}}{Dt} = 2p^{[a}\dot{z}_t^{b]} + N^{ab}$$

Final variables :

$p^a,\ S^{ab},\ \dot{z}_t^a \equiv v^a,\ J^{abcd}, \dots$

The MPD equations arise because a generalized Killing vector which is purely translational at a point becomes a combination of translations and Lorentz transformations at a later time under Killing transport. This leads to a mixing of momenta and Lorentz charges.

If ξ^a is a background exact Killing vector, $\frac{d}{dt}P_t[\xi] = \int_{B_t} d^3x\sqrt{-g}T^{ab}\mathcal{L}_\xi g_{ab} \equiv F_t[\xi]$

Then $P_t[\xi] = p^a(z_t,t)\xi_a(z_t) + \frac{1}{2}S^{ab}(z_t,t)\nabla_{[a}\xi_{b]}(z_t)$ is exactly conserved. In Kerr, there are 2 conserved quantities, as for geodesics. Analogue of Carter's constant??

In order to close the system of equations

$$\frac{Dp^a}{Dt} = \frac{1}{2}S^{bc}R_{bcd}{}^a\dot{z}_t^d + F^a$$

$$\frac{DS^{ab}}{Dt} = 2p^{[a}\dot{z}_t^{b]} + N^{ab}$$

we need equations of state to define the stresses and torques, as well as the "spin supplementary conditions" that fix the center-of-mass:

Tulczyjew: $S^{ab}p_b = 0$ (Other option: Mathisson: $S^{ab}v_b = 0$)

Proposition: If t is proper time, we can solve for $\dot{z}_t^a$ in terms of p^a, S^{ab}, F^a, N^{ab} [Steinhoff, Puetzfeld, PRD 86, 044033 (2012)]

We can then define the intrinsic angular momentum: $S_a = \frac{1}{2\mu}\epsilon_{abcd}p^bS^{cd}$, $\mu^2 \equiv -p^ap_a$

As well as the spin length: $S^2 = \frac{1}{2}S_{ab}S^{ab} = S^aS_a$

Exercise

Show that the Mathisson-Papapetrou equations reduced with the Tulczyjew condition where t is proper time

$$\frac{Dp^a}{Dt} = \frac{1}{2} S^{bc} R_{bcd}{}^{a} v^d$$

$$\frac{DS^{ab}}{Dt} = 2p^{[a}v^{b]}$$

$$S^{ab}p_b = 0$$

lead to

$$v^a = N\left[\frac{p^a}{\mu} + \frac{1}{2\mu^2\Delta} S^{ab}\frac{p^c}{\mu} R_{bcde} S^{de}\right]$$

where

$$\Delta = 1 + \frac{1}{4\mu^2} R_{abcd} S^{ab} S^{cd}$$

$$N^{-2} = 1 - \frac{1}{4\Delta^2\mu^4} S_{ab} p_c S_{de} R^{bcde} S^{af} p^g S^{hi} R_{fghi}$$

Hint: prove $R_{abcd}S^{ae}S^{bf} = \frac{1}{2} R_{abcd} S^{ab} S^{ef}$

First derived by [Ehlers, Rudolph, GRG 8, 197 (1977)]

Solution

$$\frac{Dp^a}{Dt} = \frac{1}{2}S^{bc}R_{bcd}{}^{a}v^d$$
$$\frac{DS^{ab}}{Dt} = 2p^{[a}v^{b]}$$

$$\frac{D}{Dt}(S^{ab}p_b) = 0 \longrightarrow v^a = -\frac{p_b v^b}{\mu^2}p^a - \frac{1}{2\mu^2}S^{ab}S^{cd}R_{cdfb}v^f$$

$$v^a \equiv \frac{N}{\mu}p^a + B^a{}_f v^f, \qquad B^a{}_f \equiv -\frac{1}{2\mu^2}S^{ab}S^{cd}R_{cdfb}$$

$$\longrightarrow v^a = \frac{N}{\mu}\left(p^a + B^a{}_f p^f\right) + B^a{}_f B^f{}_g v^g$$

We will solve for N later.

$$S^{ab}p_b = 0 \longrightarrow S^{ab} = \epsilon^{abcd}p_c S_d, \qquad p^a S_a = 0 \longrightarrow \epsilon_{abce}S^{ab}S^{cd} \sim (p_c S_e - p_e S_c)S^{cd} = 0$$

$$\longrightarrow S^{[ab}S^{c]d} = 0 \xrightarrow{R_{(abc)d}=0} R_{abcd}S^{ae}S^{bf} = \frac{1}{2}R_{abcd}S^{ab}S^{ef} \longrightarrow B^a{}_f B^f{}_g v^g = \frac{B^b{}_b}{2}B^a{}_f v^f$$

$$\longrightarrow v^a = \frac{N}{\mu}\left(p^a + B^a{}_f p^f \sum_{n=0}^{\infty}\frac{(B^b{}_b)^n}{2^n}\right) = \frac{N}{\mu}\left(p^a + \frac{B^a{}_f p^f}{1 - B^b{}_b/2}\right)$$

$v^a v_a = -1 \longrightarrow$ Deduce N

Note: instead of proper time, one could fix $v^a p_a = -\mu$ which results in another N [Dixon, 1970]

see e.g. [Susuki, Maeda, PRD 55(8), 4848 (1997)][Santos, Batista, PRD 101, 104049 (2020)]

Exercise

Using the Mathisson-Papapetrou equations reduced with the Tulczyjew condition

$$\frac{Dp^a}{Dt} = \frac{1}{2} S^{bc} R_{bcd}{}^a v^d$$
$$\frac{DS^{ab}}{Dt} = 2p^{[a} v^{b]}$$

$$S^{ab} p_b = 0$$

$$v^a = N \left[\frac{p^a}{\mu} + \frac{1}{2\mu^2 \Delta} S^{ab} \frac{p^c}{\mu} R_{bcde} S^{de} \right]$$

prove that $\mu^2 \equiv -p_a p^a$ as well as $S^2 \equiv \frac{1}{2} S_{ab} S^{ab}$ are conserved.

Solution

Using the Mathisson-Papapetrou equations reduced with the Tulczyjew condition

$$\frac{Dp^a}{Dt} = \frac{1}{2} S^{bc} R_{bcd}{}^{a} v^d \qquad S^{ab} p_b = 0 \qquad v^a = N \left[\frac{p^a}{\mu} + \frac{1}{2\mu^2 \Delta} S^{ab} \frac{p^c}{\mu} R_{bcde} S^{de} \right]$$

$$\frac{DS^{ab}}{Dt} = 2 p^{[a} v^{b]}$$

prove that $\mu^2 \equiv -p_a p^a$ as well as $S^2 \equiv \frac{1}{2} S_{ab} S^{ab}$ are conserved.

We have $S_{ab} \frac{DS^{ab}}{dt} = 0$ because $R_{abcd} p^c p^d = 0$

$p_a \frac{Dp^a}{dt} = 0$ because $X_a S^{ab} X_b = 0$ $\qquad X_a \equiv R_{abcd} p^b S^{cd}$

Stress-energy tensor

"Skeletization": reduce the stress-tensor to its lowest multipole moments on the worldline: [Mathisson, 1937][Schwartz, Théorie des distributions, 1950] [Tulczyjew, 1959] [Dixon, 1973]

$$T^{ab}(x) = \int_{-\infty}^{\infty} d\tau \left[p^{(a} v^{b)} \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}} - \nabla_c \left(S^{c(a} v^{b)} \frac{\delta^{(4)}(x - x_*(\tau))}{\sqrt{-g}} \right) \right]$$

$$v^a \equiv \frac{dx_*^a(\tau)}{d\tau}$$

Exercise

Prove that $\nabla_b T^{ab} = 0$ is equivalent to the Mathisson-Papapetrou equations

$$\frac{Dp^a}{Dt} = \frac{1}{2} S^{bc} R_{bcd}{}^a v^d$$

$$\frac{DS^{ab}}{Dt} = 2p^{[a} v^{b]}$$

Trick: use an arbitrary test vector ϕ_a

$$0 = \int d^4x \sqrt{-g} \nabla_a (T^{ab} \phi_b) = \int d^4x \sqrt{-g} \nabla_a T^{ab} \phi_b + \int d^4x \sqrt{-g} T^{ab} \nabla_{(a} \phi_{b)}$$

Trick: use an arbitrary test vector ϕ_a

$$0=\int d^4x\sqrt{-g}\nabla_a(T^{ab}\phi_b)=\int d^4x\sqrt{-g}\nabla_a T^{ab}\phi_b+\int d^4x\sqrt{-g}T^{ab}\nabla_{(a}\phi_{b)}$$

Therefore, conservation of the stress-tensor is equivalent to

$$0=\int d^4x\sqrt{-g}T^{ab}\nabla_{(a}\phi_{b)}=\int d\tau\left(p^{(a}v^{b)}\nabla_a\phi_b+S^{c(a}v^{b)}\nabla_c\nabla_a\phi_b\right)$$

Use $p^{(a}v^{b)}=p^{[a}v^{b]}+v^ap^b,\quad S^{ca}\nabla_c\nabla_a\phi_b=\frac{1}{2}S^{ca}R_{cab}{}^d\phi_d \qquad v^a\nabla_a=\frac{D}{D\tau}$ and integrate by parts to get

$$\int d^4x\sqrt{-g}T^{ab}\nabla_{(a}\phi_{b)}=-\frac{1}{2}\int d\tau\left[2(\frac{Dp^a}{D\tau}-\frac{1}{2}S^{bc}v^dR_{bcd}{}^a)\phi_a+(\frac{DS^{ab}}{D\tau}-2p^{[a}v^{b]})\nabla_{[a}\phi_{b]}\right]$$

The two test $\phi_a,\qquad \nabla_{[a}\phi_{b]}$ are independent on the worldine:

$$\tilde{\phi}_a=\phi_a+\nabla_a\chi,\qquad \nabla_{[a}\phi_{b]}=\nabla_{[a}\tilde{\phi}_{b]}$$

Fermi normal coordinates

$$\chi=\alpha(t)+\beta_i(t)x^i$$

This is therefore equivalent to

$$\frac{Dp^a}{Dt}=\frac{1}{2}S^{bc}R_{bcd}{}^av^d$$
$$\frac{DS^{ab}}{Dt}=2p^{[a}v^{b]}$$

[Special thanks to Harte, private communication]

Remarks

- Quadrupole and higher multipoles are not constrained by stress-tensor conservation, but by the internal dynamics [Dixon,1980]
- Dynamics of a spinning particle in Schwarzschild or Kerr is chaotic [Susuki,Maeda,1997][Hartl,2003]